Optimal arrangements of $2d$ lines in $\mathbb{C}^d$

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tl;dr

Weak $d \times 2d$ conjecture

For every d, there exists a $d \times 2d$ **equiangular tight frame** (ETF)

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Weak $d \times 2d$ conjecture

For every d, there exists a $d \times 2d$ **equiangular tight frame** (ETF)

This talk:

- two new infinite families
- conjectural construction for all d
- existence for all $d \leq 162$

Outline

1: Background

$d \times 2d$ ETF exists

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**		**12**	**13**	**14**	**15**	**16**
	18	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**		**30**	**31**	**32**
33	**34**		**36**	**37**	**38**		**40**	**41**	**42**		**44**	**45**	**46**		**48**
49	**50**	**51**	**52**		**54**	**55**	**56**	**57**	**58**		**60**	**61**	**62**	**63**	**64**
	66		**68**	**69**	**70**		**72**		**74**	**75**	**76**		**78**	**79**	**80**
	82		**84**	**85**	**86**	**87**	**88**		**90**	**91**	**92**		**94**		**96**
97	**98**	**99**	**100**		**102**		**104**		**106**		**108**		**110**		**112**
113	**114**	**115**	**116**	**117**	**118**		**120**	**121**	**122**		**124**		**126**		**128**
129	**130**		**132**		**134**	**135**	**136**		**138**	**139**	**140**	**141**	**142**		**144**
145	**146**	**147**	**148**		**150**		**152**		**154**		**156**	**157**	**158**	**159**	**160**

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49	**50**	**51**	**52**	□	**54**	**55**	**56**	**57**	**58**	□	**60**	**61**	**62**	**63**	**64**
□	**66**	□	**68**	**69**	**70**	□	**72**	□	**74**	**75**	**76**	□	**78**	**79**	**80**
□	**82**	□	**84**	**85**	**86**	**87**	**88**	□	**90**	**91**	**92**	□	**94**	□	**96**
97	**98**	**99**	**100**	□	**102**	□	**104**	□	**106**	□	**108**	□	**110**	□	**112**
113	**114**	**115**	**116**	**117**	**118**	□	**120**	**121**	**122**	□	**124**	□	**126**	□	**128**
129	**130**	□	**132**	□	**134**	**135**	**136**	□	**138**	**139**	**140**	**141**	**142**	□	**144**
145	**146**	**147**	**148**	□	**150**	□	**152**	□	**154**	□	**156**	**157**	**158**	**159**	**160**

□ = previously unknown

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2: Doubling ETFs

$d \times 2d$ ETF exists

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	18	19	20	21	22	23	24	25	26	27	28		30	31	32
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52		54	55	56	57	58	59	60	61	62	63	64
	66	67	68	69	70	71	72		74	75	76		78	79	80
	82	83	84	85	86	87	88		90	91	92		94	95	96
97	98	99	100		102	103	104		106	107	108		110	111	112
113	114	115	116	117	118	119	120	121	122	123	124		126	127	128
129	130	131	132		134	135	136		138	139	140	141	142	143	144
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17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
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49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72	73	74	75	76		78	79	80
81	82	83	84	85	86	87	88	89	90	91	92		94	95	96
97	98	99	100	101	102	103	104		106	107	108	109	110	111	112
113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128
129	130	131	132		134	135	136	137	138	139	140	141	142	143	144
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$d \times 2d$ ETF exists

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17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
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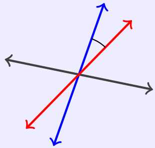
Problem

Pack n lines (1-dim subspaces) in $\mathbb{R}^d$ or $\mathbb{C}^d$ without sharp angles



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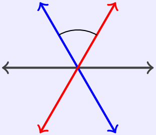

Good

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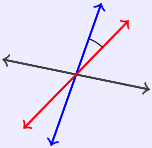

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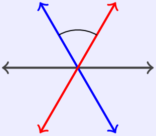

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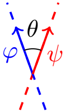


$\cos\theta = |\langle \varphi, \psi \rangle|$

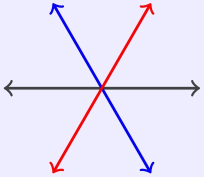
- Given lines, choose unit norm reps

$$\Phi = \begin{bmatrix} | & & | \\ \varphi_1 & \cdots & \varphi_n \\ | & & | \end{bmatrix} \in \mathbb{F}^{d \times n}$$

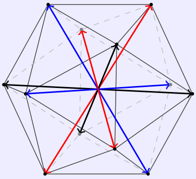
- To avoid sharp angles, minimize **coherence**

$$\mu = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|$$

Some optimal line packings



Real 2×3



Real 3×6

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 \\ 1 & 0 & -1 & \omega & 0 & -\omega^2 & \omega^2 & 0 & -\omega \\ -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 & 0 \end{bmatrix}, \quad \omega = e^{2\pi i/3}$$

Complex 3×9

How do you know it's optimal?

Theorem (Welch bound)

For n unit vectors $\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix}$ in $\mathbb{R}^d$ or $\mathbb{C}^d$,

$$\mu := \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

Equality holds iff Φ is an **equiangular tight frame** (ETF):

- Equiangular: $|\langle \varphi_i, \varphi_j \rangle| = \mu$ for all $i \neq j$
- Tight frame: $\Phi\Phi^* = \mathrm{const} \cdot I$

Welch, IEEE Trans. Inform. Theory, 1974

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equiangular tight frame (ETF) $\Longrightarrow$ optimal line packing

Welch, IEEE Trans. Inform. Theory, 1974

Certifying optimality

Example: Complex 3×9

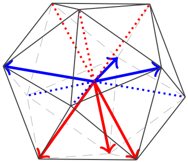
$$\Phi := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 \\ 1 & 0 & -1 & \omega & 0 & -\omega^2 & \omega^2 & 0 & -\omega \\ -1 & 1 & 0 & -\omega^2 & \omega & 0 & -\omega & \omega^2 & 0 \end{bmatrix}, \qquad \omega = e^{2\pi i/3}$$

$$\Phi^*\Phi = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & -1 & -\omega & -\omega^2 & -1 & -\omega^2 & -\omega \\ -1 & 2 & -1 & -\omega^2 & -1 & -\omega & -\omega & -1 & -\omega^2 \\ -1 & -1 & 2 & -\omega & -\omega^2 & -1 & -\omega^2 & -\omega & -1 \\ -1 & -\omega & -\omega^2 & 2 & -\omega^2 & -\omega & -1 & -1 & -1 \\ -\omega^2 & -1 & -\omega & -\omega & 2 & -\omega^2 & -1 & -1 & -1 \\ -\omega & -\omega^2 & -1 & -\omega^2 & -\omega & 2 & -1 & -1 & -1 \\ -1 & -\omega^2 & -\omega & -1 & -1 & -1 & 2 & -\omega & -\omega^2 \\ -\omega & -1 & -\omega^2 & -1 & -1 & -1 & -\omega^2 & 2 & -\omega \\ -\omega^2 & -\omega & -1 & -1 & -1 & -1 & -\omega & -\omega^2 & 2 \end{bmatrix}$$

Equiangular

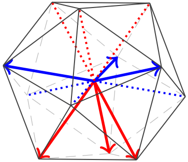
$$\Phi\Phi^* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Tight Frame



Equiangular tight frame $\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} \in \mathbb{F}^{d \times n}$

$$\Phi\Phi^* = \tfrac{n}{d} I, \qquad |\langle \varphi_i, \varphi_j \rangle| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$



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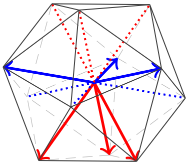
$$\Updownarrow$$

Gram matrix $G = \Phi^*\Phi \in \mathbb{F}^{n\times n}$

$$G = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{5} & - & - & + & - & - \\ - & \sqrt{5} & - & - & + & - \\ - & - & \sqrt{5} & - & - & + \\ + & - & - & \sqrt{5} & + & + \\ - & + & - & + & \sqrt{5} & + \\ - & - & + & + & + & \sqrt{5} \end{bmatrix}$$



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$$\sigma(\Phi\Phi^*) = \{ \tfrac{n}{d} \}$$

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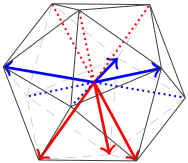
$$G^* = G, \qquad \sigma(G) = \{ \tfrac{n}{d}, 0 \},$$



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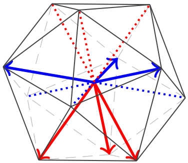

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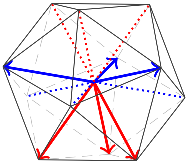
$$\Updownarrow$$

$$S = \begin{bmatrix} 0 & - & - & + & - & - \\ - & 0 & - & - & + & - \\ - & - & 0 & - & - & + \\ + & - & - & 0 & + & + \\ - & + & - & + & 0 & + \\ - & - & + & + & + & 0 \end{bmatrix}$$

Signature matrix $S = \frac{1}{\mu}(G - I) \in \mathbb{F}^{n\times n}$



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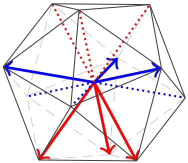

Equiangular tight frame $\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} \in \mathbb{F}^{d\times n}$

$$\Phi\Phi^* = \tfrac{n}{d} I, \qquad |\langle \varphi_i, \varphi_j \rangle| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$

$$\sigma(\Phi\Phi^*) = \{\tfrac{n}{d}\}$$

$$\Updownarrow$$

$$G = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{5} & - & - & + & - & - \\ - & \sqrt{5} & - & - & + & - \\ - & - & \sqrt{5} & - & - & + \\ + & - & - & \sqrt{5} & + & + \\ - & + & - & + & \sqrt{5} & + \\ - & - & + & + & + & \sqrt{5} \end{bmatrix}$$

Gram matrix $G = \Phi^*\Phi \in \mathbb{F}^{n\times n}$

$$G^* = G, \qquad \sigma(G) = \{\tfrac{n}{d}, 0\}, \qquad |G_{ij}| = \begin{cases} 1 & i = j \\ \mu & i \neq j \end{cases}$$

$$G^2 = \tfrac{n}{d} G$$

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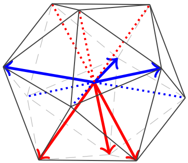

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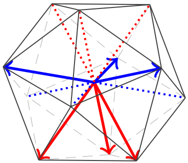

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Conference matrix $C \in \mathbb{R}^{n \times n}$:

- $C_{ij} \in \{1, -1\}$ for every $i \neq j$
- $C_{ii} = 0$ for every i
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Theorem

Conference matrix of order $2d \implies d \times 2d$ ETF

Delsarte, Goethals & Seidel, Philips Res. Repts., 1975

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Proof: $S := \begin{cases} iC & \text{if } C^\top = -C \\ C & \text{if } C^\top = C \end{cases}$

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$$\Phi = \frac{1}{\sqrt{3}} \begin{bmatrix} i\sqrt{3} & 1 & 1 & 1 \\ 0 & \sqrt{2} & \omega\sqrt{2} & \overline{\omega}\sqrt{2} \end{bmatrix}$$

$$\omega = e^{2\pi i/3}$$

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Theorem (Paley)

$$\begin{aligned} q \text{ odd prime power} &\implies \text{conference matrix of order } q+1 \\ &\implies d \times 2d \text{ ETF, } 2d = q+1 \end{aligned}$$

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Construction: Legendre $\chi\colon \mathbb{F}_q \to \{0, 1, -1\}$

$q = 3$

a	0	1	2
$\chi(a)$	0	$+$	$-$

$$\chi(a) = \begin{cases} 0 & a = 0 \\ 1 & a \neq 0 \text{ is a square} \\ -1 & \text{else} \end{cases}$$

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$$C := \begin{bmatrix} 0 & \mathbf{1}^\top \\ \pm\mathbf{1} & Q \end{bmatrix} \in \mathbb{R}^{(q+1)\times(q+1)}$$

$d \times 2d$ ETF exists

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**		**12**	**13**	**14**	**15**	**16**
	18	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**		**30**	**31**	**32**
33	**34**		**36**	**37**	**38**		**40**	**41**	**42**		**44**	**45**	**46**		**48**
49	**50**	**51**	**52**		**54**	**55**	**56**	**57**	**58**		**60**	**61**	**62**	**63**	**64**
	66		**68**	**69**	**70**		**72**		**74**	**75**	**76**		**78**	**79**	**80**
	82		**84**	**85**	**86**	**87**	**88**		**90**	**91**	**92**		**94**		**96**
97	**98**	**99**	**100**		**102**		**104**		**106**		**108**		**110**		**112**
113	**114**	**115**	**116**	**117**	**118**		**120**	**121**	**122**		**124**		**126**		**128**
129	**130**		**132**		**134**	**135**	**136**		**138**	**139**	**140**	**141**	**142**		**144**
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Outline

1: Background

2: Doubling ETFs

3: Doubling conference graphs

4: 2-circulant ETFs

$d \times 2d$ ETF exists

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	18	19	20	21	22	23	24	25	26	27	28		30	31	32
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52		54	55	56	57	58	59	60	61	62	63	64
	66	67	68	69	70	71	72		74	75	76		78	79	80
	82	83	84	85	86	87	88	89	90	91	92		94	95	96
97	98	99	100		102	103	104		106	107	108		110	111	112
113	114	115	116	117	118	119	120	121	122	123	124		126	127	128
129	130	131	132		134	135	136		138	139	140	141	142	143	144
145	146	147	148		150	151	152		154	155	156	157	158	159	160

☐ = previously unknown

Recall

Two options for conference matrix of order $n > 2$:

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Inspiration

If $C \in \mathbb{R}^{n \times n}$ is a skew-symmetric conference matrix, then so is

$$K := \begin{bmatrix} C & C+I \\ C-I & -C \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

Wallis, Bull. Austral. Math. Soc., 1971

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Check:

$$K^\top = \begin{bmatrix} C^\top & C^\top - I \\ C^\top + I & -C^\top \end{bmatrix} = \begin{bmatrix} -C & -C-I \\ -C+I & C \end{bmatrix} = -K$$

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$$\begin{aligned} KK^\top &= \begin{bmatrix} C & C+I \\ C-I & -C \end{bmatrix} \begin{bmatrix} C^\top & C^\top - I \\ C^\top + I & -C^\top \end{bmatrix} \\ &= \cdots \\ &= \begin{bmatrix} (2n-1)I & 0 \\ 0 & (2n-1)I \end{bmatrix} \end{aligned}$$

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For ETF signature matrices, doubling $S := iC \mapsto iK$ maps

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Key idea

This generalizes for other ETF signature matrices

Wallis, Bull. Austral. Math. Soc., 1971

ETF Doubling Theorem (Fallon, Iverson)

$$d \times n \text{ ETF}, \; n \in \{2d, 2d \pm 1\} \implies n \times 2n \text{ ETF}$$

via signature matrices

$$S \mapsto \begin{bmatrix} S & S + \beta I \\ S + \overline{\beta} I & -S \end{bmatrix} =: \Sigma, \qquad \beta = \beta(d, n) \in \mathbb{T}$$

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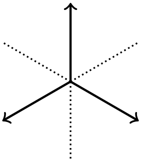
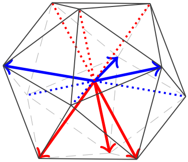


$\mapsto$



2×3 $\qquad$ 3×6

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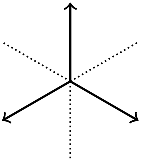
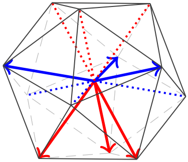


$\mapsto$



$\mapsto$ $\cdots$

2×3 $\quad$ 3×6 $\quad$ 6×12

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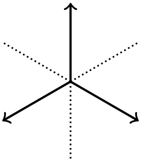
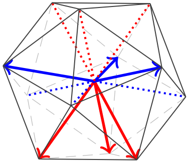


2×3

$\mapsto$



3×6

$\mapsto$ $\cdots$

6×12

$d \times 2d$ ETF $\implies$ $D \times 2D$ ETF, $D = 2^k d$

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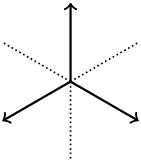

2×3

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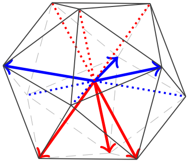

3×6

$\mapsto$ $\cdots$

6×12

$$d \times 2d \text{ ETF} \implies D \times 2D \text{ ETF}, \; D = 2^k d$$

- Sufficient: $d \times 2d$ ETF for all **odd** d

An ETF to double

Theorem (Strohmer/Renes)

Skew-conference matrix order $n \implies$ ETF size $\frac{n-2}{2} \times (n-1)$

Strohmer, Linear Algebra Appl., 2008

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Conference $\mapsto$ ETF signature:

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An ETF to double

Theorem (Strohmer/Renes)

Skew-conference matrix order $n \implies$ ETF size $\frac{n-2}{2} \times (n-1)$

Conference $\mapsto$ ETF signature:

$$C = \left[\begin{array}{c|ccc} 0 & + & + & + \\ \hline - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{array}\right]$$

Strohmer, Linear Algebra Appl., 2008
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☐ = previously unknown

Skew-conference $C \in \mathbb{R}^{n \times n}$:

- $C_{ij} \in \{+, -\}$ for every $i \neq j$
- $CC^\top = (n-1)I$
- $C^\top = -C$

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Skew-Hadamard conjecture

There is a skew-Hadamard matrix of every order divisible by 4

Wallis, J. Combinatorial Theory Ser. B, 1971

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Implications for $d \times 2d$ ETFs:

- all even d
- all $d \equiv 3 \bmod 4$

Wallis, J. Combinatorial Theory Ser. B, 1971

Outline

1: Background

2: Doubling ETFs

3: Doubling conference graphs

4: 2-circulant ETFs

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Recall

Two options for conference matrix of order $n > 2$:

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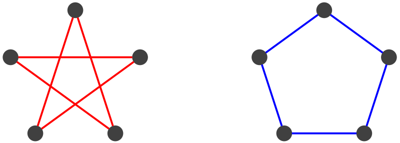

Sym conference mat order n $\iff$ **Conference graph** order $n-1$

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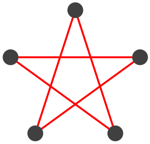
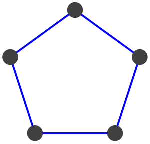
- adj verts:
  $\lambda = \frac{n-6}{2}$ common neighbors
- non-adj verts:
  $\mu = \frac{n-2}{4}$ common neighbors





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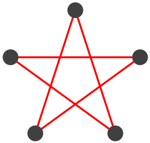

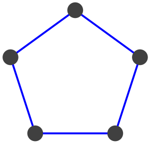

$$A^2 = \frac{n-6}{4}A + \frac{n-2}{4}B + \frac{n-2}{2}I$$

$$B^2 = \frac{n-2}{4}A + \frac{n-6}{4}B + \frac{n-2}{2}I$$

$$AB = BA = \frac{n-2}{4}A + \frac{n-2}{4}B$$

Conference Graph Doubling Theorem (Iverson, Jasper, M)

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Conference graph adj mults: $\quad S^2 = \cdots = (2n-2)I$

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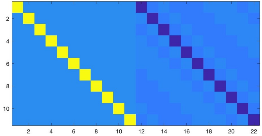
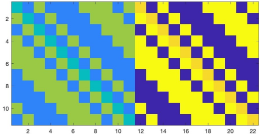
Inspiration: Results of doubling

11×22

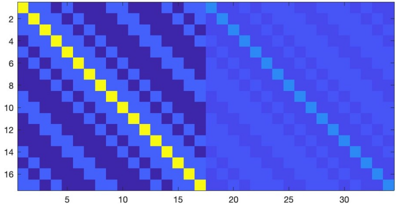


real part



imaginary part

17×34



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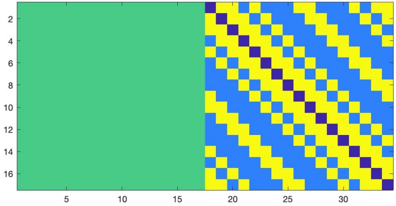

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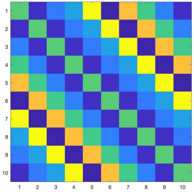
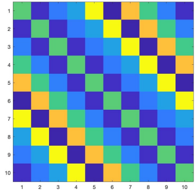


Circulant matrix:

- square
- rows cycle right
- columns cycle down

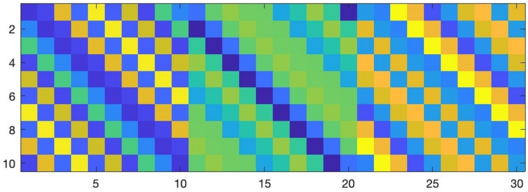


Circulant matrix:

- square
- rows cycle right
- columns cycle down

t-circulant matrix:

- $C = \begin{bmatrix} C_1 & \cdots & C_t \end{bmatrix}$
- each C_j circulant
- $d \times td$



Other ETFs are t-circulant

Gerzon's bound

$$d \times n \text{ ETF over } \mathbb{F} \implies n \leq \begin{cases} \frac{d(d+1)}{2} & \mathbb{F} = \mathbb{R} \\ d^2 & \mathbb{F} = \mathbb{C} \end{cases}$$

Lemmens & Seidel, J. Algebra, 1973

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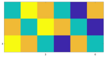
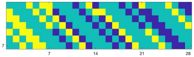
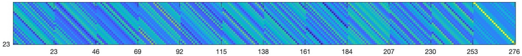


3×6 over $\mathbb{R}$



7×28 over $\mathbb{R}$



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Lemmens & Seidel, J. Algebra, 1973

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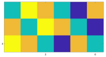

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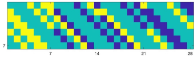

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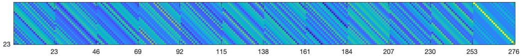

23×276 over $\mathbb{R}$

$\sim$Zauner's conjecture

For every d, there exists a $d \times d^2$ ETF over $\mathbb{C}$ that is d-circulant

---

Lemmens & Seidel, J. Algebra, 1973

Zauner, PhD thesis, 1999

Back to $d \times 2d$

$d \times 2d$ ETF from doubling is 2-circulant

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
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145	146	147	148	149	150	151	152		154	155	156	157	158	159	160

= previously unknown

Back to $d \times 2d$

Medium $d \times 2d$ conjecture

For every d, there exists a $d \times 2d$ ETF that is 2-circulant

$d \times 2d$ ETF from doubling is 2-circulant

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Infinitely many 2-circulant $d \times 2d$ ETFs

Theorem (Iverson, Jasper, M)

q odd prime power $\implies$ 2-circulant $d \times 2d$ ETF for $d = \begin{cases} \frac{1}{2}(q+1) \\ q+1 \end{cases}$

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Sketch:

1) Can detect 2-circulant from auto grp / signature mat
2) Apply for certain $d \times 2d$ ETFs:
   a) $d = \frac{1}{2}(q+1)$ from Paley conference
   b) $d = q+1$ from ETF doubling of (a)

So far

$d \times 2d$ ETF that is 2-circulant

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Not miracles

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Strong $d \times 2d$ conjecture

For every $d \neq 4$, the set of 2-circulant $d \times 2d$ ETFs contains a real manifold of dimension $\left\lceil \frac{3d}{2} \right\rceil$

Theorem (Iverson, Jasper, M)

The strong $d \times 2d$ conjecture holds for all $d \leq 162$

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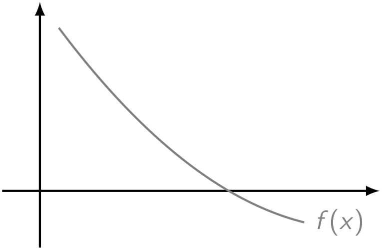

Cohn, Kumar & Minton, Geom. Topol., 2016

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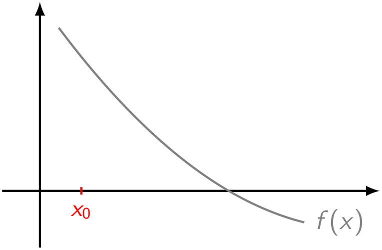

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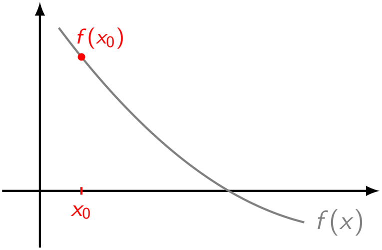

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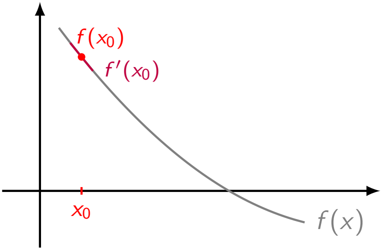

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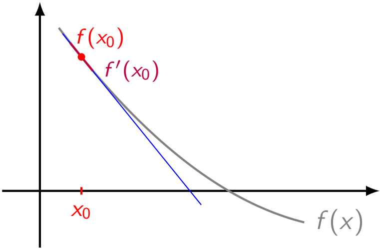

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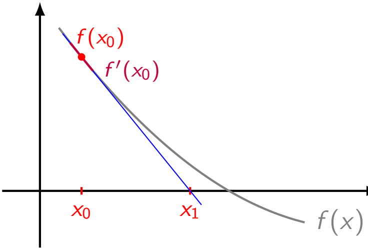

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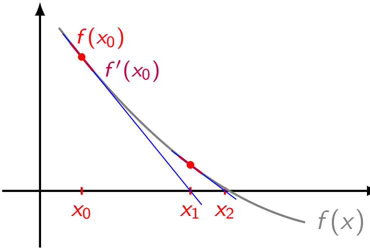

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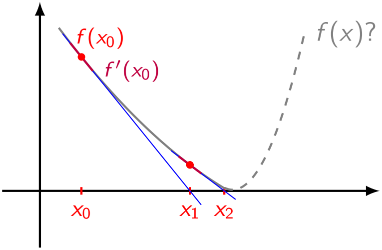

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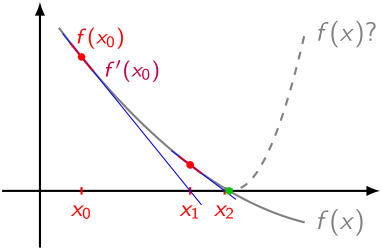

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Cohn, Kumar & Minton, Geom. Topol., 2016

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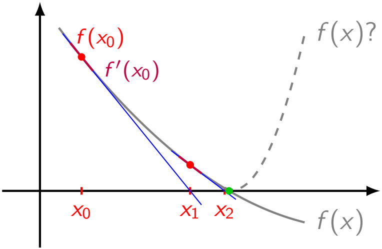

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Furthermore: (local dim) = (# vars) − (# constraints)

Cohn, Kumar & Minton, Geom. Topol., 2016

$d \times 2d$ ETF that is 2-circulant

1	2	3	4	5	6	7	8	9	10	$\boxed{11}$	12	13	14	15	16
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145	146	147	148	$\boxed{149}$	150	$\boxed{151}$	152	$\boxed{153}$	154	$\boxed{155}$	156	157	158	159	160

$\boxed{}$ = previously unknown

Open questions

- Explicit solution to $d \times 2d$?
- How to leverage vars $>$ constraints to prove existence?
- Why is $d = 4$ different?
- Skew-Hadamard conjecture $\implies$ $d \times 2d$ for $d \equiv 1 \bmod 4$?
- Generic initializations converge to 2-circulant ETFs?

$d \times 2d$ ETF that is 2-circulant

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Questions?

On the optimal arrangement of $2d$ lines in $\mathbb{C}^d$
K. Fallon, J. W. Iverson
arXiv:2312.09975

More on the optimal arrangement of $2d$ lines in $\mathbb{C}^d$
J. W. Iverson, J. Jasper, D. G. Mixon
Coming soon!

Also, google **short fat matrices** for my research blog