Quantum symmetries of Hadamard matrices

Algebraic Graph Theory Seminar

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25 September 2023

Diagrammatic categories

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Quantum symmetries of a structure defined by maps $A_1,\ldots,A_k$	$\leftrightarrow$	Category $\langle A_1,\ldots,A_k\rangle$

Finite (quantum) spaces

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- Application: All finite spaces of a given fixed size are quantum isomorphic

(Quantum) graphs

- Category $\mathscr{C} = \langle m, \psi, A \rangle = \langle$  $\rangle$, $\quad \mathscr{C}(k,l) = \{$planar labelled graphs with $k + l$ labels$\}$

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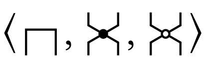
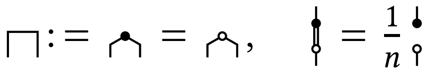
- Diagrammatic category $\mathscr{D} =$  satisfying 
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THM

Every Hadamard matrix (graph) of size $n \geq 4$ has quantum symmetries.

Any two Hadamard matrices (graphs) of the same size are quantum isomorphic.

[G. '22]

Some (open) questions

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- Duality of association schemes

halved and folded hypercube duality

Asked 2 years, 3 months ago Modified 2 years, 3 months ago Viewed 113 times

1

Notation. Consider the group $\Gamma = \mathbb{Z}_2^n$. I will denote the group operation aditively and by $\epsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ I denote the canonical generators. Let's define also $\epsilon_0 := 0$ and $\epsilon_{n+1} := (1, 1, \dots, 1) = \epsilon_1 + \cdots + \epsilon_n$. I will denote by $\tau_i \in C(\Gamma) \simeq \mathbb{C}\Gamma$ the character corresponding to ϵ_i. That is $\tau_i(\alpha) = (-1)^{\alpha_i}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma$. I will also denote $\tau_0 = 1_{C(\Gamma)}$ and $\tau_{n+1} = \tau_1 \cdots \tau_n$.

Halved hypercube graph can be defined as the Cayley graph corresponding to Γ with respect to the generating set $S = \{\epsilon_i\}_{1\le i\le n} \cup \{\epsilon_i + \epsilon_j\}_{1\le i<j\le n}$. Or maybe a bit nicer: $S = \{\epsilon_i + \epsilon_j\}_{0\le i\le j\le n}$. In general, the set of vertices in a given distance d from the vertex 0 is given by $S_d = \{\epsilon_{i_1} + \cdots + \epsilon_{i_{2d}} \mid 0 \le i_1 < i_2 \cdots < i_{2d} \le n\}$. Since it is a Cayley graph of an abelian group Γ, the eigenvectors of the adjacency matrix are exactly the characters of Γ. One can easily compute the spectrum, which is not that important, but the eigenspaces look as follows $V_d = \operatorname{span}\{\tau_{i_1} \cdots \tau_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n+1\}$.

Folded hypercube graph can be defined as the Cayley graph corresponding to Γ with respect to the generating set $S = \{\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}\}$. The set of vertices in distance d from 0 is given by $S_d = \{\epsilon_{i_1} + \cdots + \epsilon_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n+1\}$ and the eigenspaces by $V_d = \operatorname{span}\{\tau_{i_1} \cdots \tau_{i_{2d}} \mid 0 \le i_1 < \cdots < i_{2d} \le n\}$.

Observation. The two graphs are dual to each other in the sense that the set $S_d \subset \Gamma$ for one exactly matches the eigenspace $V_d \subset \mathbb{C}\Gamma$ for the other.

Questions. Does this duality have any deeper meaning? Is this a more general phenomenon? I mean, can one define a dual graph for any, let's say, Cayley graph of an abelian group such that taking the dual twice, one arives at the original one? If so, I would appreciate some reference.

graph-theory algebraic-graph-theory cayley-graphs

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asked Jun 2, 2021 at 8:12
Daniel

3 \What, you're seeing is an instance of duality on the association scheme of an abelian group. A more helpful comment is that the case you have ($\mathbb{Z}_2^n$) can be described by the Hadamard transform from coding theory. – Chris Godsil Jun 2, 2021 at 13:53

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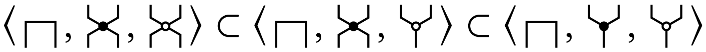
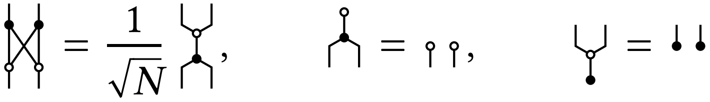
  What about



  imposing



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  Some clues for Walsh matrices:

A206707 Order of largest automorphism group of a Hadamard matrix of order 4n. 0

192, 21504, 190080, 10321920, 6840, 760320, 58968, 20478689280 (list; graph; refs; listen; history; text; internal format)

A028368 a(n) = (Product_{j=1..n-1} (2^j-1)) * 2^binomial(n+1,2). 0

1, 2, 8, 192, 21504, 10321920, 20478689280, 165140150353920, 5369036568306647040, 700981414358115837542400, 366798338802685125615786393600, 768480666818860817418136536376934400 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

LINKS Table of n, a(n) for n=0..11.

I. Strazdins, Universal affine classification of Boolean functions, Acta Applic. Math. 46 (1997), 147-167.

THEOREM 1. *For* $n \geqq 3$ *there exist* 13 *different fundamental groups (see Table* I):

$$C_n, C_n^{(1)}, C_n^{(0)}, C'_n, K_n, K'_n, L_n, L_n^{(1)}, L_n^{(0)}, M_n, M'_n, T_n, S_n.$$

Table I gives the minimal n-restricted Post class for each group and order of the group, where

$$p_n = 2^{\binom{n+1}{2}}, \qquad q_n = \prod_{j=1}^{n}(2^j - 1).$$

Table I. Fundamental groups

No.	Group	Notation	Post class	Order
1	Universal	C_n	C_1	$2^n!$
2	Preserving 1	$C_n^{(1)}$	C_2	$\left.\right\} (2^n - 1)!$
3	Preserving 0	$C_n^{(0)}$	C_3	
4	Preserving 1 and 0	C'_n	C_4	$(2^n - 2)!$
5	Selfdual	K_n	D_3	$2^{n-1}!2^{2^{n-1}}$
6	Selfdual, preserving 1 and 0	K'_n	D_1	$(2^{n-1} - 1)!2^{2^{n-1}-1}$
7	Affine	L_n	L_1	$p_n q_n$
8	Dual linear	$L_n^{(1)}$	L_2	$\left.\right\} p_{n-1} q_n$
9	Linear	$L_n^{(0)}$	L_3	
10	Affine self-dual	M_n	L_5	$p_n q_{n-1}$
11	Linear self-dual	M'_n	L_4	$p_{n-1} q_{n-1}$
12	Renaming	T_n	O_4	$n!2^n$
13	Permutation	S_n	O_1	$n!$

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- Anything else?