Fractional Sudoku

A coherent configuration and fractional completion threshold

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Introduction

Partial latin squares

A *latin square* of order n is an $n \times n$ array with entries from $[n] := \{1, 2, \ldots, n\}$ such that each symbol appears exactly once in every row and every column.

A *partial latin square* of order n is an $n \times n$ array whose cells are either empty or filled with one of n symbols in such a way that each symbol appears at most once in every row and every column.

A *completion* of a partial latin square P is a latin square which contains every entry of P.

Examples

Here is a partial latin square and a completion of it to a latin square of order 5.

1				
	2	4		
			3	
			1	

1	3	5	2	4
3	2	4	5	1
4	1	2	3	5
2	5	1	4	3
5	4	3	1	2

We can't use any row, column, or symbol very often if we want a completion to exist:

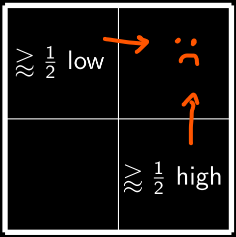
1	2	3	4	
				5

1				
2				
3				
4				
	5			

1				
	1			
		1		
			1	
				2

Barriers

We can't use any row, column, or symbol very often if we want a completion to exist:



Completion threshold

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Theorem

For sufficiently large n, every ϵ-dense partial latin square of order n has a completion.

- Prehistory: $\epsilon \to 0$
- 1991: $\epsilon = 10^{-7}$ (Gustavsson)
- 2013: $\epsilon \approx 10^{-4}$ (Bartlett)
- 2019: $\epsilon \approx 0.04$ (Bowditch & D.)

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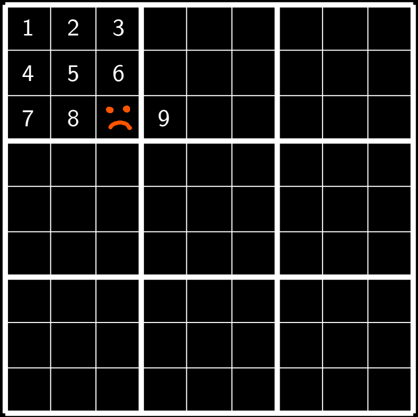
- 2019: $\epsilon \approx 0.04$ (Bowditch & D.)
- Conjecture: $\epsilon = 0.25$

A *Sudoku latin square* of type (h, w) is a latin square of order $n = hw$ divided into a $w \times h$ pattern of $h \times w$ sub-arrays (boxes), each of which contains every symbol exactly once.

A partial Sudoku is a partial latin square in which each symbol appears at most once in every box.

Completion is defined analogously as for partial latin squares.

Stronger sparseness assumptions



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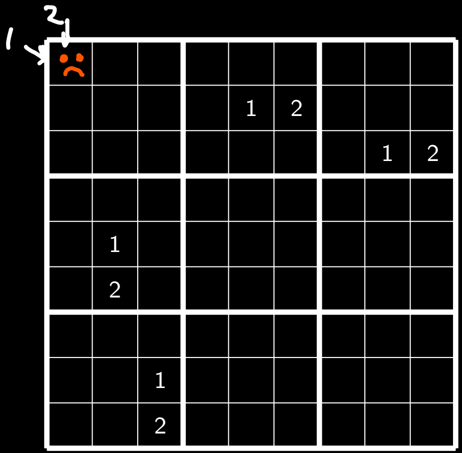

Stronger sparseness assumptions

Definition

Let us say that a Sudoku of type (h, w) is *ϵ-dense* if:

- each row, column, and box has at most ϵn filled cells; AND
- each symbol occurs at most ϵh times in any bundle of h rows corresponding to the box partition, and likewise at most ϵw times in any bundle of w columns.

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Question: Can ϵ-dense Sudoku always be completed for some $\epsilon > 0$?

The linear system

Linear equations

Consider an $n \times n$ Sudoku. Let the rows, columns, symbols and boxes be denoted r_i, c_j, s_k, b_ℓ, respectively, where $i, j, k, \ell \in [n]$.

Let x_{ijk} denote the number/fraction of symbols s_k placed in cell (i, j). Sudoku constraints correspond to linear equations on these variables:

- every cell has exactly one symbol: $\sum_k x_{ijk} = 1$ for each $(i, j) \in [n]^2$.
- every row has every symbol once: $\sum_j x_{ijk} = 1$ for each $(i, k) \in [n]^2$.
- " column " " : $\sum_i x_{ijk} = 1$ for each $(j, k) \in [n]^2$.

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- " column " " : $\sum_i x_{ijk} = 1$ for each $(j, k) \in [n]^2$.
- every box contains every symbol once:

  $$\sum_{(i,j)\in \mathrm{box}(\ell)} x_{ijk} = 1$$

  for each $(k, \ell) \in [n]^2$.

Linear system

This results in a $4n^2 \times n^3$ linear system

$$W\vec{\mathbf{x}} = \vec{\mathbf{1}} \quad \text{(all-ones vector)}.$$

There is a naïve solution: $\vec{\mathbf{x}} = \frac{1}{n}\vec{\mathbf{1}}$.

If some entries have been pre-filled, a similar linear system can be used. We can either adjust the right side, or delete those variables x_{ijk} which are unavailable.

If we get a $\{0, 1\}$-valued solution $\vec{\mathbf{x}}$, this leads to a completion of the corresponding partial Sudoku.

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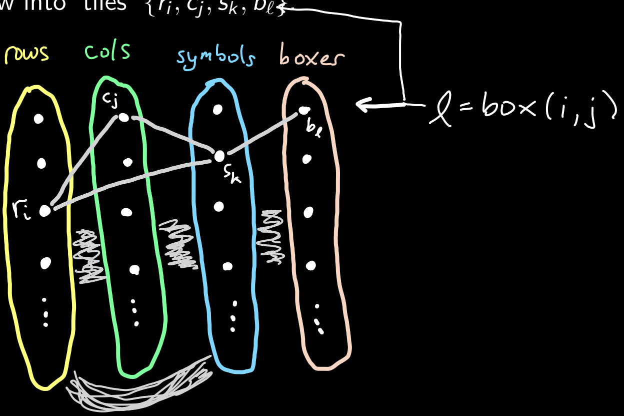
If we get a $[0,1]$-valued solution $\vec{\mathbf{x}}$, this leads to a *fractional* completion of the corresponding partial Sudoku.

Graph decomposition model

We can think of a Sudoku as an edge-decomposition of the 4-partite graph shown below into 'tiles' $\{r_i, c_j, s_k, b_\ell\}$.



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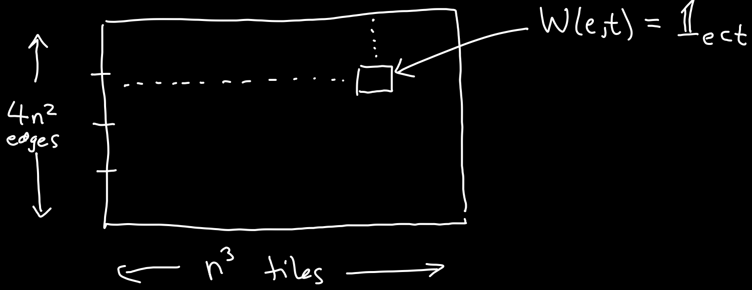
W is the $\{0, 1\}$ inclusion matrix of edges versus tiles.



The normal system

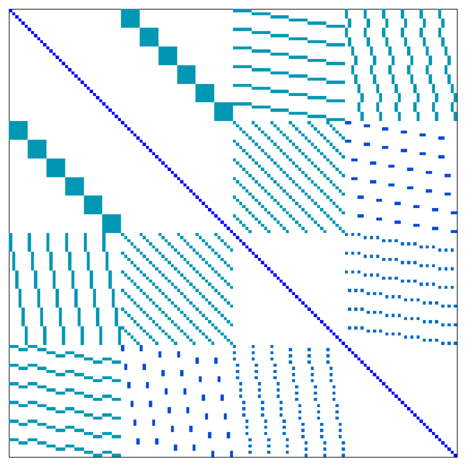
Let $M = WW^\top$ and consider instead the normal system $M\vec{\mathbf{x}} = \vec{\mathbf{1}}$.

Rows and columns of M are indexed by edges $r_i c_j, r_i s_k, c_j s_k, b_\ell s_k$. Entries tell us how many tiles contain two given edges.

That is,

$$M(e,f) = \begin{cases} n & \text{if } e = f \\ h & \text{if } e \cup f = \{c_j, s_k, b_\ell\} \text{ where } c_j \text{ meets } b_\ell \\ w & \text{if } e \cup f = \{r_i, s_k, b_\ell\} \text{ where } r_i \text{ meets } b_\ell \\ 1 & \text{if } e \cup f \text{ has exactly one of each } r_i, c_j, s_k \\ 0 & \text{otherwise.} \end{cases}$$

Coefficient matrix

$M =$ 

Coherent configuration

Using the symmetries present in Sudoku, we can express M in an algebra of fixed dimension, independent of h and w.

- Rows can be equal, unequal in the same bundle, or in different bundles. Same with columns.
- Boxes can be equal, row-adjacent, column-adjacent, or neither.
- Symbols can be equal or unequal.

These lead to 69 relations on edges.

Relations


row-col
row-symbol
col-symbol
box-symbol
row-col
row-symbol
col-symbol
box-symbol

Structure constants

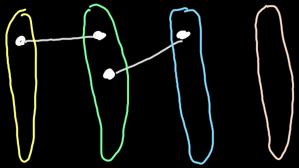
We found and stored **symbolic** structure constants for this coherent configuration using the following procedure:

- argue that they are all polynomials of degree ≤ 2 in each of h, w;
- directly compute all structure constants for the nine cases $2 \leq h, w \leq 4$;
- interpolate to arrive at symbolic expressions.

Possible values

Fix two edges e, f which are related in some way. In picking a third edge having prescribed relations with each of e, f, we multiply two of:

- row choices $\in \{0, 1, h, h-1, h-2, n-h, n-2h\}$;
- column choices $\in \{0, 1, w, w-1, w-2, n-w, n-2w\}$;
- symbol choices $\in \{0, 1, n, n-1, n-2\}$;
- box choices $\in \{0, 1, h, h-1, h-2\} * \{0, 1, w, w-1, w-2\}$.

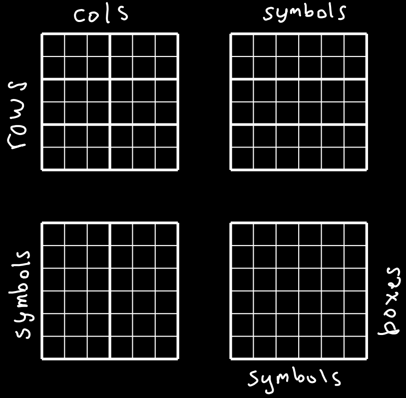


Spectral decomposition

Summary

Proposition

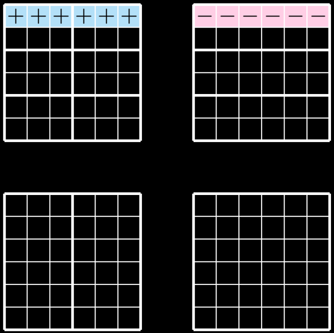
The eigenvalues of M are $\theta_j = jn$, $j = 0, 1, \ldots, 4$. *Each eigenspace has a basis of eigenvectors consisting of vectors with entries in* $\{0, \pm 1\}$.



Kernel of M

- $\theta_0 = 0$; kernel dimension $3n + (h + w)(n - 1)$

(A)



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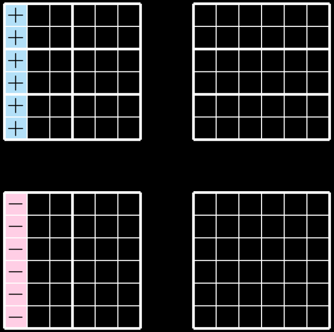

Kernel of M

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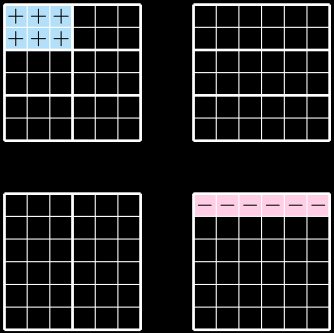
(B)



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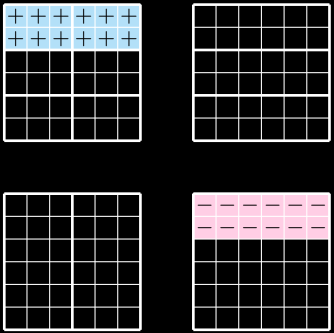
(C)



Other eigenvalues and eigenvectors

- $\theta_1 = n$; eigenspace dimension $4n^2 - (2n-3)(h+w) - 5n - 1$

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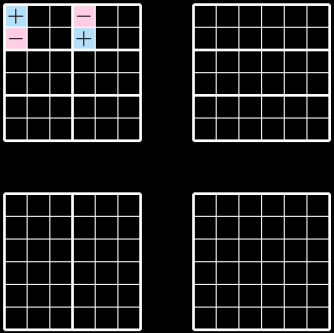

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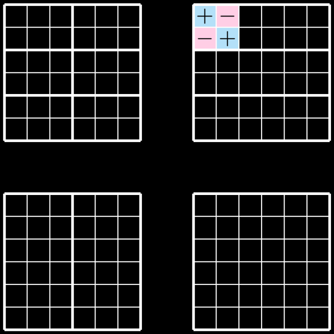

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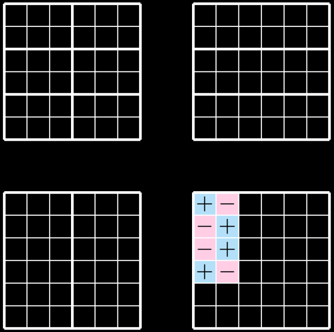
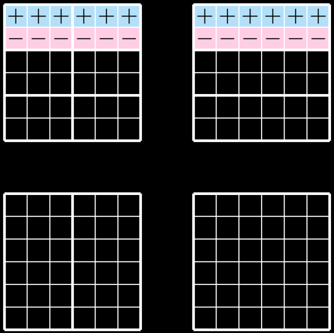


Other eigenvalues and eigenvectors

- $\theta_2 = 2n$; eigenspace dimension $(n-3)(h+w-1)+2n$

(A)



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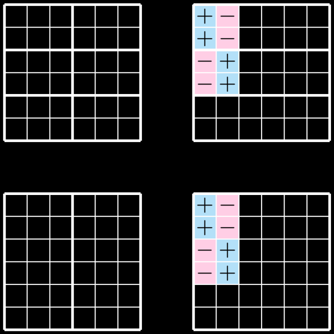

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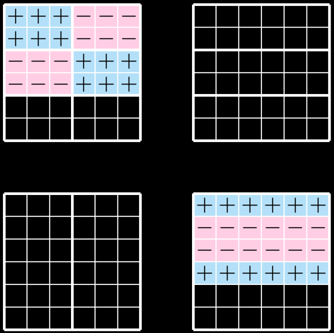
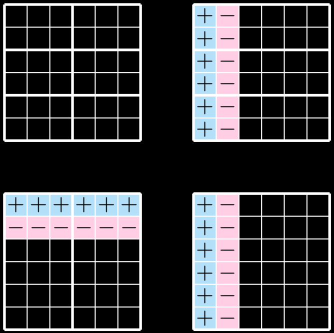


Other eigenvalues and eigenvectors

- $\theta_3 = 3n$; eigenspace dimension $n + h + w - 3$

(A)



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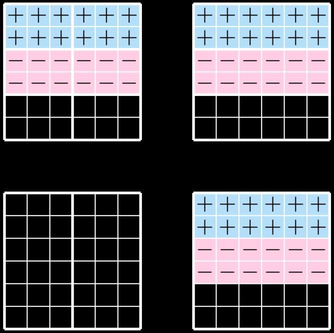
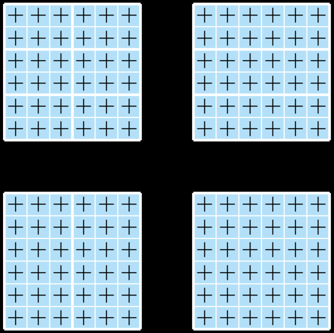


Other eigenvalues and eigenvectors

- $\theta_4 = 4n$; eigenspace dimension 1



Projectors

Using orthogonal projections onto the eigenspaces, we can write

$$M = nE_1 + 2nE_2 + 3nE_3 + 4nE_4.$$

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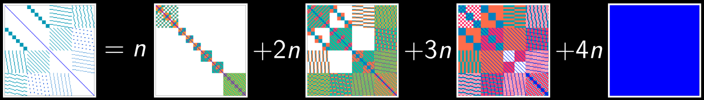

A generalized inverse

Let K denote projection onto the kernel. For $x \in \mathbb{R}$, we can invert the additive shift $A = M + \frac{n}{x}K$ as

$$A^{-1} = \frac{1}{n}\left(xK + \sum_{j=1}^{4} \frac{1}{j}E_j\right).$$

With the help of computer, the choice $x = 3/2$ minimizes

$$\|A^{-1}\|_\infty = \frac{15}{4n} - \frac{7(h+w)}{8n^2} - \frac{4}{9n^2} + \frac{31(h+w) - 21}{72n^3} < \frac{15}{4n}.$$

Perturbation

Perturbed linear systems

Lemma

Let A be an $N \times N$ invertible matrix over the reals. Suppose $A - \Delta A$ is a perturbation. Then

- *$A - \Delta A$ is invertible provided $\|A^{-1}\Delta A\|_\infty < 1$; and*
- *the solution $\vec{\mathbf{x}}$ to $(A - \Delta A)\vec{\mathbf{x}} = A\vec{\mathbf{1}}$ is entrywise nonnegative provided $\|A^{-1}\Delta A\|_\infty \leq \frac{1}{2}$.*

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Proof idea.

Use the series expansion $(A - \Delta A)^{-1} = \sum_{k=0}^{\infty}(A^{-1}\Delta A)^k A^{-1}$. □

A perturbation of M

Let S be an ϵ-dense partial Sudoku of type (h, w), where $hw = n$.

Define M_S similarly to M, so that $M_S(e, f)$ records the number of available tiles $\{r_i, c_j, s_k, b_\ell\}$ containing $e \cup f$.

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Here is a white lie*, but morally true:

Proposition

$\|M - M_S\|_\infty < 12\epsilon n.$

*: we need to use a border to make M_S and M have the same dimensions

Main result

Theorem

Let $\epsilon < 1/101$. For sufficiently large h and w, every ϵ-dense partial Sudoku of type (h, w) has a **fractional** *completion, that is, an assignment of positive rational frequencies to symbols in unfilled cells so that the Sudoku conditions hold.*

Main result

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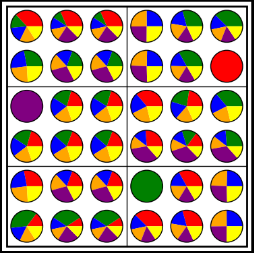

Proof sketch

We show $M_S\vec{\mathbf{x}} = \vec{\mathbf{1}}$ has an entrywise nonnegative solution $\vec{\mathbf{x}}$.

We shift the coefficient matrix by a multiple of K and view it as a perturbation of A (the 'empty' Sudoku).

Letting ΔA be this perturbation, we succeed when

$$\|A^{-1}\Delta A\|_\infty < \frac{15}{4n} \times 12\epsilon n + \frac{11}{2}\epsilon + o(1) = \frac{101}{2}\epsilon + o(1) < \frac{1}{2}.$$

Wrap-up

Concluding remarks

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Concluding remarks

- Some details on bordering our matrices have been suppressed for clarity.
- If the boxes are asymptotically thin (say w fixed and $n = hw$ large), then the bundle condition can be dropped.
- A structure of wiggly but ‘near-rectangular’ boxes can be handled via a secondary perturbation.
- Can this sparsity threshold for fractional completion be converted into something for actual completion?

Thank you!

References

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