New strongly regular graphs from real line packings

https://slides.com/johnjasper/waterlooagts/

John Jasper

AFIT Air Force Institute of Technology

University of Waterloo

Algebraic Graph Theory Seminar

Outline

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find vectors maximally "spread out"

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special optimizers have nice repn's

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find vectors maximally "spread out"

special optimizers have nice repn's

nice repn's $\Rightarrow$ rare graphs

Background

Measuring how "spread out" vectors are

Definition. Given unit vectors $\Phi = (\varphi_i)_{i=1}^N$, we define the **coherence**

$$\mu(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|.$$

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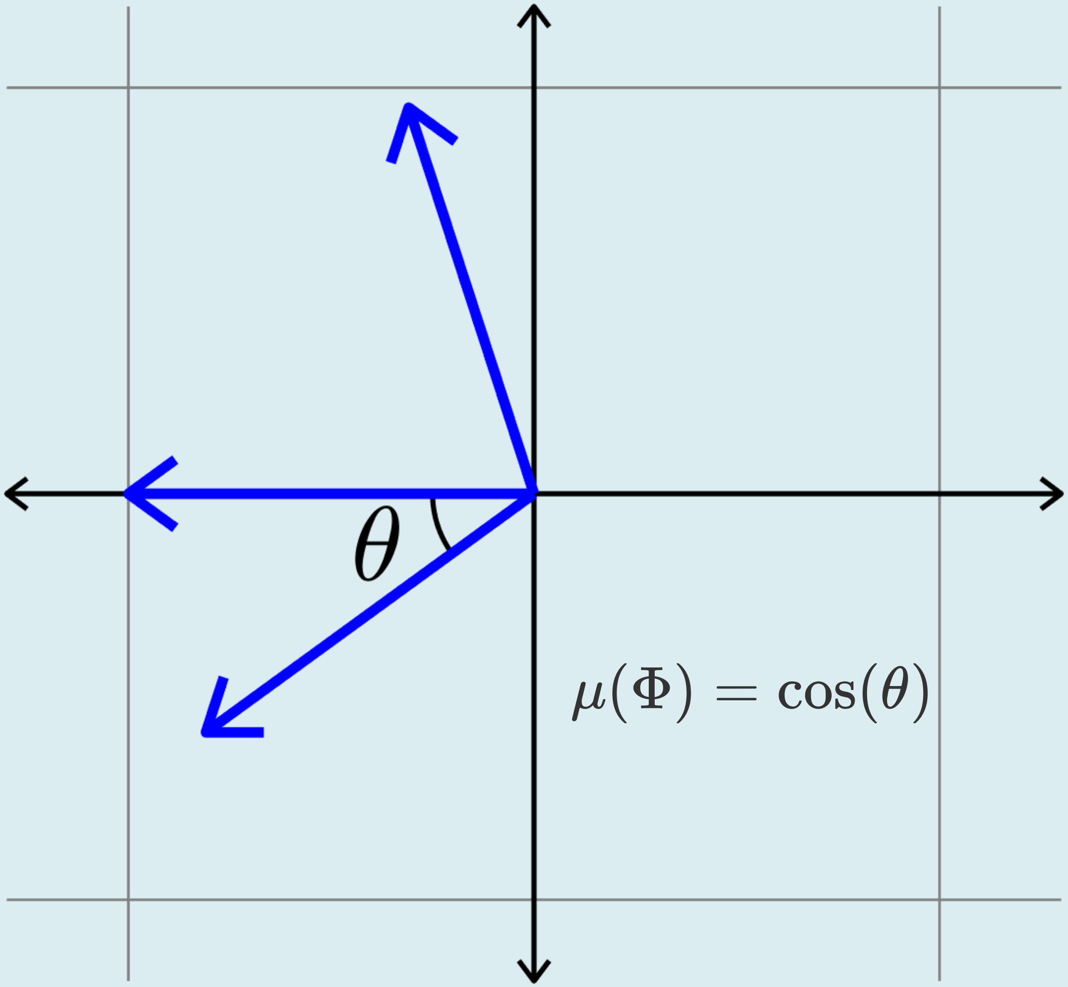

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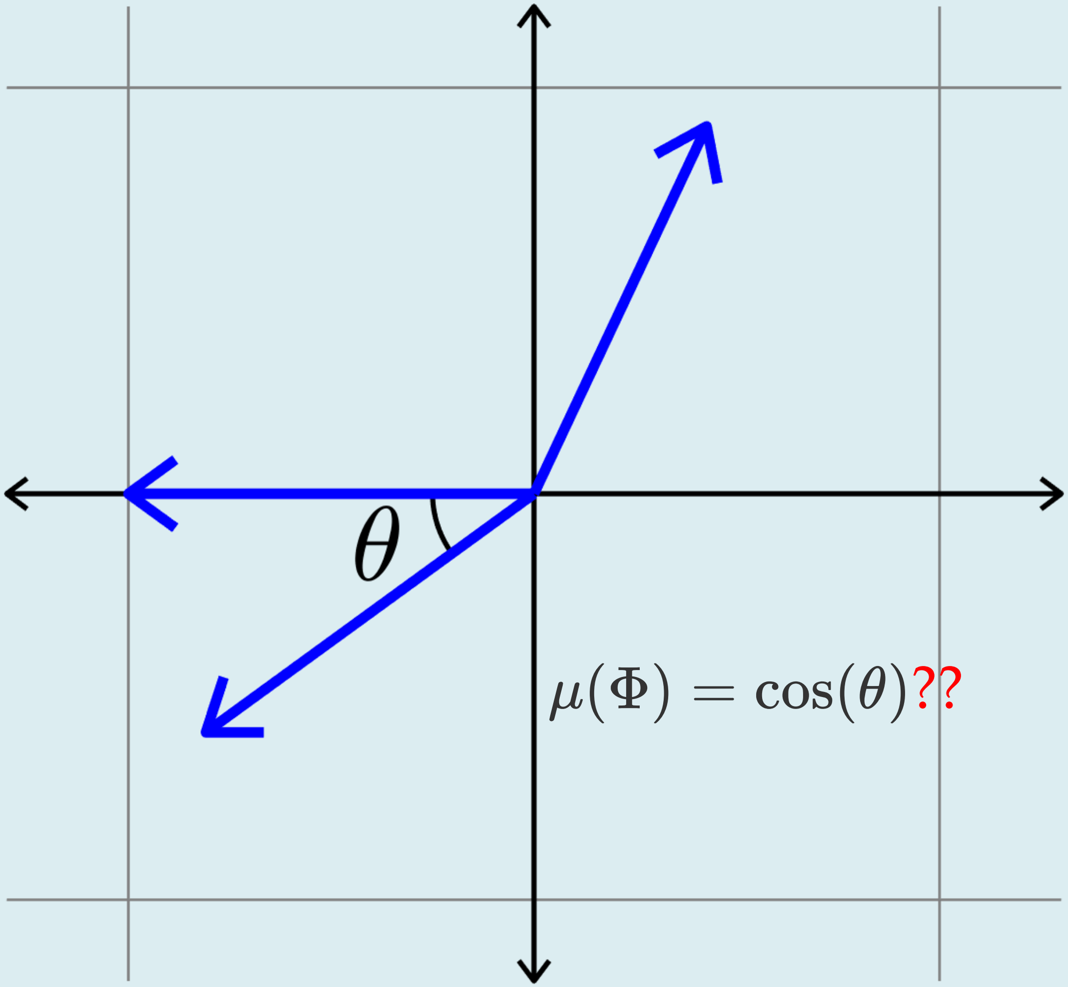

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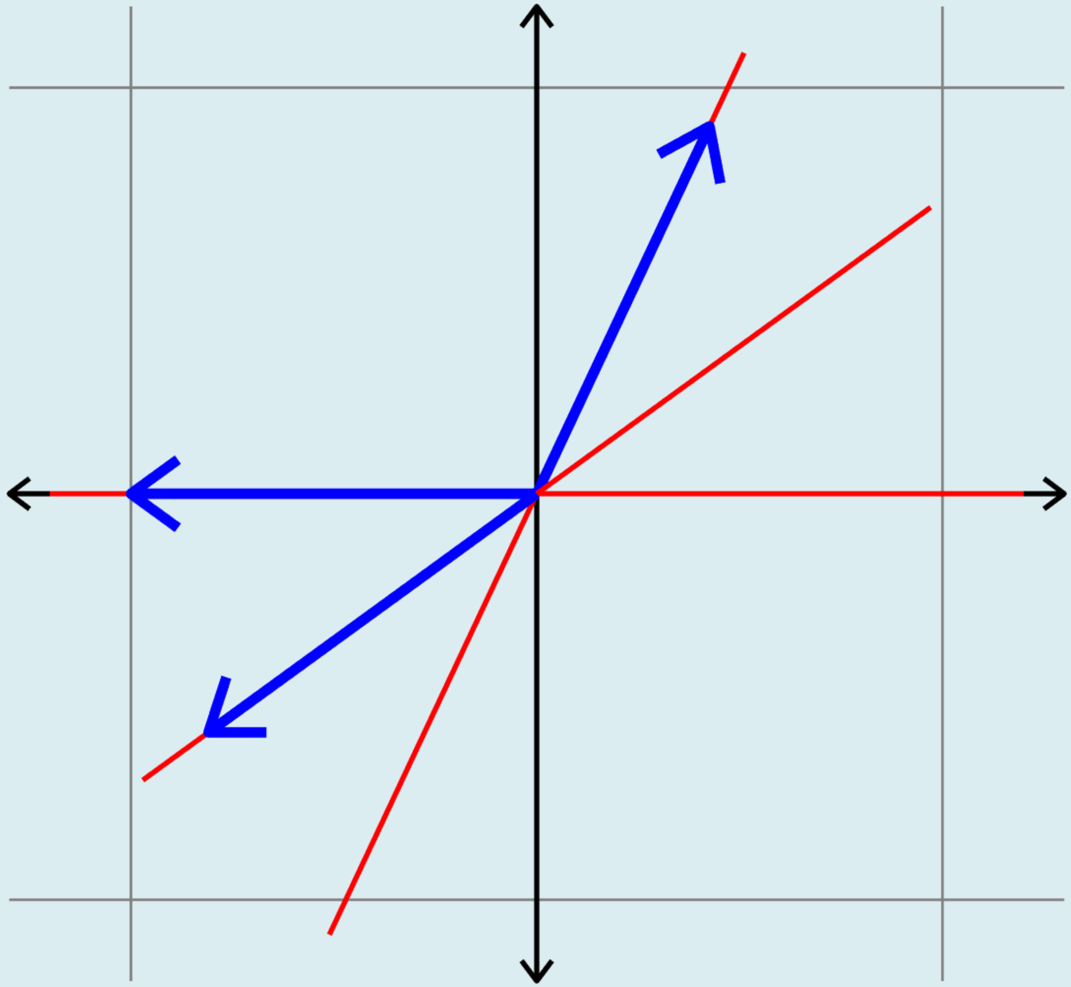

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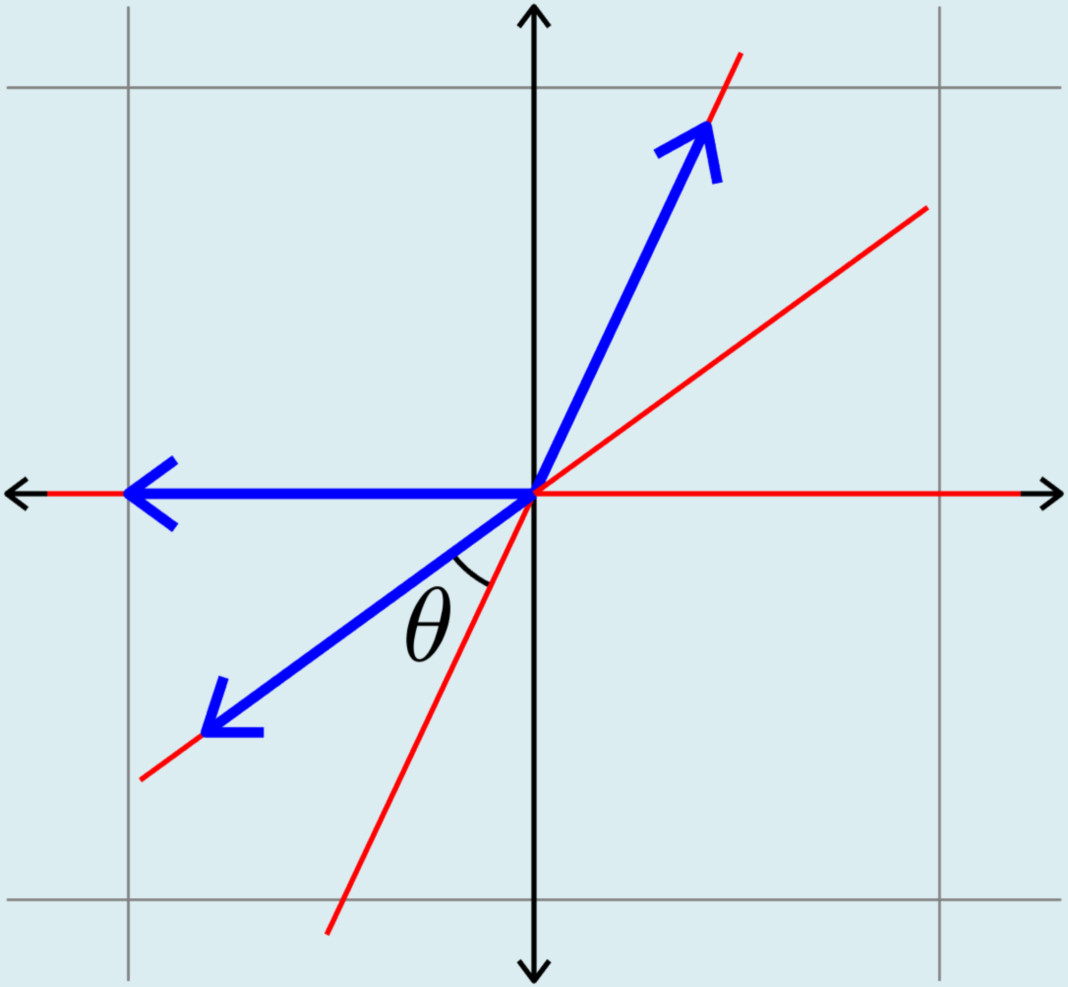

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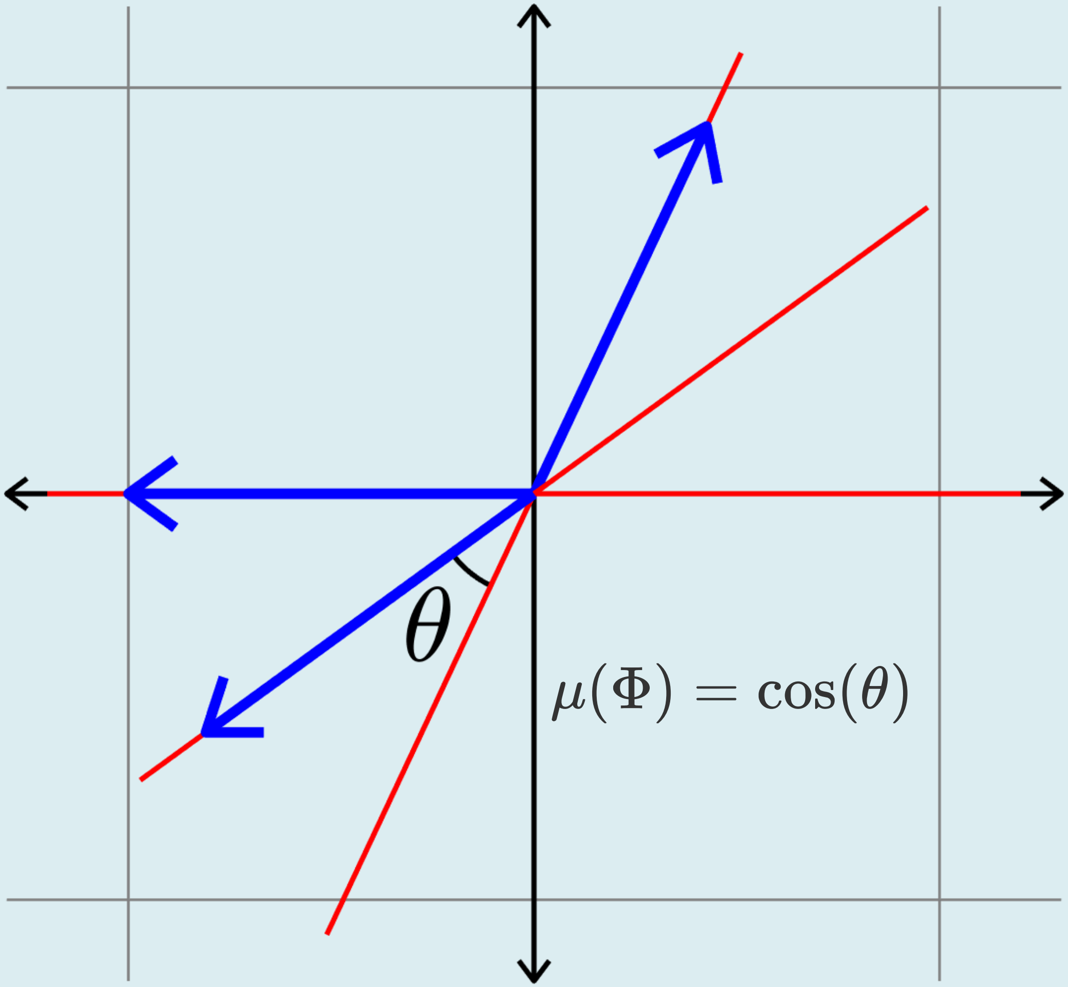

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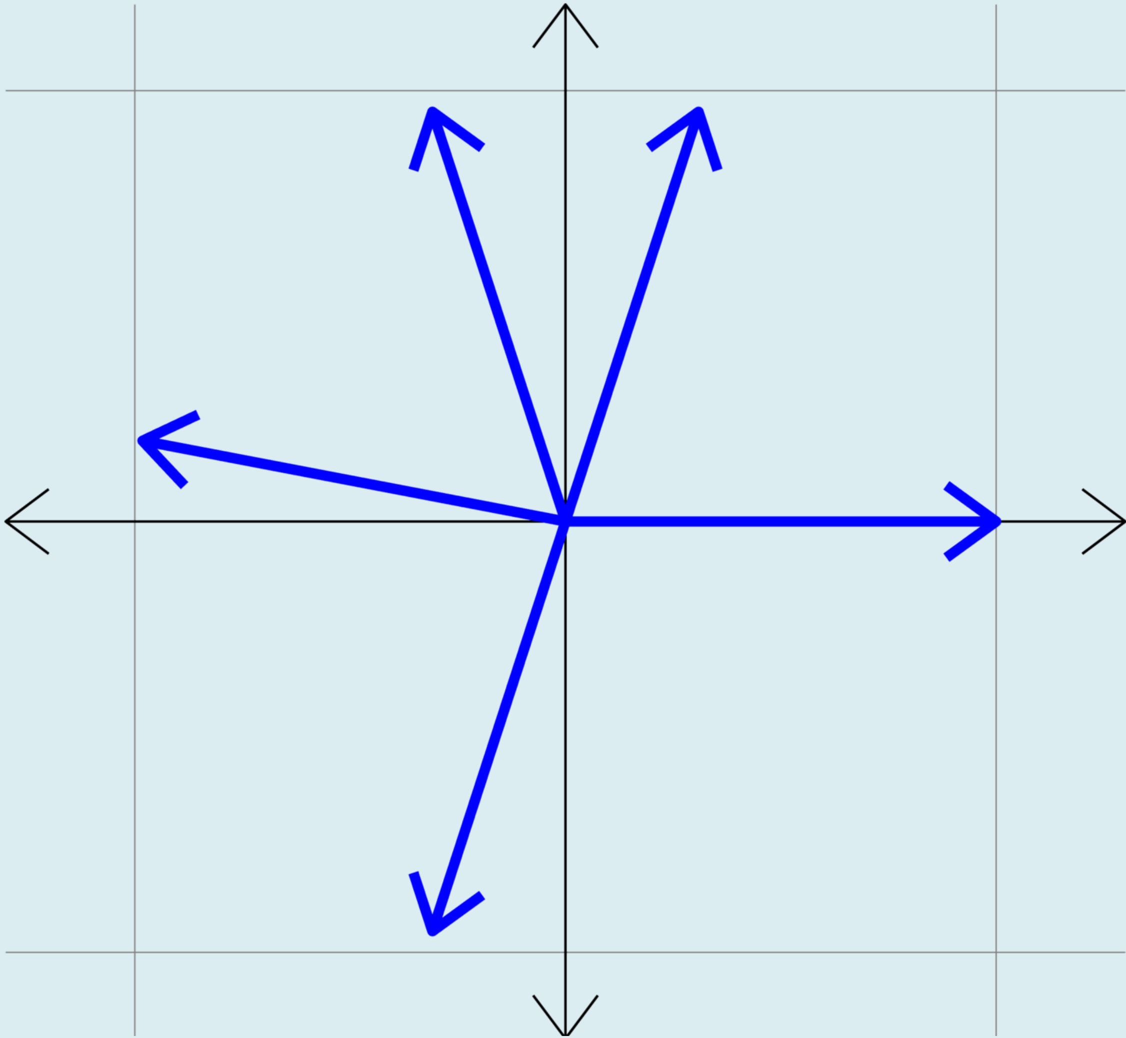
Minimizing coherence

between vectors

$\Updownarrow$

Maximizing min. angle

between lines



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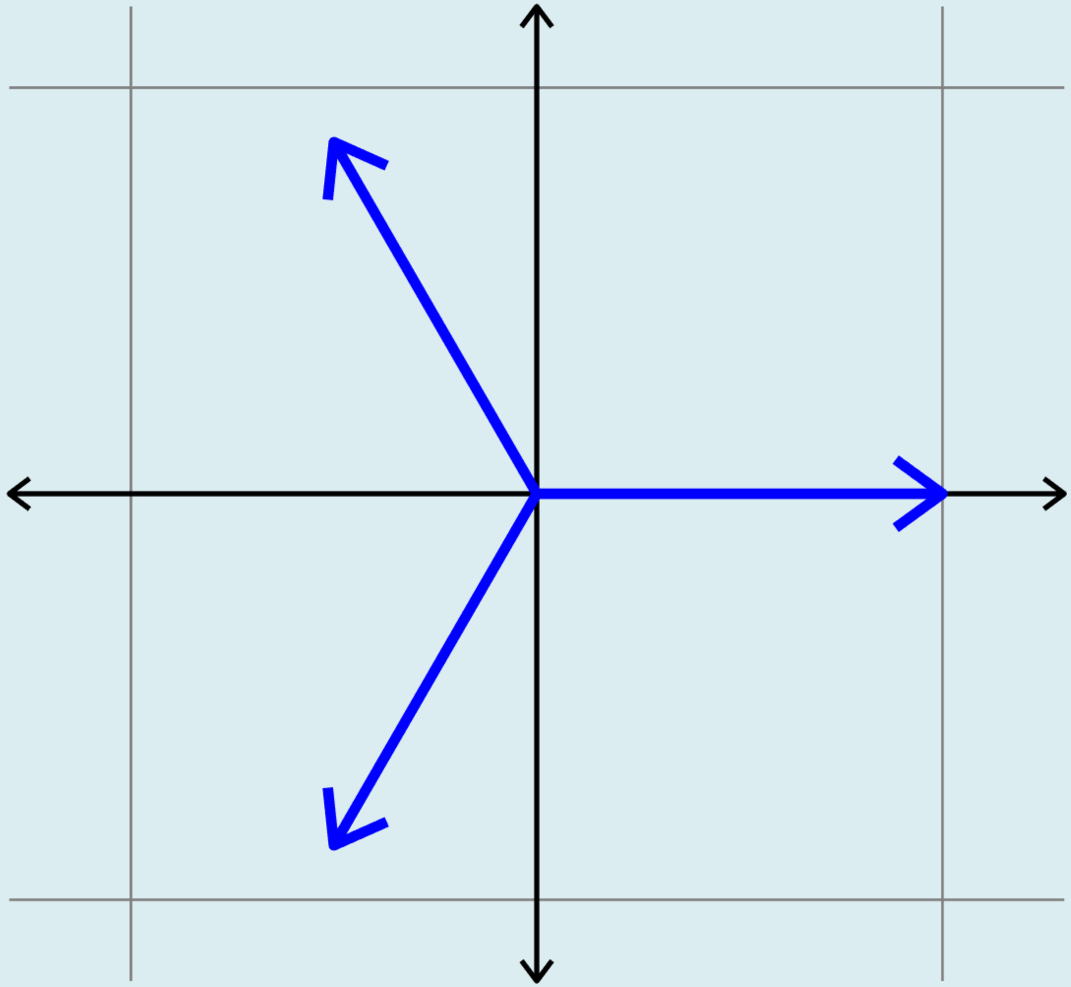

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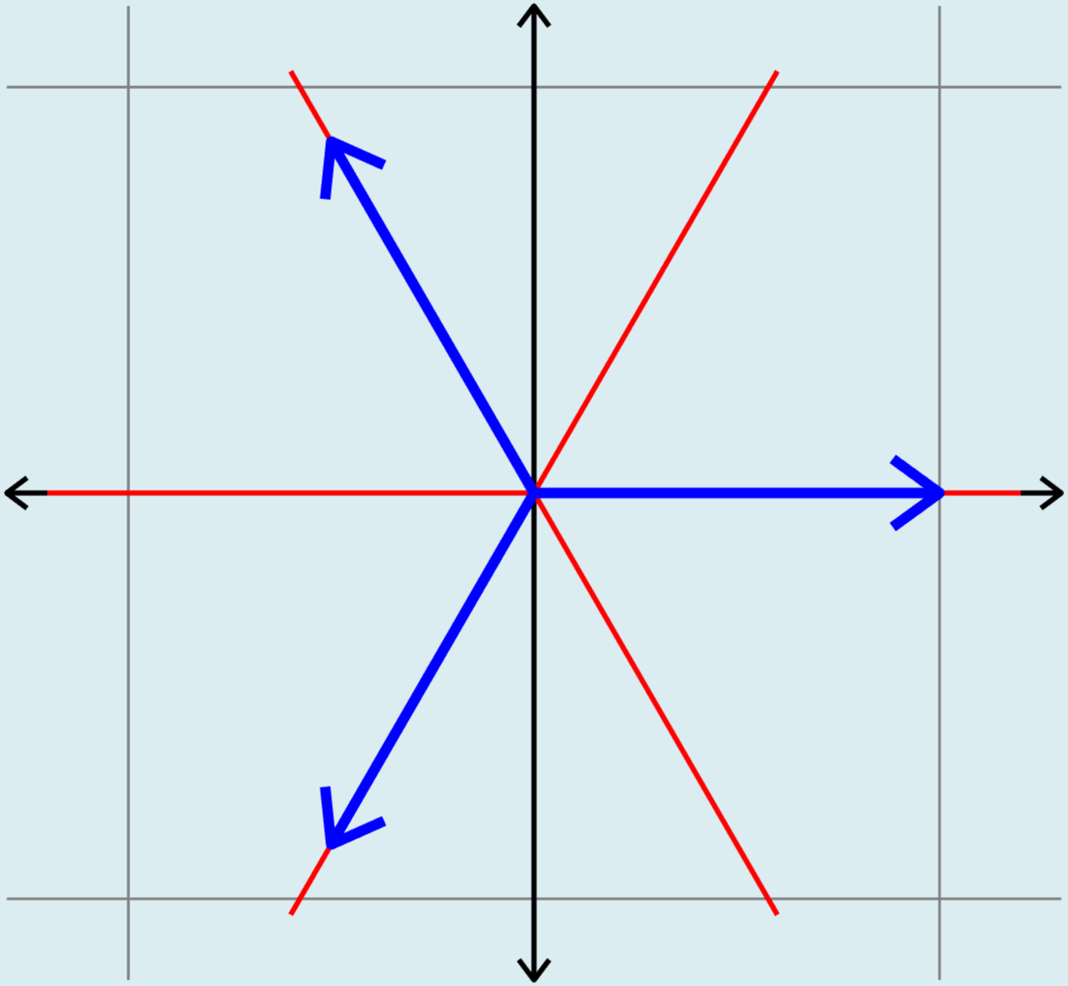

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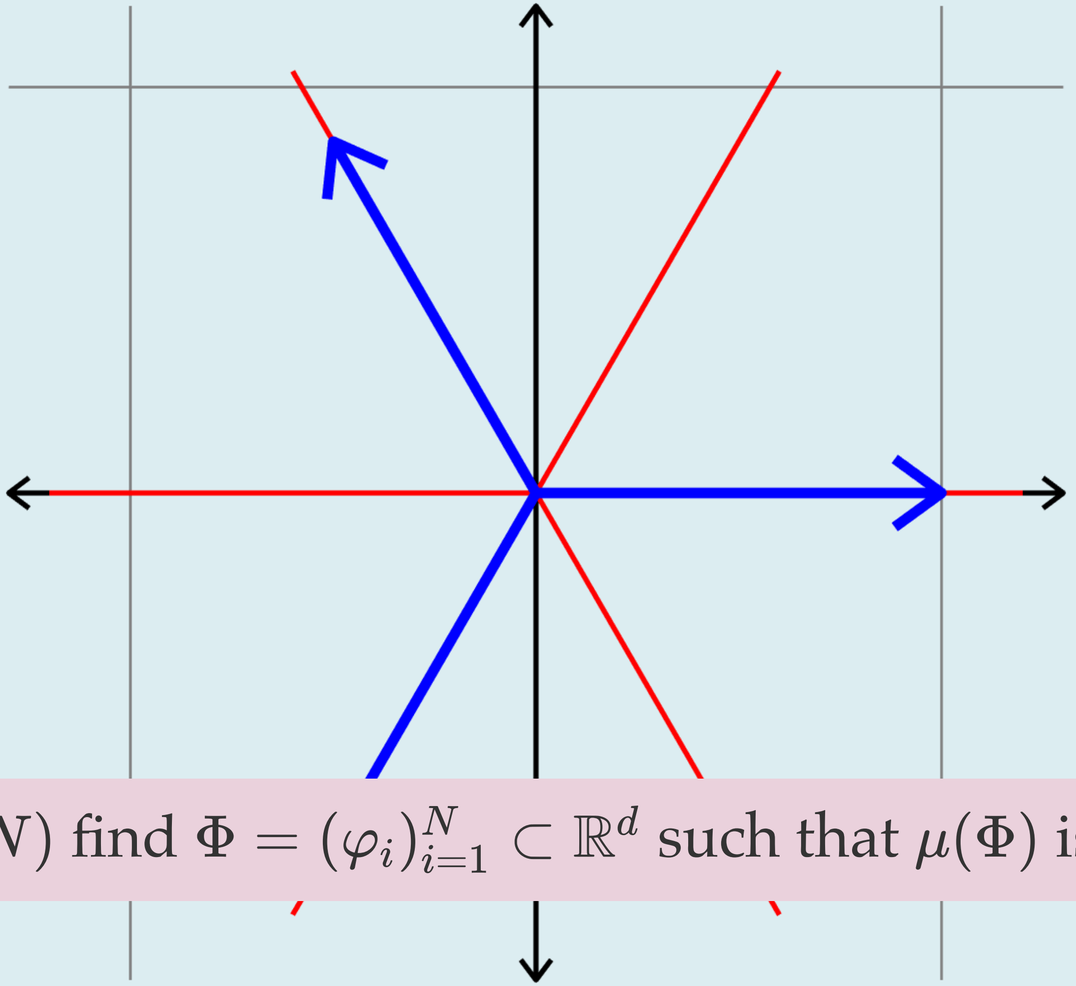

Goal:

Given (d, N) find $\Phi = (\varphi_i)_{i=1}^N \subset \mathbb{R}^d$ such that $\mu(\Phi)$ is minimal.

Vectors that are as spread out as possible

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Theorem (the Welch bound). For unit vectors $\Phi = (\varphi_i)_{i=1}^N$ in $\mathbb{R}^d$

$$\mu(\Phi) \geq \sqrt{\frac{N-d}{d(N-1)}}.$$

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Welch bound equality $\Longleftrightarrow$ equiangular tight frame (ETF)

ETF Gram matrix

Definition. Let

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_N \end{bmatrix} \in \mathbb{R}^{d \times N},$$

be a rank d matrix where each column φ_n is unit norm

$$\|\varphi_n\|^2 = 1.$$

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If both 1) and 2) hold, then $(\varphi_n)_{n=1}^N$ is an **ETF(d, N)**.

$$\Phi^\top \Phi = \begin{bmatrix} 1 & \varphi_1^\top \varphi_2 & \cdots & \varphi_1^\top \varphi_N \\ \varphi_2^\top \varphi_1 & 1 & \cdots & \varphi_2^\top \varphi_N \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N^\top \varphi_1 & \varphi_N^\top \varphi_2 & \cdots & 1 \end{bmatrix}$$

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1) $\Phi^\top \Phi \propto$ projection

2) $|\varphi_m^\top \varphi_n|$ constant

Examples of equiangular tight frames

Example 1. Consider the (multiple of a) unitary matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & -\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix}$$

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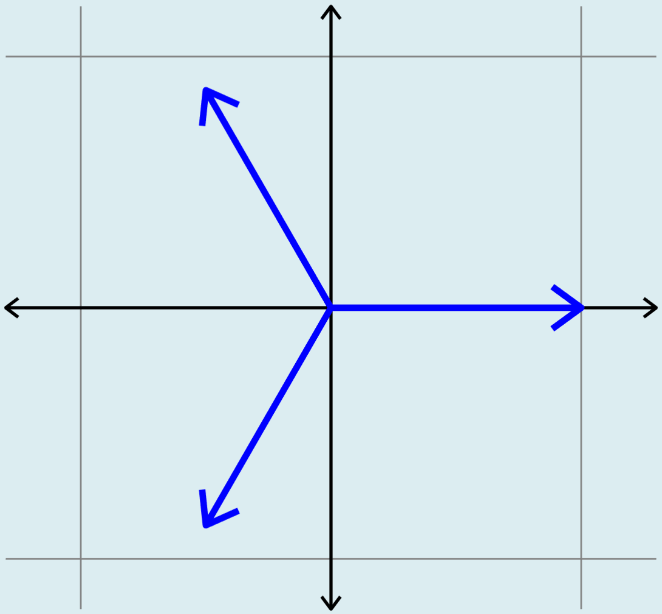

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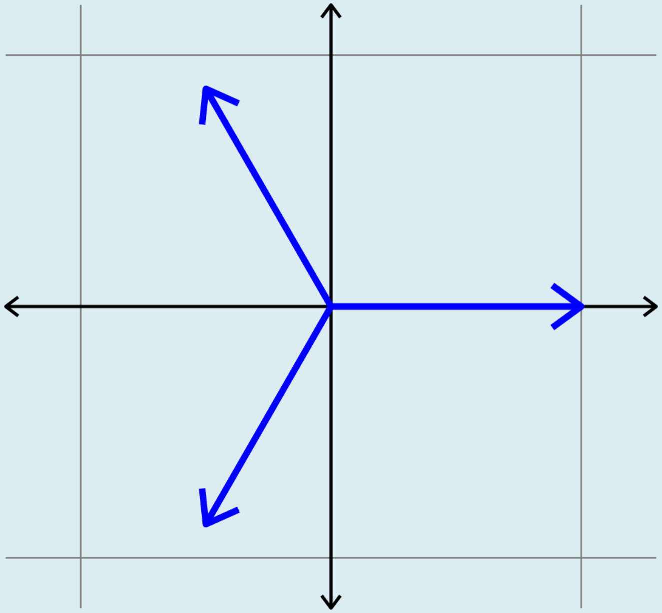

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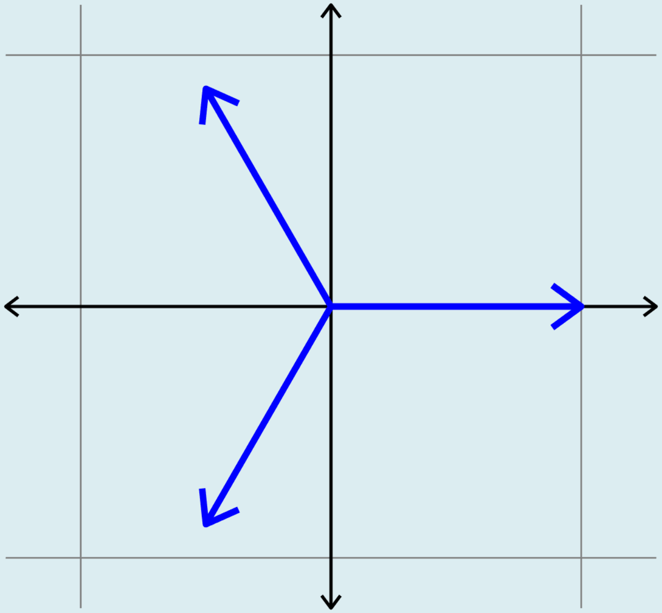

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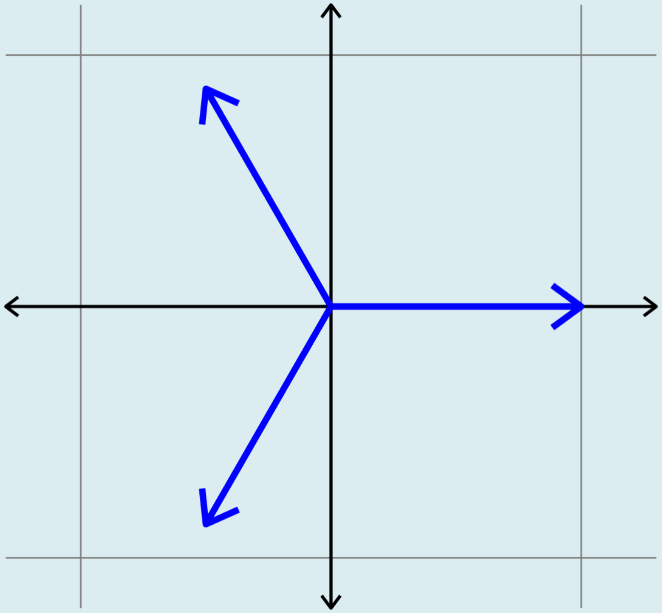

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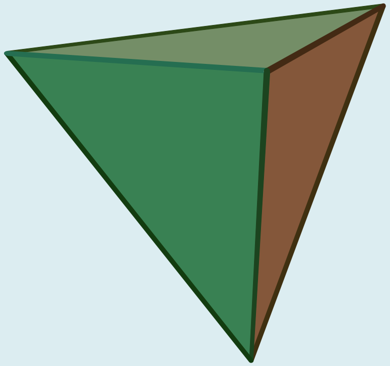
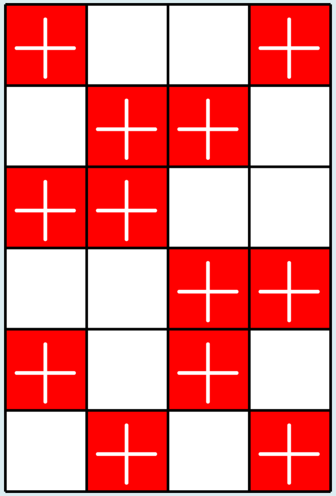


Example 3.

Steiner system

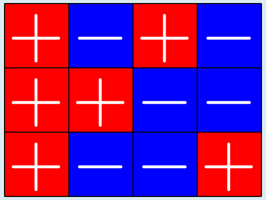
r ones per column



$\star$

$r \times (r+1)$ ETF with

unimodular entries



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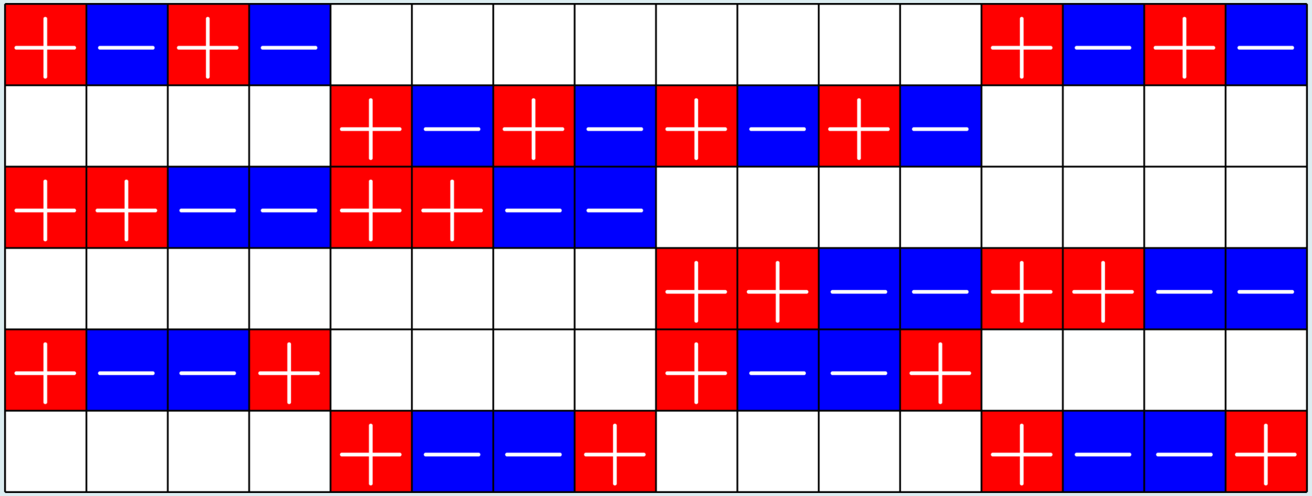

"Steiner" ETF

Goethals, Seidel, Can. J. Math. 1970

A McFarland difference set

$$D = \{(0,0,0,1),(0,0,1,0),(0,0,1,1),(0,1,1,1),(1,0,1,0),(1,1,0,1)\}$$

is a (McFarland) difference set in $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

The subgroup

$$H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0 \times 0 \leqslant G$$

is **disjoint from *D*.**

	$(0,0,0,1)$	$(1,1,0,1)$	$(0,0,1,0)$	$(1,0,1,0)$	$(0,0,1,1)$	$(0,1,1,1)$
$(0,0,0,1)$	$(0,0,0,0)$	$(1,1,0,0)$	$(0,0,1,1)$	$(1,0,1,1)$	$(0,0,1,0)$	$(0,1,1,0)$
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Ex:

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

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(0,0,0,0)	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
(0,0,0,1)	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
(0,0,1,0)	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
(0,0,1,1)	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
(0,1,0,0)	+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
(0,1,0,1)	+	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
(0,1,1,0)	+	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
(0,1,1,1)	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
(1,0,0,0)	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
(1,0,0,1)	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
(1,0,1,0)	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
(1,0,1,1)	+	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
(1,1,0,0)	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
(1,1,0,1)	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
(1,1,1,0)	+	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
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(0,0,1,0)	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
(0,0,1,1)	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
(0,1,0,0)	+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
(0,1,0,1)	+	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
(0,1,1,0)	+	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
(0,1,1,1)	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
(1,0,0,0)	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
(1,0,0,1)	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
(1,0,1,0)	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
(1,0,1,1)	+	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
(1,1,0,0)	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
(1,1,0,1)	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
(1,1,1,0)	+	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
(1,1,1,1)	+	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+

A McFarland difference set

	(0, 0, 0, 0)	(0, 0, 0, 1)	(0, 0, 1, 0)	(0, 0, 1, 1)	(0, 1, 0, 0)	(0, 1, 0, 1)	(0, 1, 1, 0)	(0, 1, 1, 1)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 0, 1, 0)	(1, 0, 1, 1)	(1, 1, 0, 0)	(1, 1, 0, 1)	(1, 1, 1, 0)	(1, 1, 1, 1)
(0, 0, 0, 1)	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
(0, 0, 1, 0)	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
(0, 0, 1, 1)	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
(0, 1, 1, 1)	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
(1, 0, 1, 0)	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
(1, 1, 0, 1)	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−

A McFarland difference set

	(0,0,0,0)	(0,0,0,1)	(0,0,1,0)	(0,0,1,1)	(0,1,0,0)	(0,1,0,1)	(0,1,1,0)	(0,1,1,1)	(1,0,0,0)	(1,0,0,1)	(1,0,1,0)	(1,0,1,1)	(1,1,0,0)	(1,1,0,1)	(1,1,1,0)	(1,1,1,1)
$(0,0,0,1)$	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
$(0,0,1,0)$	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
$(0,0,1,1)$	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
$(0,1,1,1)$	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
$(1,0,1,0)$	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
$(1,1,0,1)$	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−

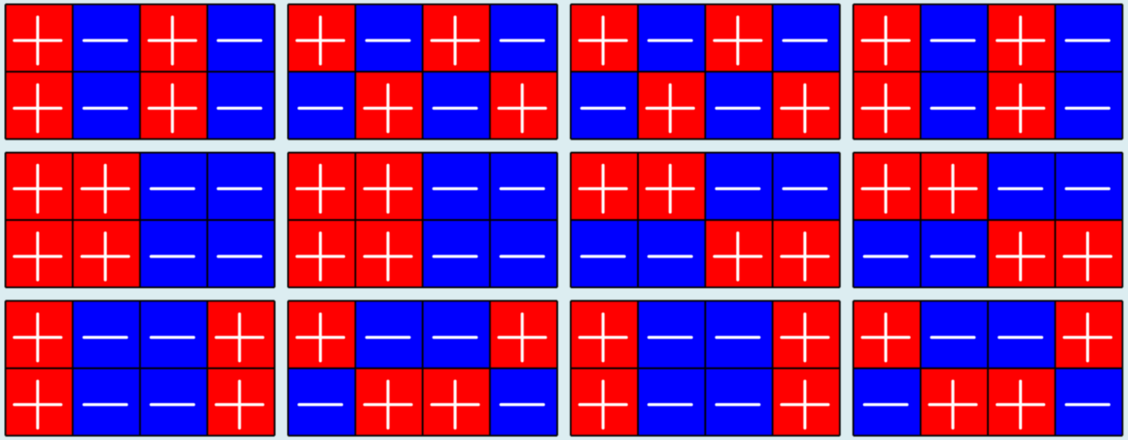
A McFarland difference set

	(0,0,0,0)	(0,0,0,1)	(0,0,1,0)	(0,0,1,1)	(0,1,0,0)	(0,1,0,1)	(0,1,1,0)	(0,1,1,1)	(1,0,0,0)	(1,0,0,1)	(1,0,1,0)	(1,0,1,1)	(1,1,0,0)	(1,1,0,1)	(1,1,1,0)	(1,1,1,1)
(0,0,0,1)	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
(1,1,0,1)	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
(0,0,1,0)	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
(1,0,1,0)	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
(0,0,1,1)	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
(0,1,1,1)	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−

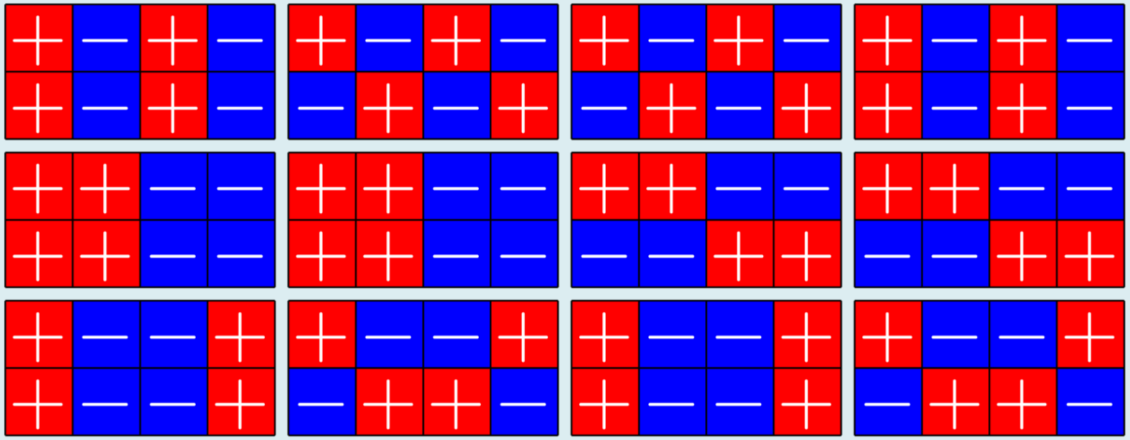
A McFarland difference set

	(0, 0, 0, 0)	(0, 0, 0, 1)	(0, 0, 1, 0)	(0, 0, 1, 1)	(0, 1, 0, 0)	(0, 1, 0, 1)	(0, 1, 1, 0)	(0, 1, 1, 1)	(1, 0, 0, 0)	(1, 0, 0, 1)	(1, 0, 1, 0)	(1, 0, 1, 1)	(1, 1, 0, 0)	(1, 1, 0, 1)	(1, 1, 1, 0)	(1, 1, 1, 1)
(0, 0, 0, 1)	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
(1, 1, 0, 1)	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
(0, 0, 1, 0)	+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
(1, 0, 1, 0)	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
(0, 0, 1, 1)	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
(0, 1, 1, 1)	+	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−

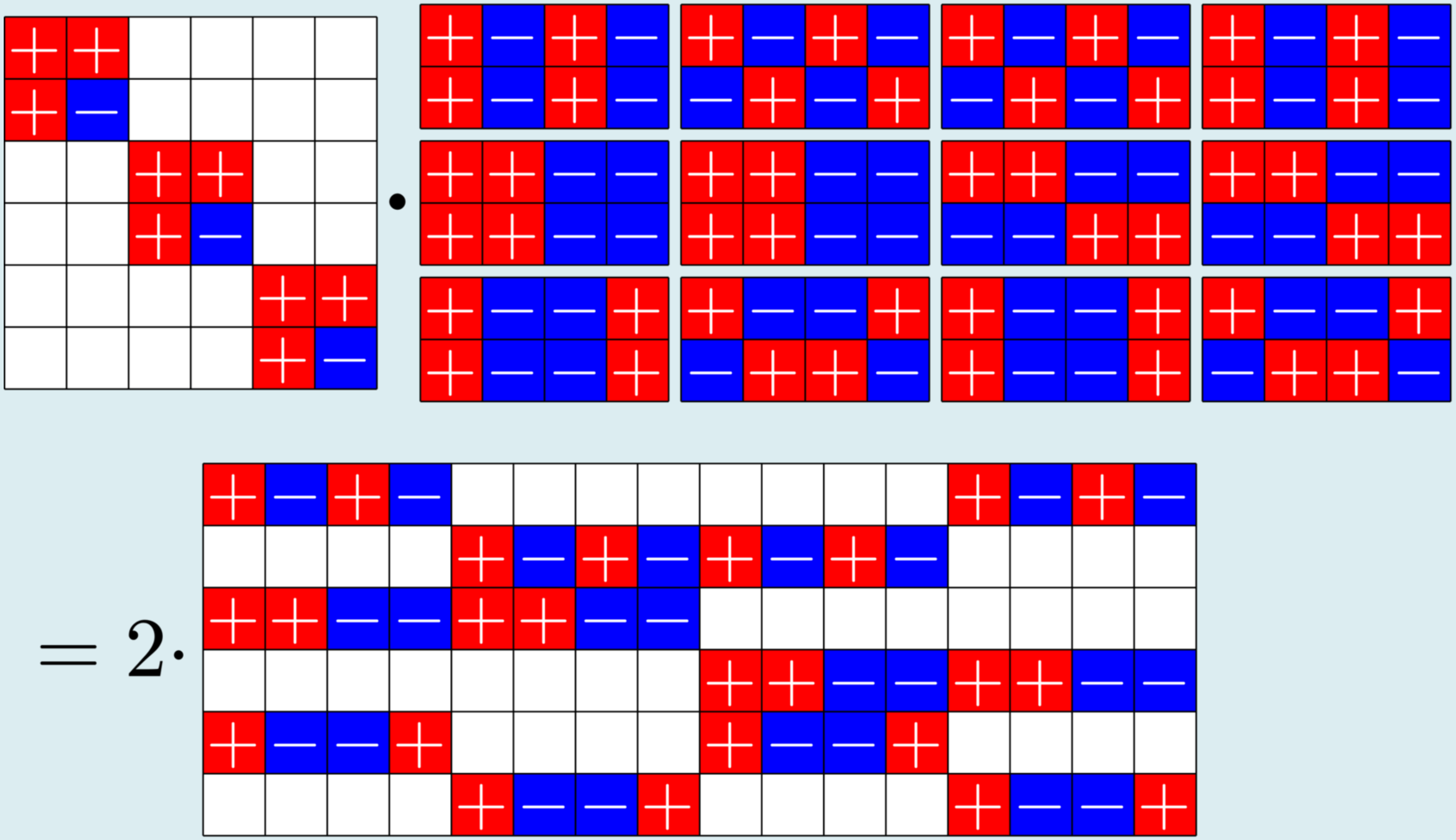
A McFarland difference set



A McFarland difference set



A McFarland difference set



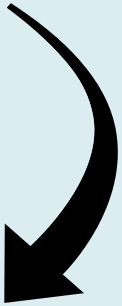
Graphs from ETFs

16 vectors in $\mathbb{R}^6$

$$\Phi = \begin{bmatrix}
+ & - & + & - & & & & & & & & & + & - & + & - \\
& & & & + & - & + & - & + & - & + & - & & & & \\
+ & + & - & - & + & + & - & - & & & & & & & & \\
& & & & & & & & + & + & - & - & + & + & - & - \\
+ & - & - & + & & & & & + & - & - & + & & & & \\
& & & & + & - & - & + & & & & & + & - & - & +
\end{bmatrix}$$

16 vectors in $\mathbb{R}^6$

$$\Phi = \begin{bmatrix}
+ & - & + & - & & & & & & & & & + & - & + & - \\
 & & & & + & - & + & - & + & - & + & - & & & & \\
+ & + & - & - & + & + & - & - & & & & & & & & \\
 & & & & & & & & + & + & - & - & + & + & - & - \\
+ & - & - & + & & & & & + & - & - & + & & & & \\
 & & & & + & - & - & + & & & & & + & - & - & +
\end{bmatrix}$$



$$\Phi^\top \Phi = \begin{bmatrix}
3 & - & - & - & + & + & - & - & + & - & - & + & + & - & + & - \\
- & 3 & - & - & + & + & - & - & - & + & + & - & - & + & - & + \\
- & - & 3 & - & - & - & + & + & - & + & + & - & + & - & + & - \\
- & - & - & 3 & - & - & + & + & + & - & - & + & - & + & - & + \\
+ & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + \\
+ & + & - & - & - & 3 & - & - & - & + & - & + & - & + & + & - \\
- & - & + & + & - & - & 3 & - & + & - & + & - & - & + & + & - \\
- & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + \\
+ & - & - & + & + & - & + & - & 3 & - & - & - & + & + & - & - \\
- & + & + & - & - & + & - & + & - & 3 & - & - & + & + & - & - \\
- & + & + & - & + & - & + & - & - & - & 3 & - & - & - & + & + \\
+ & - & - & + & - & + & - & + & - & - & - & 3 & - & - & + & + \\
+ & - & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - \\
- & + & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - \\
+ & - & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - \\
- & + & - & + & + & - & - & + & - & - & + & + & - & - & - & 3
\end{bmatrix}$$

Gramians are forgetful

$$
\Phi^\top \Phi = \begin{bmatrix}
3 & - & - & - & + & + & - & - & + & - & - & + & + & - & + & - \\
- & 3 & - & - & + & + & - & - & - & + & + & - & - & + & - & + \\
- & - & 3 & - & - & - & + & + & - & + & + & - & + & - & + & - \\
- & - & - & 3 & - & - & + & + & + & - & - & + & - & + & - & + \\
+ & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + \\
+ & + & - & - & - & 3 & - & - & - & + & - & + & - & + & + & - \\
- & - & + & + & - & - & 3 & - & + & - & + & - & - & + & + & - \\
- & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + \\
+ & - & - & + & + & - & + & - & 3 & - & - & - & + & + & - & - \\
- & + & + & - & - & + & - & + & - & 3 & - & - & + & + & - & - \\
- & + & + & - & + & - & + & - & - & - & 3 & - & - & - & + & + \\
+ & - & - & + & - & + & - & + & - & - & - & 3 & - & - & + & + \\
+ & - & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - \\
- & + & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - \\
+ & - & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - \\
- & + & - & + & + & - & - & + & - & - & + & + & - & - & - & 3
\end{bmatrix}
$$

Given $\Phi^\top \Phi$, Φ is only determined up to a unitary

Gramians are forgetful

$$\Phi =$$

$$\begin{bmatrix}
0.0726 & 0.6120 & -0.7461 & -0.2067 & 0.0624 & 0.6018 & 0.1621 & 0.7015 & 0.0124 & 1.5703 & -0.8064 & 0.7516 & -0.8959 & 0.6621 & -0.7962 & 0.7618 \\
0.3319 & 0.1969 & 0.3534 & 0.2184 & -0.7400 & -0.8750 & -0.8007 & -0.9357 & -1.3141 & -0.3831 & -1.2926 & -0.3616 & -0.1600 & 0.7710 & -0.2207 & 0.7103 \\
0.2495 & 1.1812 & -0.9988 & -0.0670 & -0.1924 & 0.7394 & -1.1314 & -0.1996 & 0.7834 & 0.0729 & -0.4648 & -1.1754 & 0.9160 & 0.2055 & -0.0230 & -0.7336 \\
-1.4067 & -0.0454 & -0.0988 & 1.2626 & -0.3747 & 0.9867 & -0.9742 & 0.3872 & -0.7656 & -0.5298 & 0.5423 & 0.7781 & 0.1098 & 0.3456 & -0.4898 & -0.2539 \\
-0.8974 & -0.3692 & -1.1365 & -0.6083 & -1.0782 & -0.5501 & -0.1197 & 0.4084 & -0.1298 & -0.3009 & -0.3689 & -0.5400 & -1.1466 & -1.3176 & -0.1881 & -0.3591 \\
-0.1953 & -1.0262 & -0.1392 & -0.9700 & 1.0528 & 0.2220 & -0.2986 & -1.1295 & -0.2369 & 0.1040 & -0.1807 & 0.1602 & -0.0774 & 0.2635 & -1.4288 & -1.0879
\end{bmatrix}$$

$$\Phi^\top \Phi = \begin{bmatrix}
3 & - & - & - & + & + & - & - & + & - & - & + & + & - & + & - \\
- & 3 & - & - & + & + & - & - & - & + & + & - & - & + & - & + \\
- & - & 3 & - & - & - & + & + & - & + & + & - & + & - & + & - \\
- & - & - & 3 & - & - & + & + & + & - & - & + & - & + & - & + \\
+ & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + \\
+ & + & - & - & - & 3 & - & - & - & + & - & + & - & + & + & - \\
- & - & + & + & - & - & 3 & - & + & - & + & - & - & + & + & - \\
- & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + \\
+ & - & - & + & + & - & + & - & 3 & - & - & - & + & + & - & - \\
- & + & + & - & - & + & - & + & - & 3 & - & - & + & + & - & - \\
- & + & + & - & + & - & + & - & - & - & 3 & - & - & - & + & + \\
+ & - & - & + & - & + & - & + & - & - & - & 3 & - & - & + & + \\
+ & - & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - \\
- & + & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - \\
+ & - & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - \\
- & + & - & + & + & - & - & + & - & - & + & + & - & - & - & 3
\end{bmatrix}$$

Given $\Phi^\top \Phi$, Φ is only determined up to a unitary

ETFs $\Rightarrow$ Graphs

Gram matrix of ETF:

$$\Phi = \begin{bmatrix} + & - & + & - & & & & & & & & & + & - & + & - \\ & & & & + & - & + & - & + & - & + & - & & & & \\ + & + & - & - & + & + & - & - & & & & & & & & \\ & & & & & & & & + & + & - & - & + & + & - & - \\ + & - & - & + & & & & & + & - & - & + & & & & \\ & & & & + & - & - & + & & & & & + & - & - & + \end{bmatrix}$$

$$\Phi^*\Phi = \begin{bmatrix} 3 & - & - & - & + & + & - & - & + & - & - & + & + & - & + & - \\ - & 3 & - & - & + & + & - & - & - & + & + & - & - & + & - & + \\ - & - & 3 & - & - & - & + & + & - & + & + & - & + & - & + & - \\ - & - & - & 3 & - & - & + & + & + & - & - & + & - & + & - & + \\ + & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + \\ + & + & - & - & - & 3 & - & - & - & + & - & + & - & + & + & - \\ - & - & + & + & - & - & 3 & - & + & - & + & - & - & + & + & - \\ - & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + \\ + & - & - & + & + & - & + & - & 3 & - & - & - & + & + & - & - \\ - & + & + & - & - & + & - & + & - & 3 & - & - & + & + & - & - \\ - & + & + & - & + & - & + & - & - & - & 3 & - & - & - & + & + \\ + & - & - & + & - & + & - & + & - & - & - & 3 & - & - & + & + \\ + & - & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - \\ - & + & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - \\ + & - & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - \\ - & + & - & + & + & - & - & + & - & - & + & + & - & - & - & 3 \end{bmatrix}$$

ETFs ⇒ Graphs

Gram matrix of ETF:

$\Phi =$

+	−	+	−									+	−	+	−
				+	−	+	−	+	−	+	−				
+	+	−	−	+	+	−	−								
								+	+	−	−	+	+	−	−
+	−	−	+					+	−	−	+				
				+	−	−	+					+	−	−	+

$\Phi^*\Phi =$

3	−	−	−	+	+	−	−	+	−	−	+	+	−	+	−
−	3	−	−	+	+	−	−	−	+	+	−	−	+	−	+
−	−	3	−	−	−	+	+	−	+	+	−	+	−	+	−
−	−	−	3	−	−	+	+	+	−	−	+	−	+	−	+
+	+	−	−	3	−	−	−	+	−	+	−	+	−	−	+
+	+	−	−	−	3	−	−	−	+	−	+	−	+	+	−
−	−	+	+	−	−	3	−	+	−	+	−	−	+	+	−
−	−	+	+	−	−	−	3	−	+	−	+	+	−	−	+
+	−	−	+	+	−	+	−	3	−	−	−	+	+	−	−
−	+	+	−	−	+	−	+	−	3	−	−	+	+	−	−
−	+	+	−	+	−	+	−	−	−	3	−	−	−	+	+
+	−	−	+	−	+	−	+	−	−	−	3	−	−	+	+
+	−	+	−	+	−	−	+	+	+	−	−	3	−	−	−
−	+	−	+	−	+	+	−	+	+	−	−	−	3	−	−
+	−	+	−	−	+	+	−	−	−	+	+	−	−	3	−
−	+	−	+	+	−	−	+	−	−	+	+	−	−	−	3

Adjacency matrix of graph:

Replace diagonal 1's with 0's

Replace -1's with 1's

Zero out the diagonal

$G =$

0	1	1	1	0	0	1	1	0	1	1	0	0	1	0	1
1	0	1	1	0	0	1	1	1	0	0	1	1	0	1	0
1	1	0	1	1	1	0	0	1	0	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	1	1	0	1	0	1	0
0	0	1	1	0	1	1	1	0	1	0	1	0	1	1	0
0	0	1	1	1	0	1	1	1	0	1	0	1	0	0	1
1	1	0	0	1	1	0	1	0	1	0	1	1	0	0	1
1	1	0	0	1	1	1	0	1	0	1	0	0	1	1	0
0	1	1	0	0	1	0	1	0	1	1	1	0	0	1	1
1	0	0	1	1	0	1	0	1	0	1	1	0	0	1	1
1	0	0	1	0	1	0	1	1	1	0	1	1	1	0	0
0	1	1	0	1	0	1	0	1	1	1	0	1	1	0	0
0	1	0	1	0	1	1	0	0	0	1	1	0	1	1	1
1	0	1	0	1	0	0	1	0	0	1	1	1	0	1	1
0	1	0	1	1	0	0	1	1	1	0	0	1	1	0	1
1	0	1	0	0	1	1	0	1	1	0	0	1	1	1	0

ETFs ⇒ Graphs

Gram matrix of ETF:

$$\Phi = \begin{bmatrix} + & - & + & - & & & & & & & & & + & - & + & - \\ & & & & + & - & + & - & + & - & + & - & & & & \\ + & + & - & - & + & + & - & - & & & & & & & & \\ & & & & & & & & + & + & - & - & + & + & - & - \\ + & - & - & + & & & & & + & - & - & + & & & & \\ & & & & + & - & - & + & & & & & + & - & - & + \end{bmatrix}$$

$$\Phi^*\Phi = \begin{bmatrix} 3 & - & - & - & + & + & - & - & + & - & - & + & + & - & + & - \\ - & 3 & - & - & + & + & - & - & - & + & + & - & - & + & - & + \\ - & - & 3 & - & - & - & + & + & - & + & + & - & + & - & + & - \\ - & - & - & 3 & - & - & + & + & + & - & - & + & - & + & - & + \\ + & + & - & - & 3 & - & - & - & + & - & + & - & + & - & - & + \\ + & + & - & - & - & 3 & - & - & - & + & - & + & - & + & + & - \\ - & - & + & + & - & - & 3 & - & + & - & + & - & - & + & + & - \\ - & - & + & + & - & - & - & 3 & - & + & - & + & + & - & - & + \\ + & - & - & + & + & - & + & - & 3 & - & - & - & + & + & - & - \\ - & + & + & - & - & + & - & + & - & 3 & - & - & + & + & - & - \\ - & + & + & - & + & - & + & - & - & - & 3 & - & - & - & + & + \\ + & - & - & + & - & + & - & + & - & - & - & 3 & - & - & + & + \\ + & - & + & - & + & - & - & + & + & + & - & - & 3 & - & - & - \\ - & + & - & + & - & + & + & - & + & + & - & - & - & 3 & - & - \\ + & - & + & - & - & + & + & - & - & - & + & + & - & - & 3 & - \\ - & + & - & + & + & - & - & + & - & - & + & + & - & - & - & 3 \end{bmatrix}$$

Adjacency matrix of graph:

- Every vertex has 9 neighbors
- Adjacent vertices have 4 common neighbors
- Non-adjacent vertices have 6 common neighbors

$$G = \begin{bmatrix} 0&1&1&1&0&0&1&1&0&1&1&0&0&1&0&1 \\ 1&0&1&1&0&0&1&1&1&0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&0&0&1&0&0&1&0&1&0&1 \\ 1&1&1&0&1&1&0&0&0&1&1&0&1&0&1&0 \\ 0&0&1&1&0&1&1&1&0&1&0&1&0&1&1&0 \\ 0&0&1&1&1&0&1&1&1&0&1&0&1&0&0&1 \\ 1&1&0&0&1&1&0&1&0&1&0&1&1&0&0&1 \\ 1&1&0&0&1&1&1&0&1&0&1&0&0&1&1&0 \\ 0&1&1&0&0&1&0&1&0&1&1&1&0&0&1&1 \\ 1&0&0&1&1&0&1&0&1&0&1&1&0&0&1&1 \\ 1&0&0&1&0&1&0&1&1&1&0&1&1&1&0&0 \\ 0&1&1&0&1&0&1&0&1&1&1&0&1&1&0&0 \\ 0&1&0&1&0&1&1&0&0&0&1&1&0&1&1&1 \\ 1&0&1&0&1&0&0&1&0&0&1&1&1&0&1&1 \\ 0&1&0&1&1&0&0&1&1&1&0&0&1&1&0&1 \\ 1&0&1&0&0&1&1&0&1&1&0&0&1&1&1&0 \end{bmatrix}$$

ETFs $\Rightarrow$ Graphs

Gram matrix of ETF:

$\Phi =$

+	−	+	−									+	−	+	−
				+	−	+	−	+	−	+	−				
+	+	−	−	+	+	−	−								
								+	+	−	−	+	+	−	−
+	−	−	+					+	−	−	+				
				+	−	−	+					+	−	−	+

$\Phi^*\Phi =$

3	−	−	−	+	+	−	−	+	−	−	+	+	−	+	−
−	3	−	−	+	+	−	−	−	+	+	−	−	+	−	+
−	−	3	−	−	−	+	+	−	+	+	−	+	−	+	−
−	−	−	3	−	−	+	+	+	−	−	+	−	+	−	+
+	+	−	−	3	−	−	−	+	−	+	−	+	−	−	+
+	+	−	−	−	3	−	−	−	+	−	+	−	+	+	−
−	−	+	+	−	−	3	−	+	−	+	−	−	+	+	−
−	−	+	+	−	−	−	3	−	+	−	+	+	−	−	+
+	−	−	+	+	−	+	−	3	−	−	−	+	+	−	−
−	+	+	−	−	+	−	+	−	3	−	−	+	+	−	−
−	+	+	−	+	−	+	−	−	−	3	−	−	−	+	+
+	−	−	+	−	+	−	+	−	−	−	3	−	−	+	+
+	−	+	−	+	−	−	+	+	+	−	−	3	−	−	−
−	+	−	+	−	+	+	−	+	+	−	−	−	3	−	−
+	−	+	−	−	+	+	−	−	−	+	+	−	−	3	−
−	+	−	+	+	−	−	+	−	−	+	+	−	−	−	3

Adjacency matrix of graph:

$G =$

0	1	1	1	0	0	1	1	0	1	1	0	0	1	0	1
1	0	1	1	0	0	1	1	1	0	0	1	1	0	1	0
1	1	0	1	1	1	0	0	1	0	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	1	1	0	1	0	1	0
0	0	1	1	0	1	1	1	0	1	0	1	0	1	1	0
0	0	1	1	1	0	1	1	1	0	1	0	1	0	0	1
1	1	0	0	1	1	0	1	0	1	0	1	1	0	0	1
1	1	0	0	1	1	1	0	1	0	1	0	0	1	1	0
0	1	1	0	0	1	0	1	0	1	1	1	0	0	1	1
1	0	0	1	1	0	1	0	1	0	1	1	0	0	1	1
1	0	0	1	0	1	0	1	1	1	0	1	1	1	0	0
0	1	1	0	1	0	1	0	1	1	1	0	1	1	0	0
0	1	0	1	0	1	1	0	0	0	1	1	0	1	1	1
1	0	1	0	1	0	0	1	0	0	1	1	1	0	1	1
0	1	0	1	1	0	0	1	1	1	0	0	1	1	0	1
1	0	1	0	0	1	1	0	1	1	0	0	1	1	1	0

ETFs $\Rightarrow$ Graphs

Gram matrix of ETF:

Mult. by -1

$$\Phi = \begin{bmatrix}
+ & - & + & - & & & & & & & & & + & - & + & - \\
 & & & & + & - & - & - & + & - & + & - & & & & \\
+ & + & - & - & + & + & + & - & & & & & & & & \\
 & & & & & & & & + & + & - & - & + & + & - & - \\
+ & - & - & + & & & & & + & - & - & + & & & & \\
 & & & & + & - & + & + & & & & & + & - & - & +
\end{bmatrix}$$

$$\Phi^*\Phi = \begin{bmatrix}
3 & - & - & - & + & + & + & - & + & - & - & + & + & - & + & - \\
- & 3 & - & - & + & + & + & - & - & + & + & - & - & + & - & + \\
- & - & 3 & - & - & - & - & + & - & + & + & - & + & - & + & - \\
- & - & - & 3 & - & - & - & + & + & - & - & + & - & + & - & + \\
+ & + & - & - & 3 & - & + & - & + & - & + & - & + & - & - & + \\
+ & + & - & - & - & 3 & + & - & - & + & - & + & - & + & + & - \\
+ & + & - & - & + & + & 3 & + & - & + & - & + & + & - & - & + \\
- & - & + & + & - & - & + & 3 & - & + & - & + & + & - & - & + \\
+ & - & - & + & + & - & - & - & 3 & - & - & - & + & + & - & - \\
- & + & + & - & - & + & + & + & - & 3 & - & - & + & + & - & - \\
- & + & + & - & + & - & - & - & - & - & 3 & - & - & - & + & + \\
+ & - & - & + & - & + & + & + & - & - & - & 3 & - & - & + & + \\
+ & - & + & - & + & - & + & + & + & + & - & - & 3 & - & - & - \\
- & + & - & + & - & + & - & - & + & + & - & - & - & 3 & - & - \\
+ & - & + & - & - & + & - & - & - & - & + & + & - & - & 3 & - \\
- & + & - & + & + & - & + & + & - & - & + & + & - & - & - & 3
\end{bmatrix}$$

Adjacency matrix of graph:

Not regular!

$$G = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}$$

Strongly Regular Graphs

Definition. An n-vertex graph is called ***strongly regular*** if

- every vertex has k neighbors
- adjacent vertices have λ common neighbors
- **non**adjacent vertices have μ common neighbors

Such a graph is called an SRG(n, k, λ, μ).

Equivalently, the adjacency matrix G satisfies

$$G^2 = kI + \lambda G + \mu(J - I - G)$$ (where J is the all-ones matrix.)

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$$G^2 = kI + \lambda G + \mu(J - I - G)$$ (where J is the all-ones matrix.)

Φ ETF $\Rightarrow$ "quadratic relation":

$$(\Phi^\top \Phi)^2 = A\,\Phi^\top \Phi.$$

But how many -1's in each row ???

ETF ⇒ SRG

Suppose Φ is an ETF:

- $\mathbf{1} := (1, 1, \ldots, 1) \in \ker \Phi$
- $\mathbf{1} \in$ row space of Φ, or

$\Bigg\} \Leftrightarrow$ $\mathbf{1}$ is an eigenvector of $\Phi^\top \Phi$.

$\Rightarrow$ the associated graph is regular and thus **strongly regular**

ETF $\Rightarrow$ SRG

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$\Rightarrow$ the associated graph is regular and thus **strongly regular**

Example.

$$\Phi = \begin{bmatrix} + & + & + & + & & & & & & & & & + & + & + & + \\ & & & & + & + & + & + & + & + & + & + & & & & \\ + & - & - & + & + & - & - & + & & & & & & & & \\ & & & & & & & & + & - & - & + & + & - & - & + \\ + & + & - & - & & & & & + & + & - & - & & & & \\ & & & & + & + & - & - & & & & & + & + & - & - \end{bmatrix}$$

$\mathbf{1}$ in row space

SRG$(16, 5, 0, 2)$

$$\Phi = \begin{bmatrix} + & - & + & - & & & & & & & & & + & - & + & - \\ & & & & + & - & + & - & + & - & + & - & & & & \\ + & + & - & - & + & + & - & - & & & & & & & & \\ & & & & & & & & + & + & - & - & + & + & - & - \\ + & - & - & + & & & & & + & - & - & + & & & & \\ & & & & + & - & - & + & & & & & + & - & - & + \end{bmatrix}$$

$\mathbf{1} \in \ker \Phi$

SRG$(16, 9, 4, 6)$

ETF $\Rightarrow$ SRG

Suppose Φ is an ETF:

- $\mathbf{1} := (1, 1, \ldots, 1) \in \ker \Phi$
- $\mathbf{1} \in$ row space of Φ, or

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$\Rightarrow$ the associated graph is regular and thus **strongly regular**

Example.

$\Phi =$

+	+	+	+									+	+	+	+
				+	+	+	+	+	+	+	+				
+	−	−	+	+	−	−	+								
								+	−	−	+	+	−	−	+
+	+	−	−					+	+	−	−				
				+	+	−	−					+	+	−	−

$\mathbf{1}$ in row space

SRG$(16, 5, 0, 2)$

$\Phi =$

+	−	+	−									+	−	+	−
				+	−	+	−	+	−	+	−				
+	+	−	−	+	+	−	−								
								+	+	−	−	+	+	−	−
+	−	−	+					+	−	−	+				
				+	−	−	+					+	−	−	+

$\mathbf{1} \in \ker \Phi$

SRG$(16, 9, 4, 6)$

Nice ETF representation $\Rightarrow$ new SRGs!

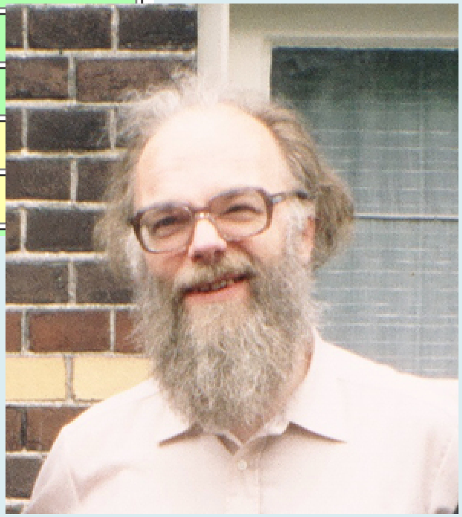
Andries Brouwer's Table of SRGs

	v	k	λ	μ	r^f	s^g	comments
+	81	24	9	6	6^{24}	-3^{56}	OA(9,3); Wallis (AR(3,1)+S(2,3,9)); $VNO^+(4,3)$ affine polar graph; from a partial spread: projective ternary [12,4] code with weights 6, 9
		56	37	42	2^{56}	-7^{24}	OA(9,7)
+	81	30	9	12	3^{50}	-6^{30}	Mesner; pg(5,5,2) - van Lint & Schrijver; $VNO^-(4,3)$ affine polar graph; Hamada-Helleseth: projective ternary [15,4] code with weights 9, 12
		50	31	30	5^{30}	-4^{50}	
+	81	32	13	12	5^{32}	-4^{48}	OA(9,4); $Bilin_{2x2}(3)$; vanLint-Schrijver(2); Wallis2 (AR(3,1)+S(2,3,9)); $VO^+(4,3)$ affine polar graph; from a partial spread: projective ternary [16,4] code with weights 9, 12
		48	27	30	3^{48}	-6^{32}	OA(9,6); vanLint-Schrijver(3)
-	81	40	13	26	1^{72}	-14^{8}	Absolute bound
		40	25	14	13^{8}	-2^{72}	Absolute bound
+	81	40	19	20	4^{40}	-5^{40}	Paley(81); OA(9,5); 2-graph*
+	82	36	15	16	4^{41}	-5^{40}	switch OA(9,5)+*; 2-graph
		45	24	25	4^{40}	-5^{41}	S(2,5,41); 2-graph
?	85	14	3	2	4^{34}	-3^{50}	
		70	57	60	2^{50}	-5^{34}	

= ∄

= ???

= ∃



Andries Brouwer's Table of SRGs

	v	k	λ	μ	r^f	s^g	comments
+	81	24	9	6	6^{24}	-3^{56}	OA(9,3); Wallis (AR(3,1)+S(2,3,9)); $VNO^+(4,3)$ affine polar graph; from a partial spread: projective ternary [12,4] code with weights 6, 9
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		50	31	30	5^{30}	-4^{50}	
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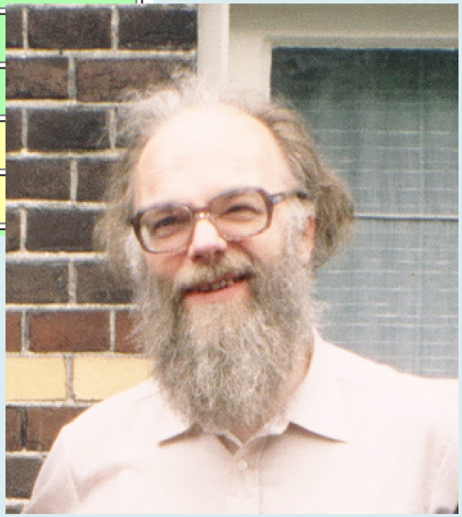
+	82	36	15	16	4^{41}	-5^{40}	switch OA(9,5)+*; 2-graph
		45	24	25	4^{40}	-5^{41}	S(2,5,41); 2-graph
?	85	14	3	2	4^{34}	-3^{50}	
		70	57	60	2^{50}	-5^{34}	

(red) = ∄

(yellow) = ???

(green) = ∃

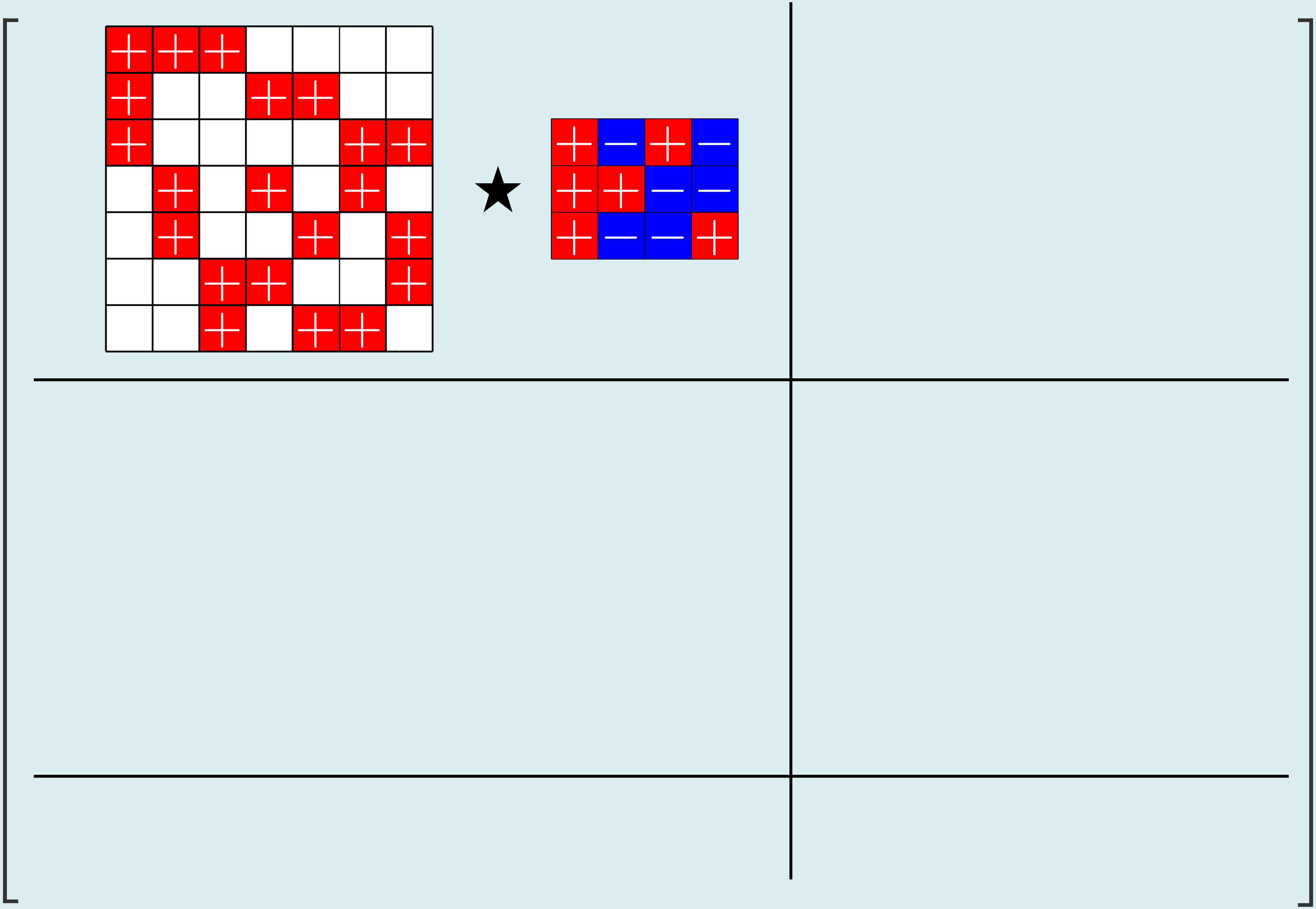


Goal: (yellow) ↦ (green)

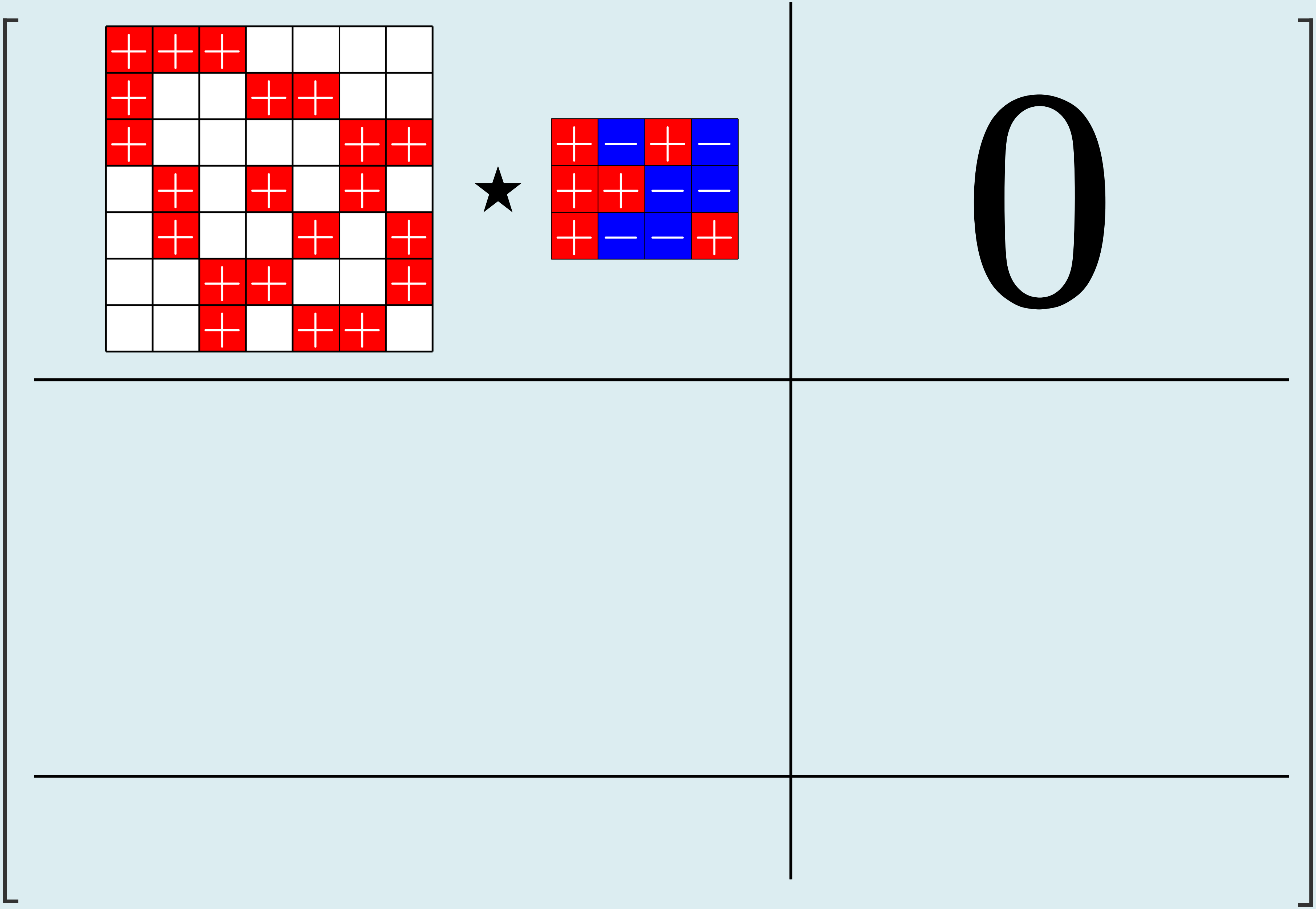
Prototype

Tremain ETFs

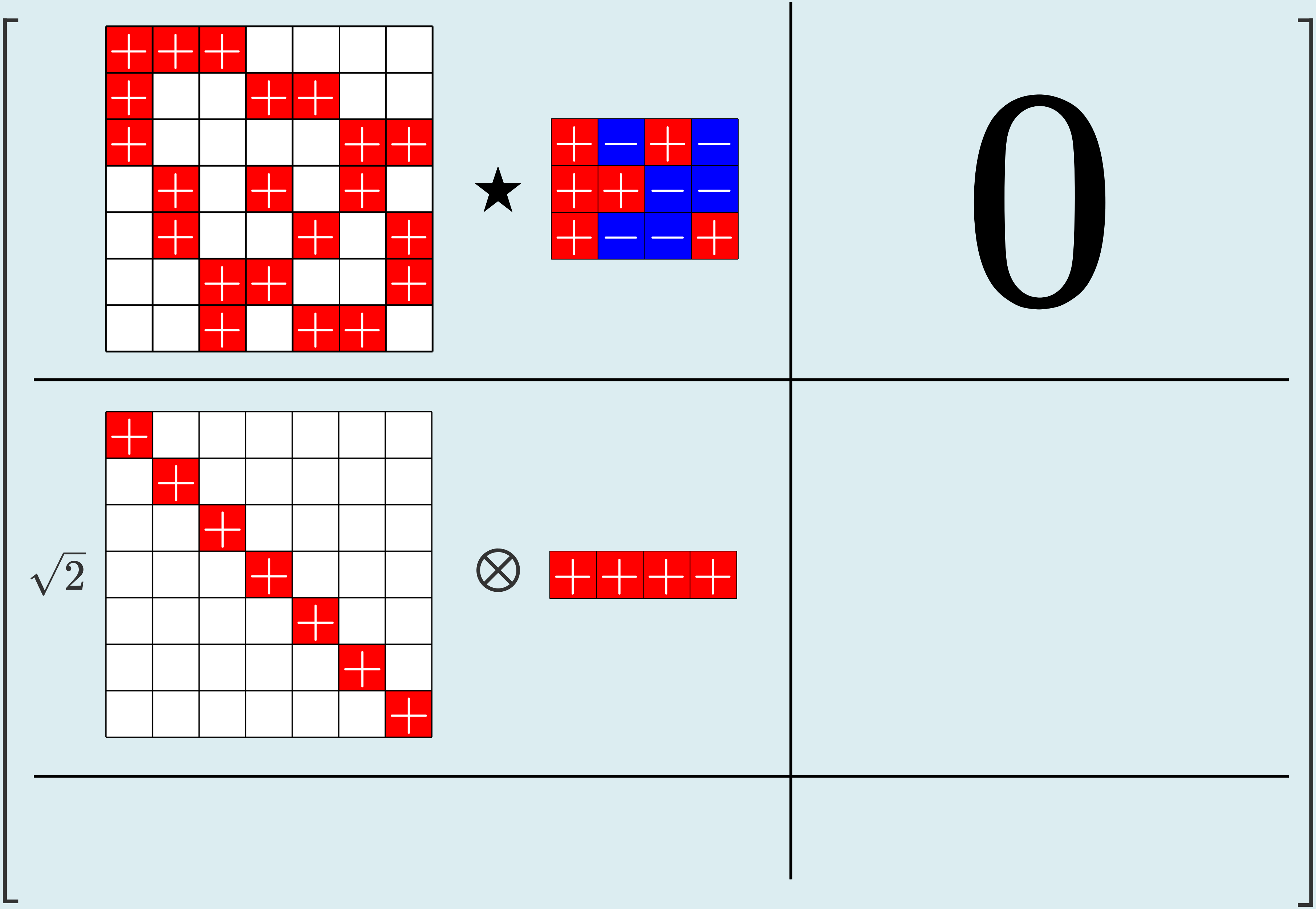
Tremain ETFs



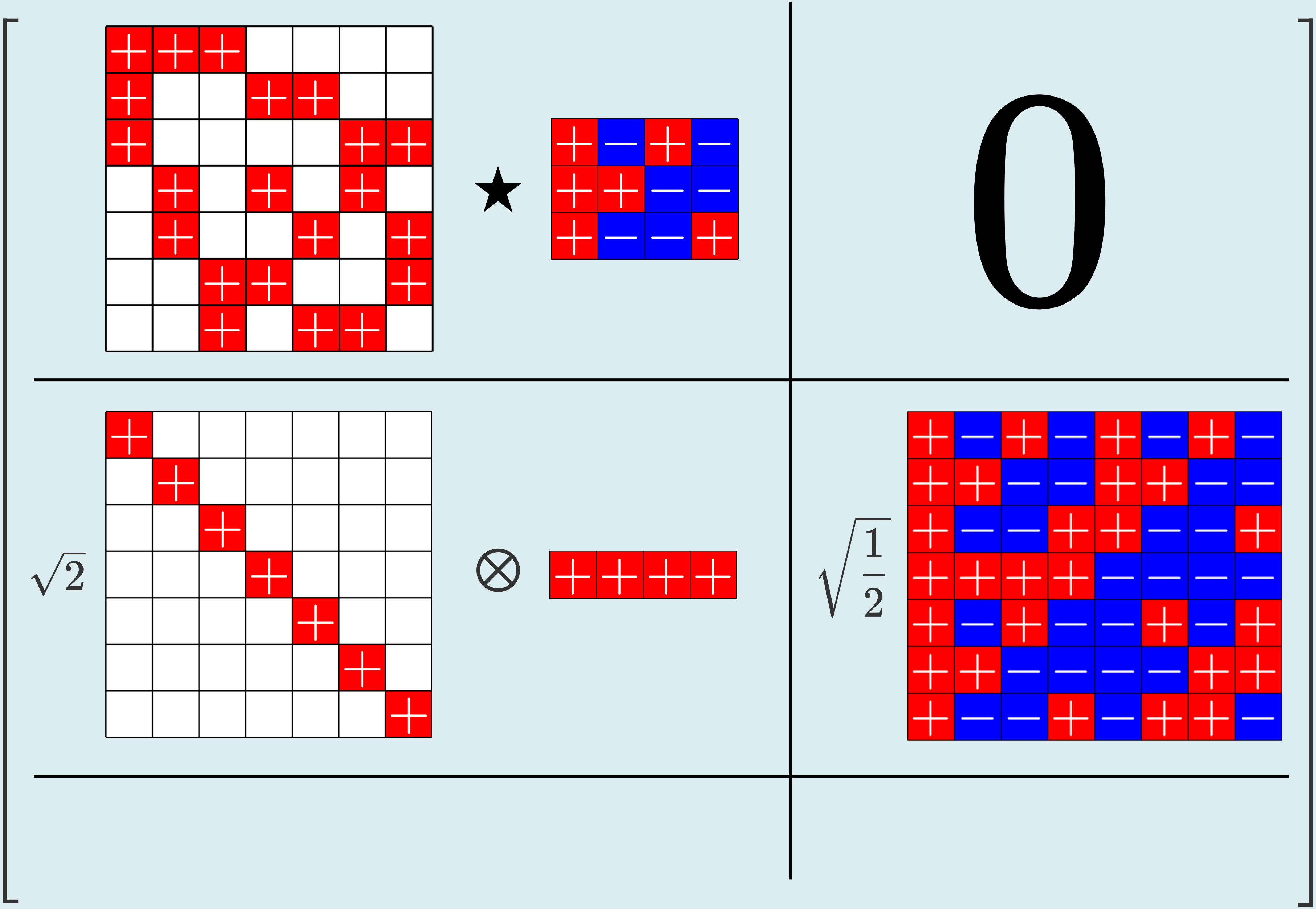
Tremain ETFs



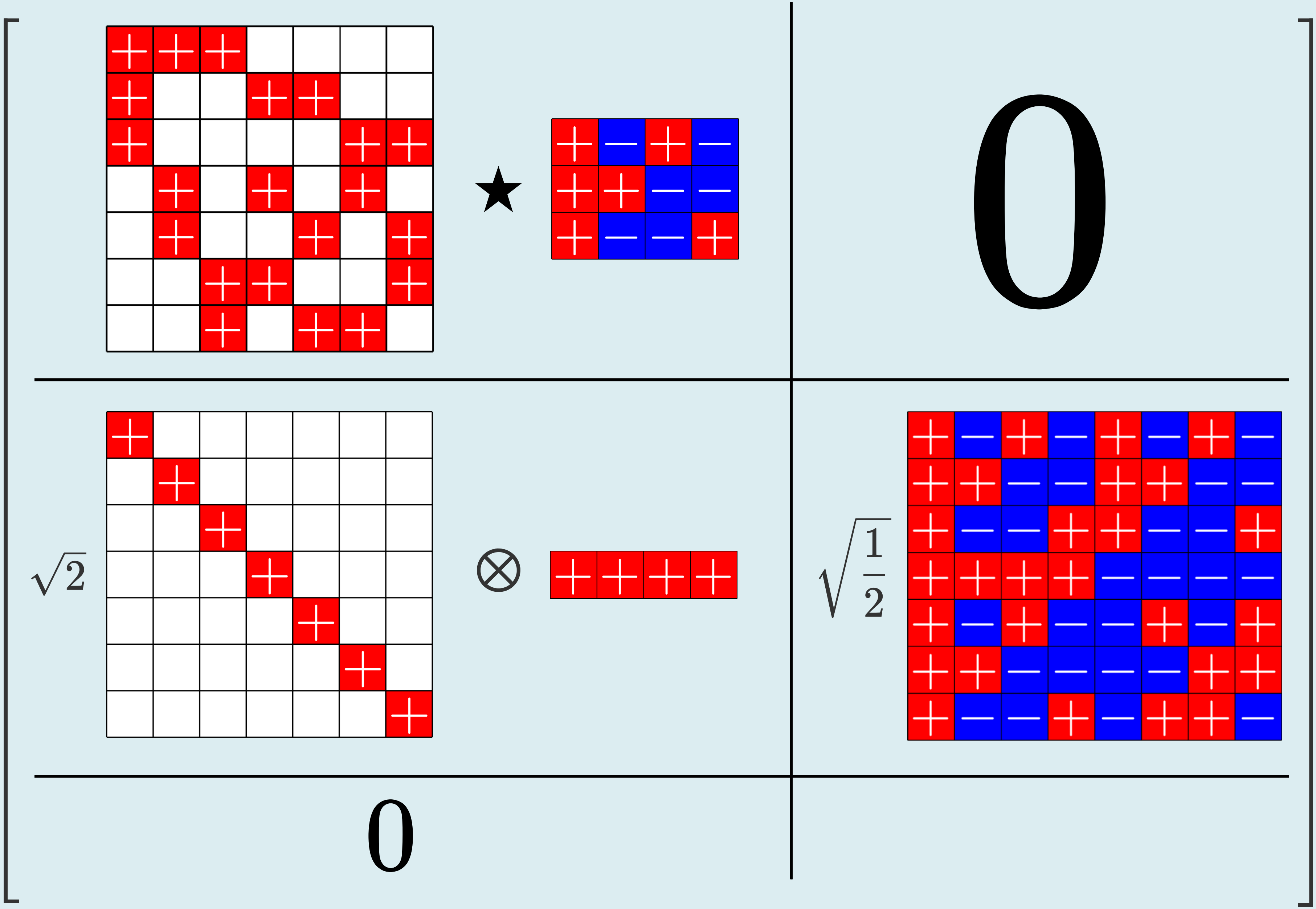
Tremain ETFs



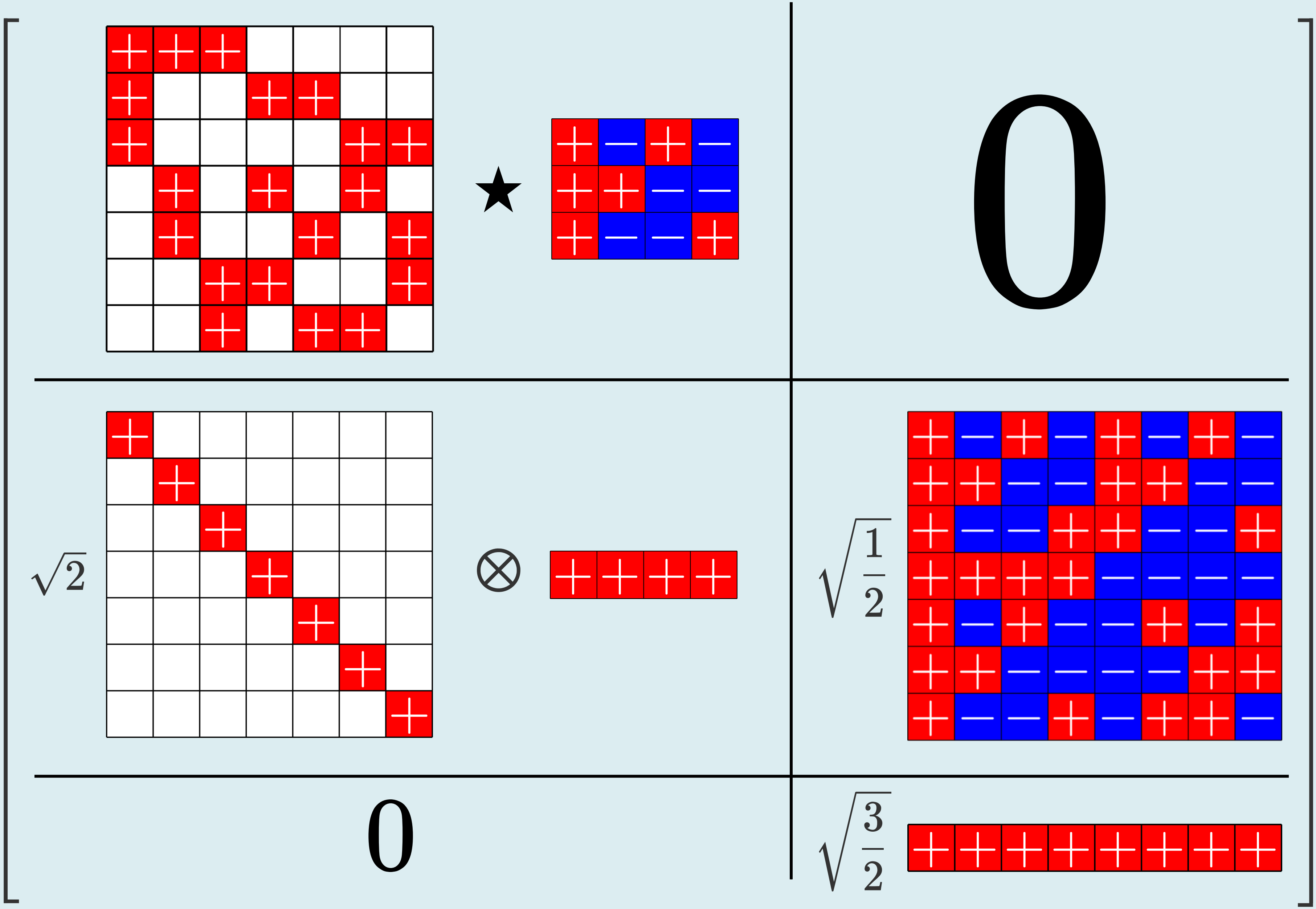
Tremain ETFs

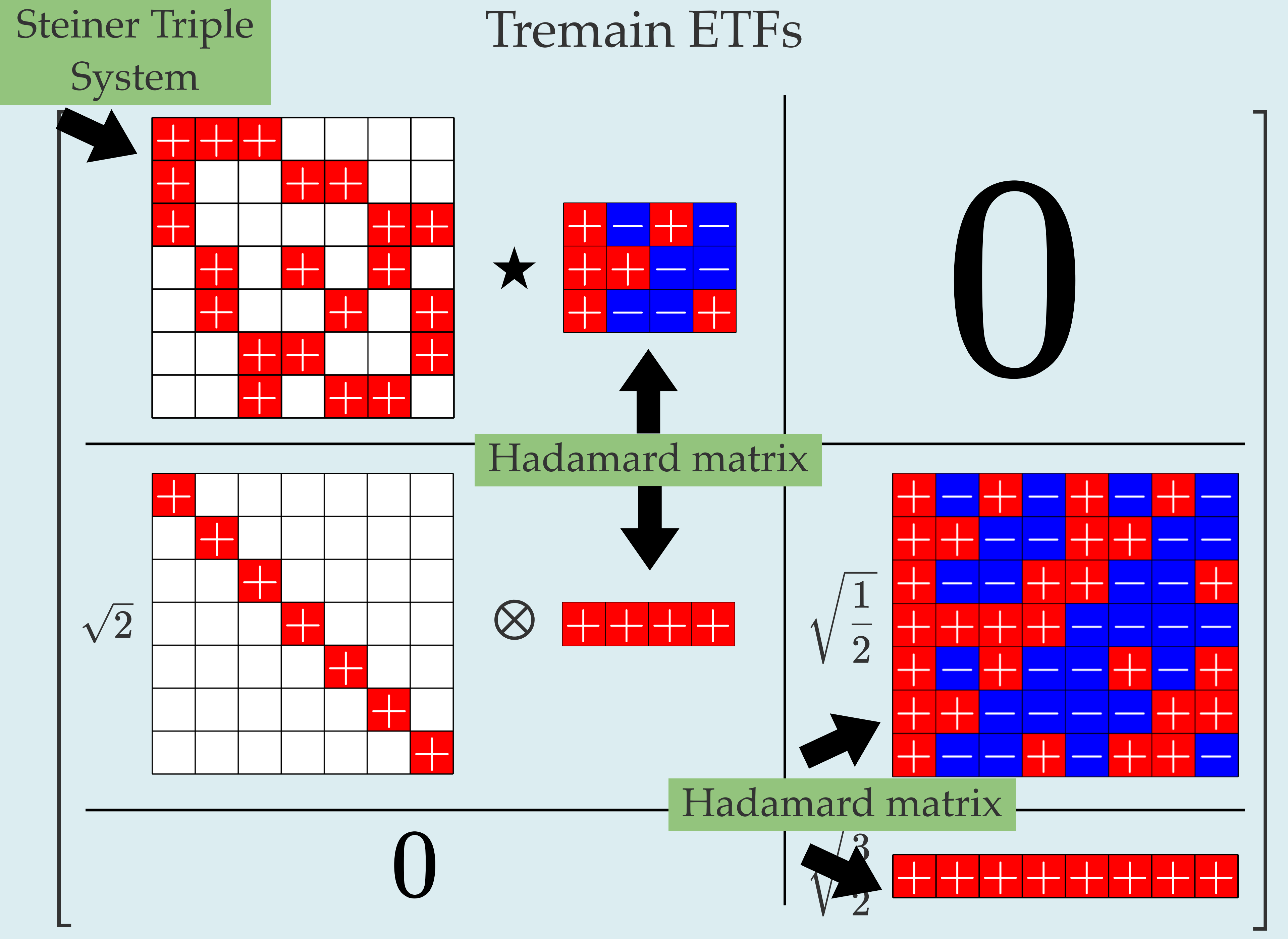


Tremain ETFs

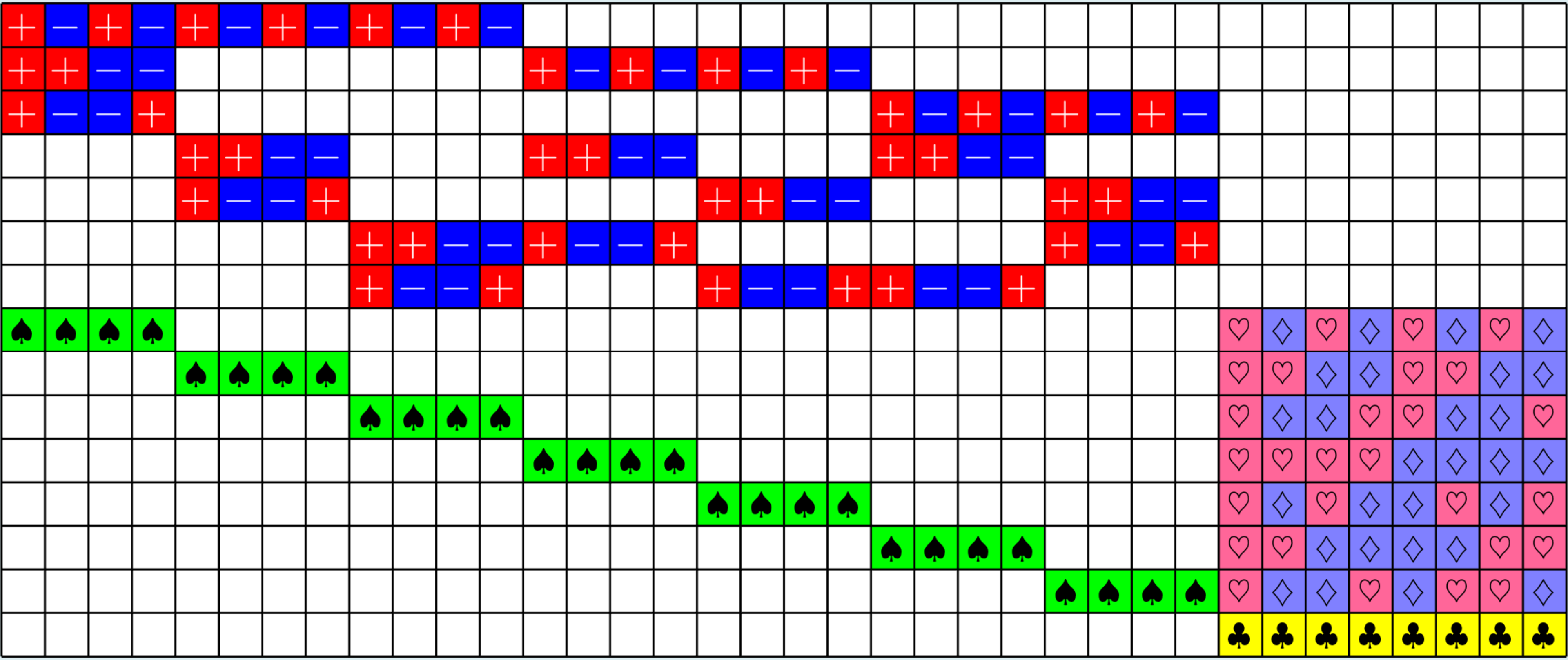


Tremain ETFs




Steiner Triple System
Tremain ETFs
Hadamard matrix
Hadamard matrix

Tremain ETFs



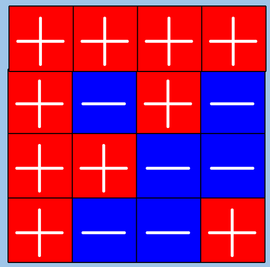
$+ = +1$ $\quad - = -1$ $\quad \spadesuit = \sqrt{2}$

$\heartsuit = +\sqrt{\frac{1}{2}}$ $\quad \diamondsuit = -\sqrt{\frac{1}{2}}$ $\quad \clubsuit = \sqrt{\frac{3}{2}}$

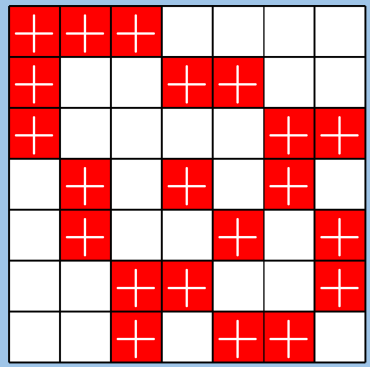
Tremain ETFs:

Theorem (Fickus, J, Mixon, Peterson '18). If there exists an

$h \times h$ Hadamard matrix with $h \equiv 1$ or $2 \bmod 3$,



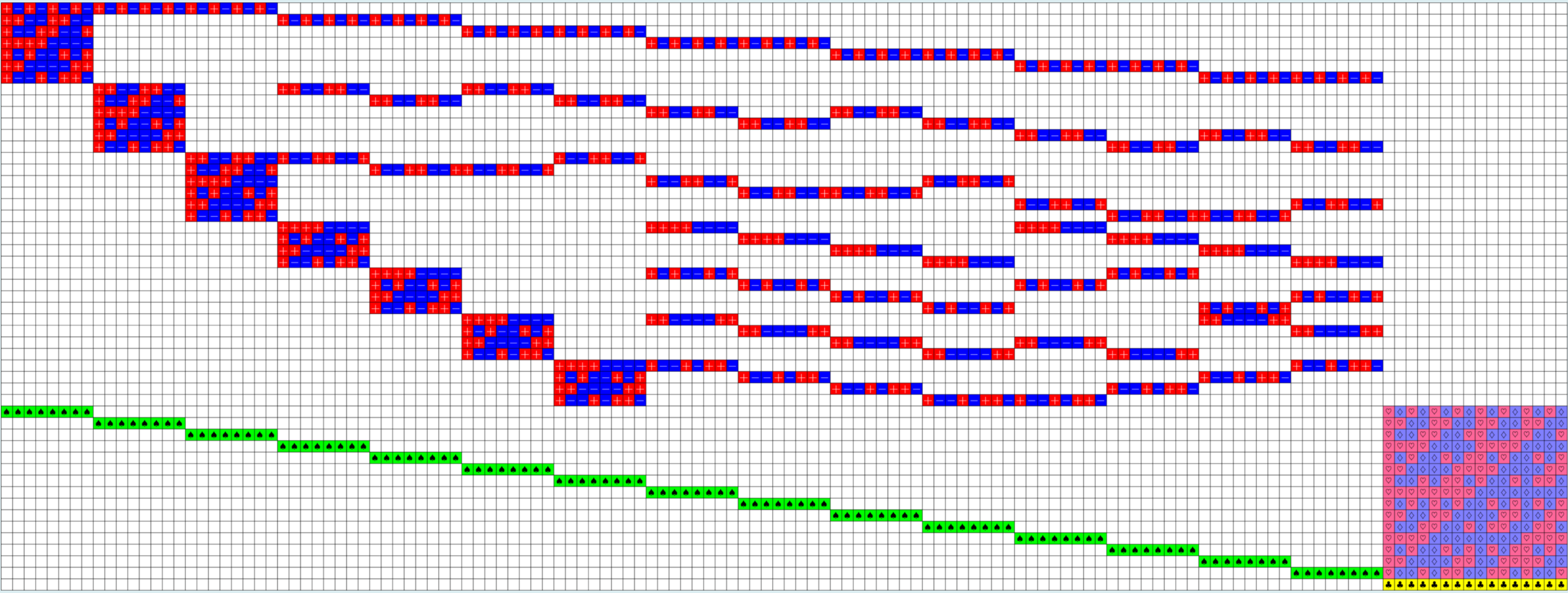
then there exists a $(2, 3, 2h-1)$-Steiner system



and by the Tremain construction there exists a real $d \times N$ ETF where

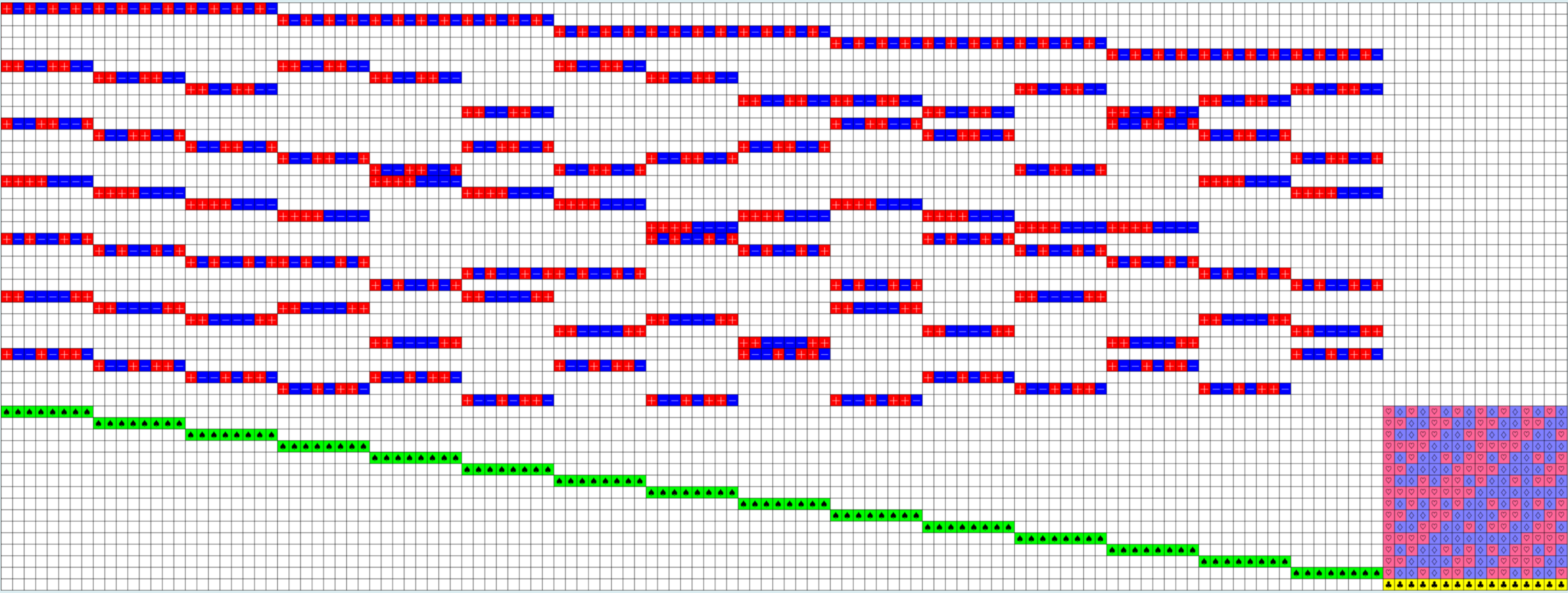
$$d = \frac{1}{3}(h+1)(2h+1), \qquad N = h(2h+1).$$

51×136 Tremain ETF



- + $= +1$
- − $= -1$
- ♠ $= \sqrt{2}$
- ♤ $= -\sqrt{2}$
- ♡ $= +\sqrt{\frac{1}{2}}$
- ♢ $= -\sqrt{\frac{1}{2}}$
- ♣ $= \sqrt{\frac{3}{2}}$
- ♧ $= -\sqrt{\frac{3}{2}}$

51 × 136 Tremain ETF



Red + = $+1$

Blue − = -1

Green ♠ (black) = $\sqrt{2}$

Green ♠ (white) = $-\sqrt{2}$

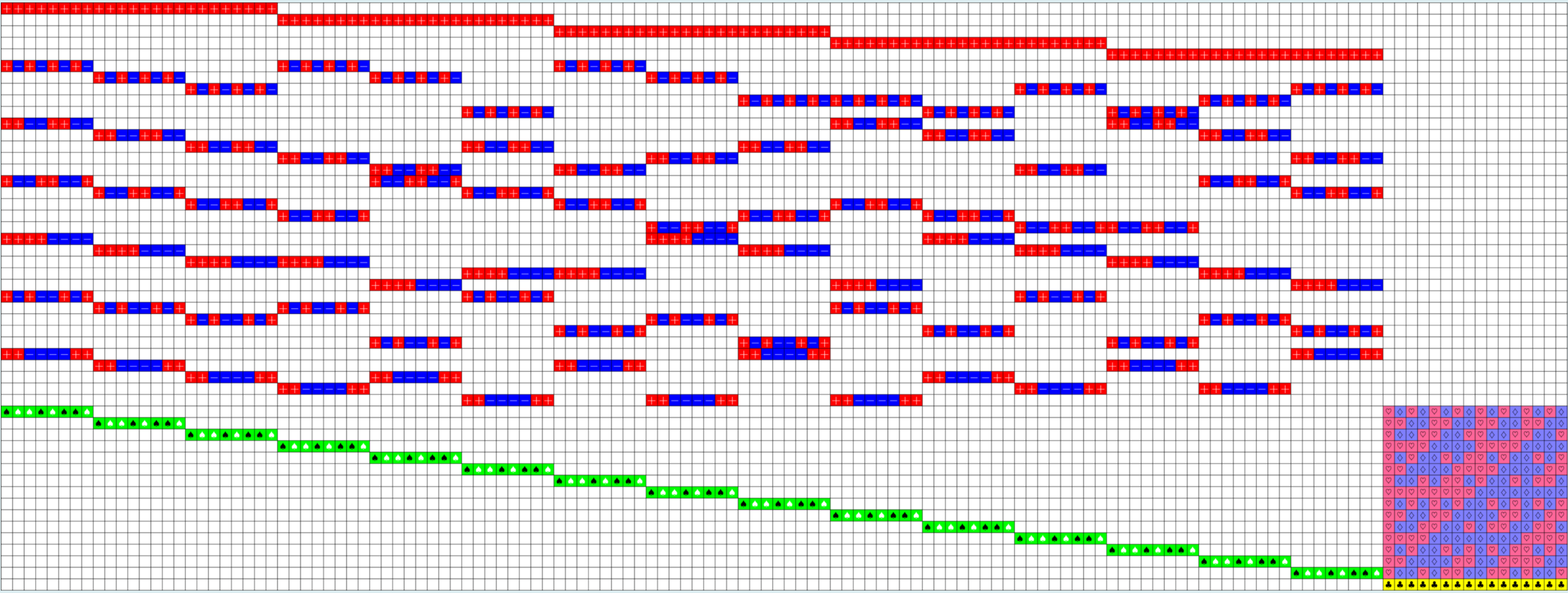
Pink ♡ = $+\sqrt{\frac{1}{2}}$

Purple ◇ = $-\sqrt{\frac{1}{2}}$

Yellow ♣ (black) = $\sqrt{\frac{3}{2}}$

Yellow ♣ (white) = $-\sqrt{\frac{3}{2}}$

51×136 Tremain ETF



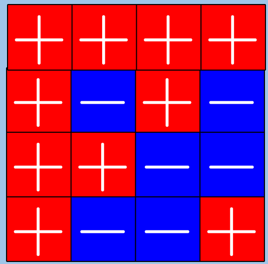
$+$ $= +1$ $-$ $= -1$ ♠ $= \sqrt{2}$ ♤ $= -\sqrt{2}$

♡ $= +\sqrt{\frac{1}{2}}$ ♢ $= -\sqrt{\frac{1}{2}}$ ♣ $= \sqrt{\frac{3}{2}}$ ♧ $= -\sqrt{\frac{3}{2}}$

Axial Tremain ETFs:

Theorem (Fickus, J, Mixon, Peterson '18). If there exists an

$h \times h$ Hadamard matrix with $h \equiv 2 \bmod 3$,



then there exists a strongly regular graph with parameters:

$$v = h(2h+1), \; k = \frac{(h+2)(2h-1)}{2}, \; \lambda = \frac{(h-1)(h+4)}{2}, \; \mu = \frac{h(h+2)}{2}$$

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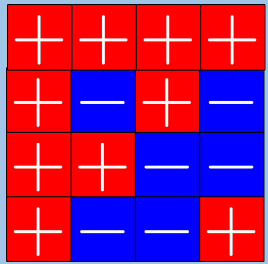
Theorem (Fickus, J, Mixon, Peterson '18). If there exists an $h \times h$ Hadamard matrix with $h \equiv 2 \bmod 3$,



then there exists a strongly regular graph with parameters:

$$v = h(2h+1),\ k = \frac{(h+2)(2h-1)}{2},\ \lambda = \frac{(h-1)(h+4)}{2},\ \mu = \frac{h(h+2)}{2}$$

This gave us a new ETF!

From Brouwer's table online:

+	820	390	180	190	10^{533}	-20^{286}	2-graph
		429	228	220	19^{286}	-11^{533}	from ETF Fickus et al.; 2-graph
?	820	399	198	190	19^{287}	-11^{532}	2-graph?
		420	210	220	10^{532}	-20^{287}	2-graph?

Axial Tremain ETFs:

Theorem (Fickus, J, Mixon, Peterson '18). If there exists an

$h \times h$ Hadamard matrix with $h \equiv 2 \bmod 3$,

+	+	+	+
+	−	+	−
+	+	−	−
+	−	−	+

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This gave us a new ETF!

Let's replicate this success!

From Brouwer's table online:

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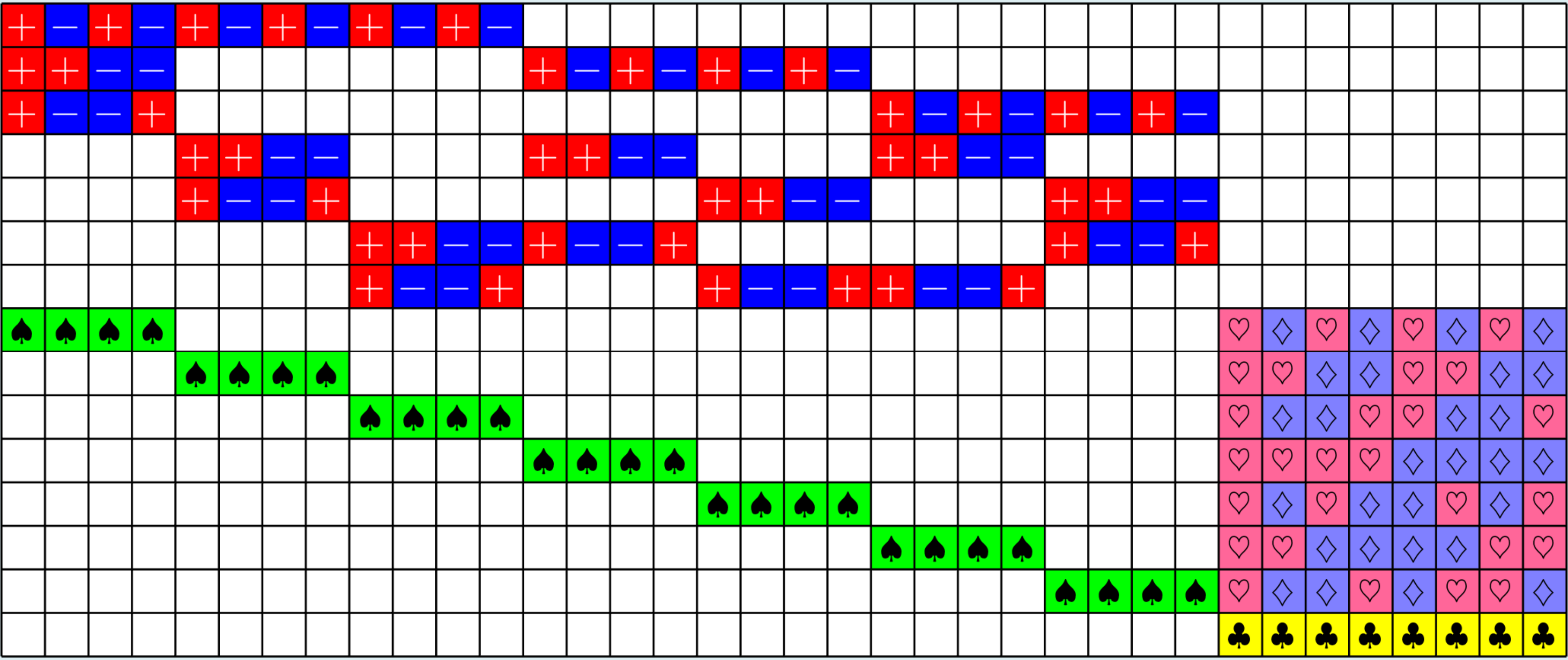
New Results

How can I make these vectors sum to zero?



+ $= +1$ − $= -1$ ♠ $= \sqrt{2}$

♡ $= +\sqrt{\frac{1}{2}}$ ♢ $= -\sqrt{\frac{1}{2}}$ ♣ $= \sqrt{\frac{3}{2}}$

Back to Brouwer's Table

	v	k	λ	μ	r^f	s^g	comments
		18	9	9	3^{14}	-3^{20}	S(2,3,15); lines in PG(3,2); $O^+(6,2)$; from ETF Fickus et al.; 2-graph*
!	36	10	4	2	4^{10}	-2^{25}	6^2
		25	16	20	1^{25}	-5^{10}	OA(6,5) does not exist (Tarry)
180!	36	14	4	6	2^{21}	-4^{14}	$U_3(3).2$ / $L_2(7).2$ - subconstituent of Hall-Janko graph; complete enumeration by McKay & Spence; $RSHCD^-$; 2-graph
		21	12	12	3^{14}	-3^{21}	2-graph
!	36	14	7	4	5^8	-2^{27}	Triangular graph T(9)
		21	10	15	1^{27}	-6^8	
32548!	36	15	6	6	3^{15}	-3^{20}	complete enumeration by McKay & Spence; OA(6,3); $NO^-(6,2)$; $RSHCD^+$; 2-graph
		20	10	12	2^{20}	-4^{15}	$NO^-(5,3)$; OA(6,4) does not exist (Tarry); 2-graph
+	37	18	8	9	2.541^{18}	-3.541^{18}	partial enumeration by McKay & Spence; see also Crnković-Maksimović and Maksimović-Rukavina; Paley(37); 2-graph*
28!	40	12	2	4	2^{24}	-4^{15}	complete enumeration by Spence; O(5,3) Sp(4,3); GQ(3,3)
		27	18	18	3^{15}	-3^{24}	NU(4,2)

Back to Brouwer's Table

	v	k	λ	μ	r^f	s^g	comments
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28!	40	12	2	4	2^{24}	-4^{15}	complete enumeration by Spence; O(5,3) Sp(4,3); GQ(3,3)
		27	18	18	3^{15}	-3^{24}	NU(4,3)

Graph from a 15×36 ETF with **1** the kernel

Back to Brouwer's Table

	v	k	λ	μ	r^f	s^g	comments
		18	9	9	3^{14}	-3^{20}	S(2,3,15); lines in PG(3,2); $O^+(6,2)$; from ETF Fickus et al.; 2-graph*
!	36	10	4	2	4^{10}	-2^{25}	6^2
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		21	12	12	3^{14}	-3^{21}	2-graph
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		21	10	15	1^{27}	-6^8	
32548!	36	15	6	6	3^{15}	-3^{20}	complete enumeration by McKay & Spence; OA(6,3); $NO^-(6,2)$; $RSHCD^+$; 2-graph
		20					
+	37	18	8	9	2.541^{18}	-3.541^{18}	partial enumeration by McKay & Spence; see also Crnković-Maksimović and Maksimović-Rukavina; Paley(37); 2-graph*
28!	40	12	2	4	2^{24}	-4^{15}	complete enumeration by Spence; O(5,3) Sp(4,3); GQ(3,3)
		27	18	18	3^{15}	-3^{24}	NU(4,2)

Graph from a 15×36 ETF with **1** the kernel

Nice short fat repn ?

$NO_6^-(2)$

0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
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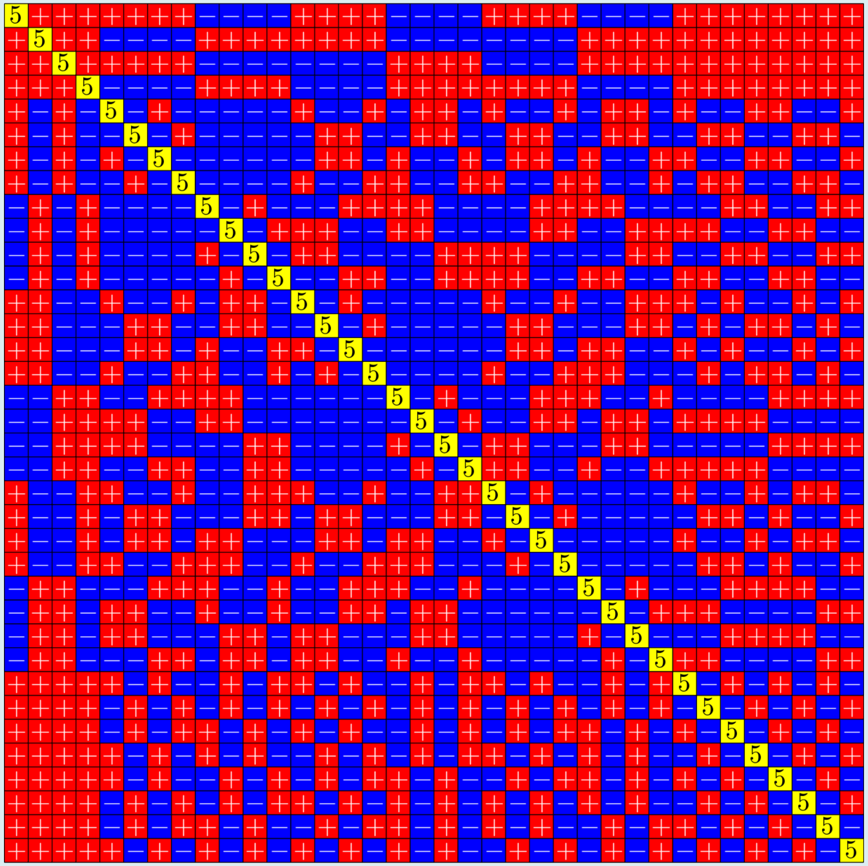
$NO_6^-(2)$

$$\Phi^\top \Phi =$$



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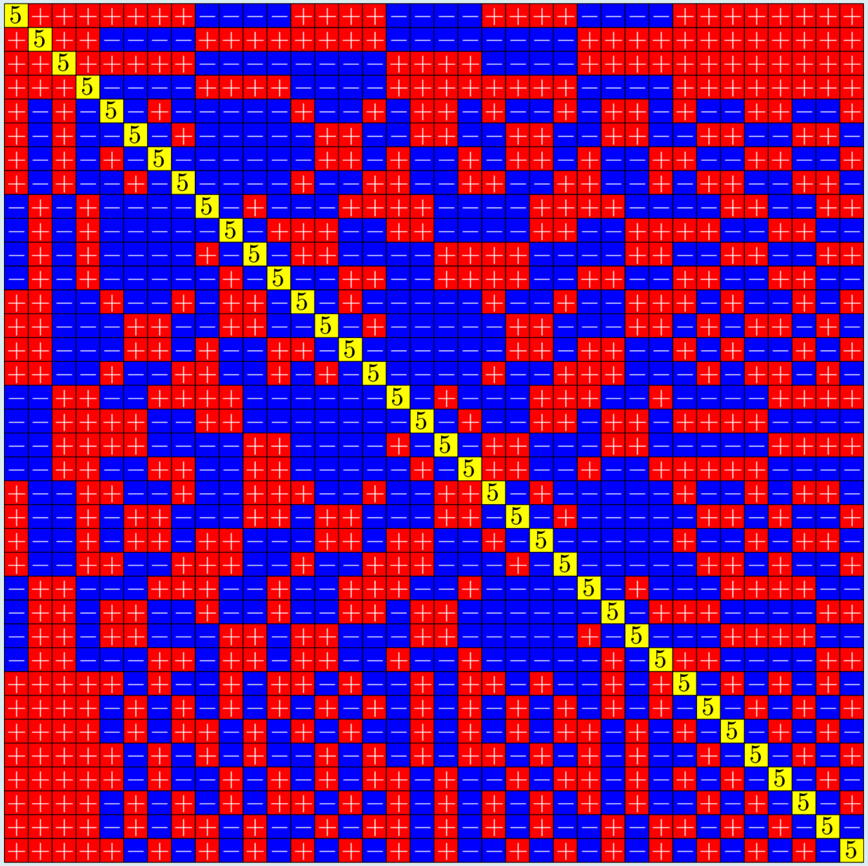

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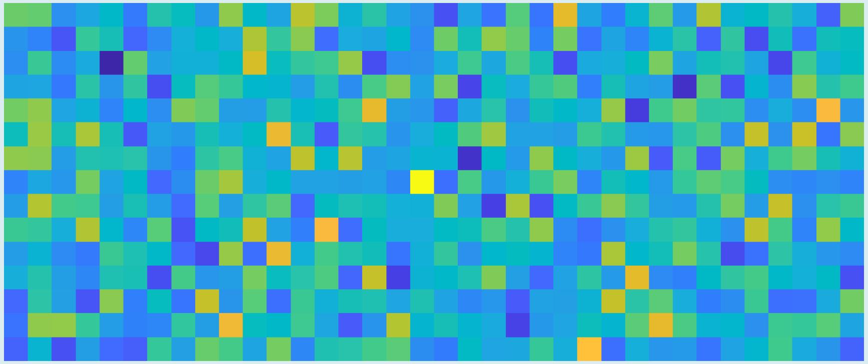

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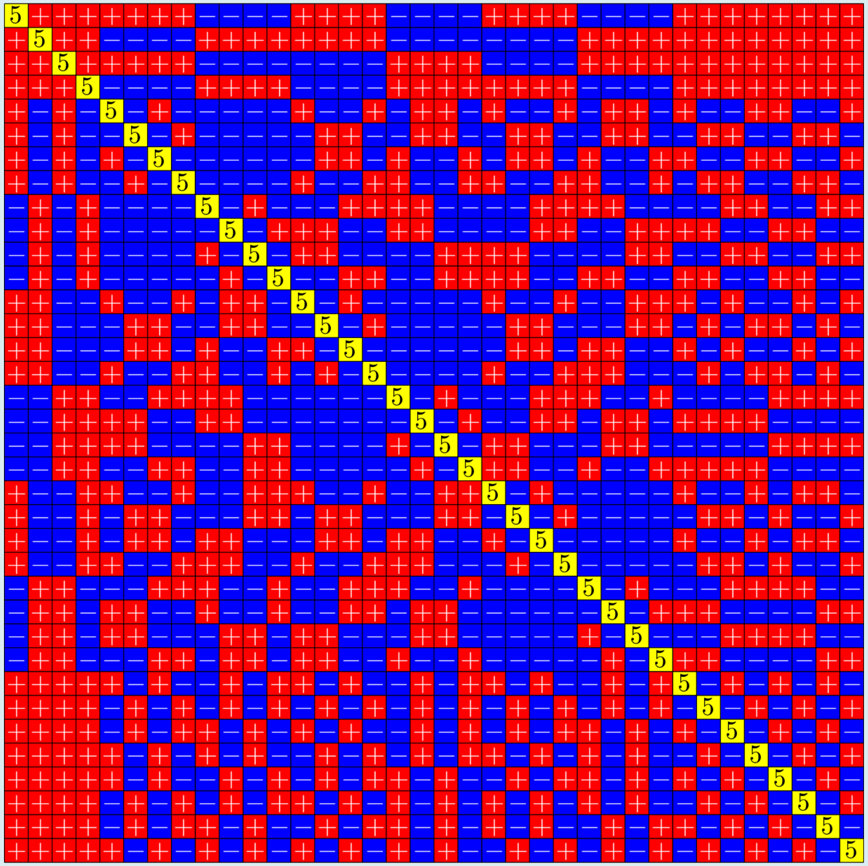

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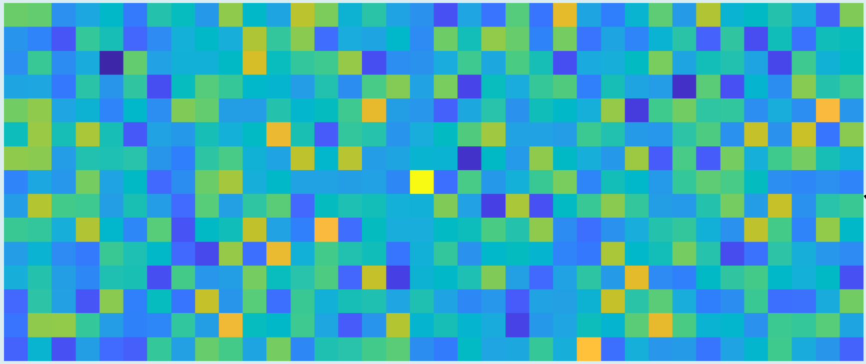
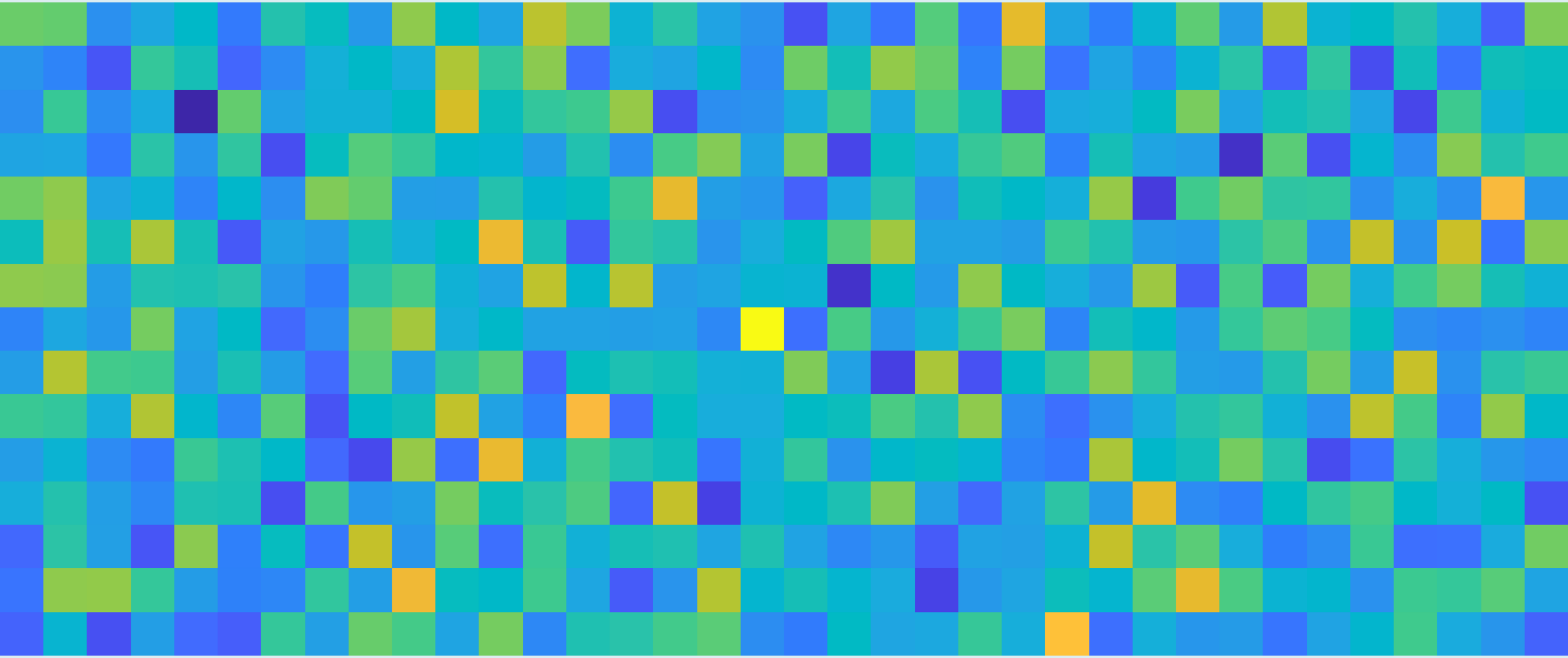


Remember: Gramians are forgetful ☹

$NO_6^-(2)$



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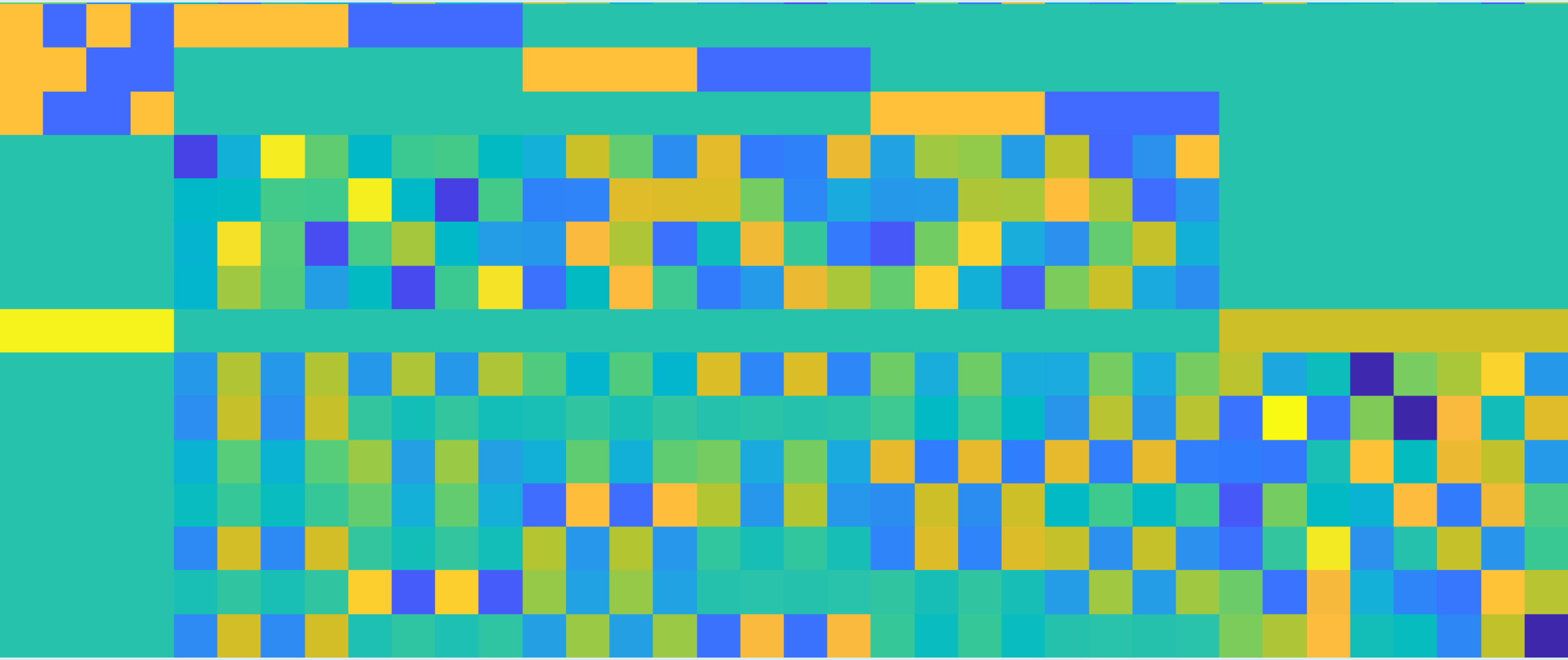

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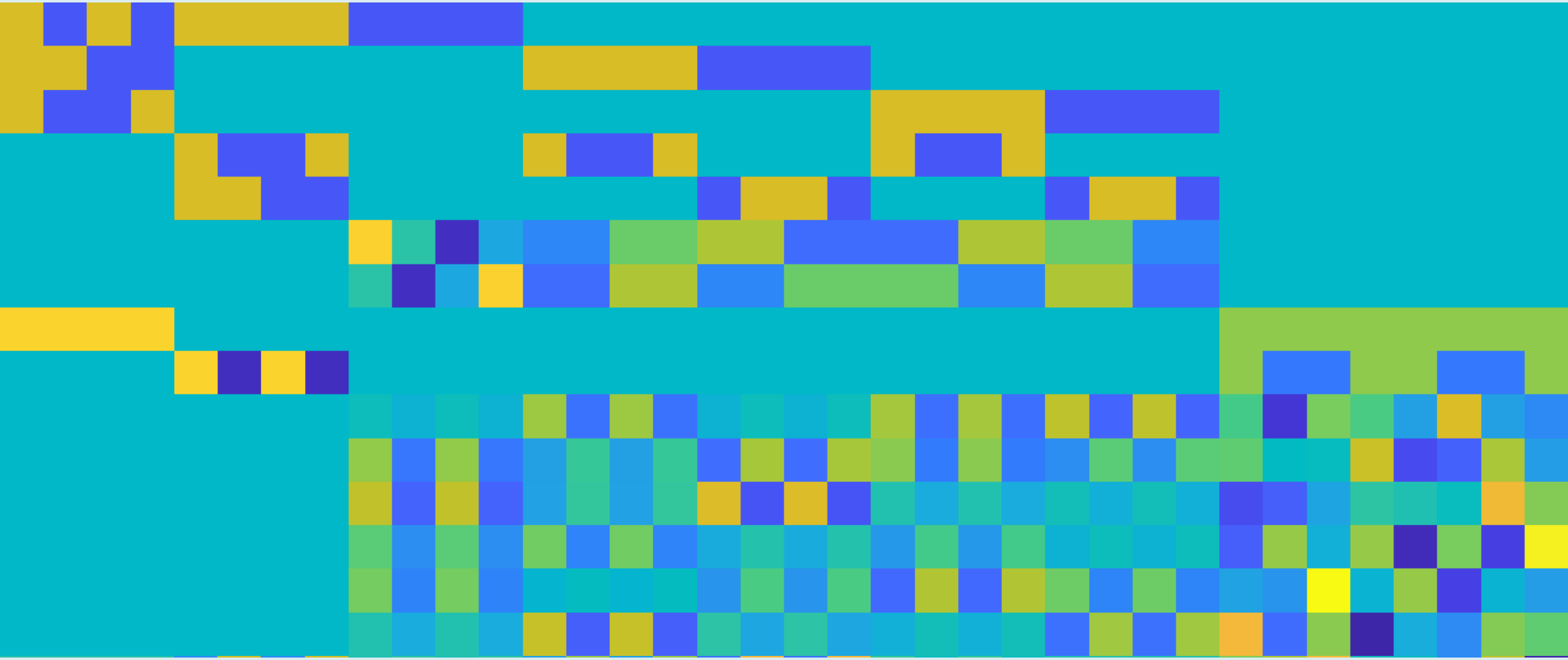

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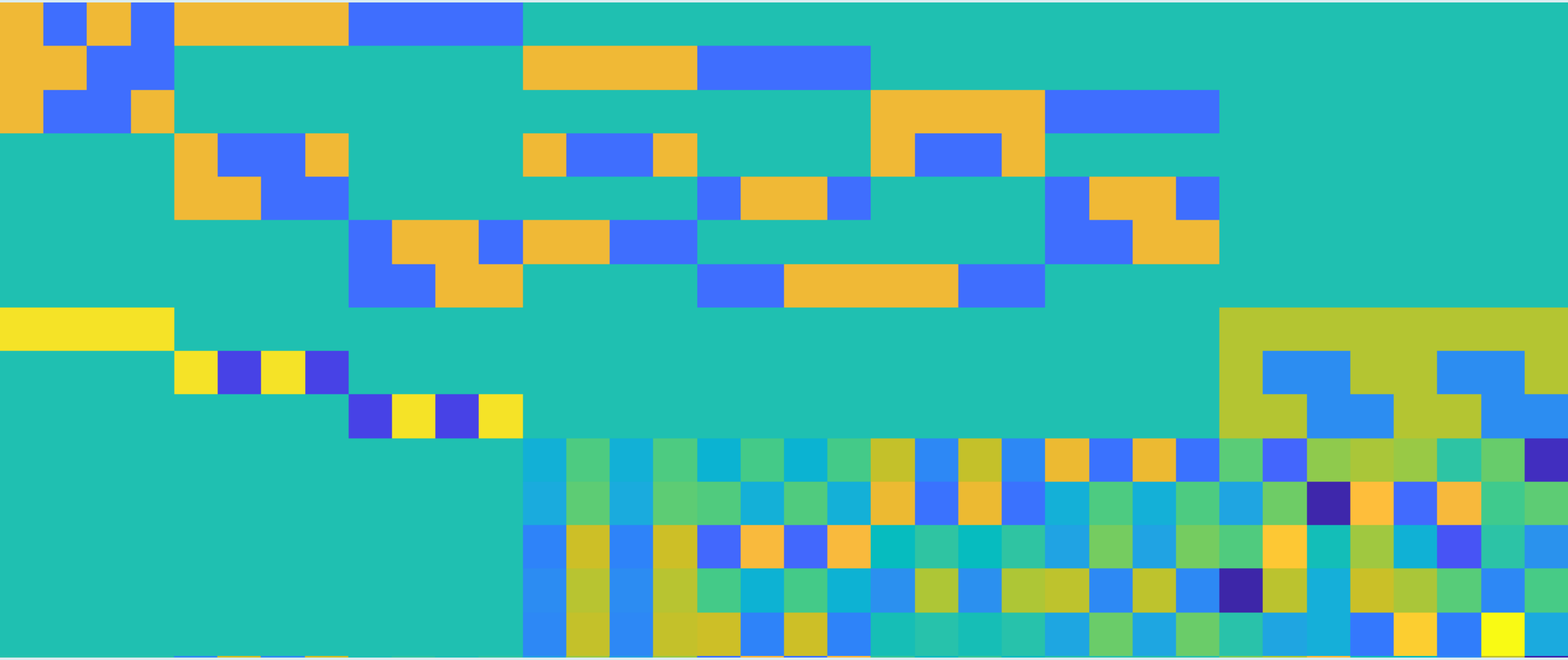

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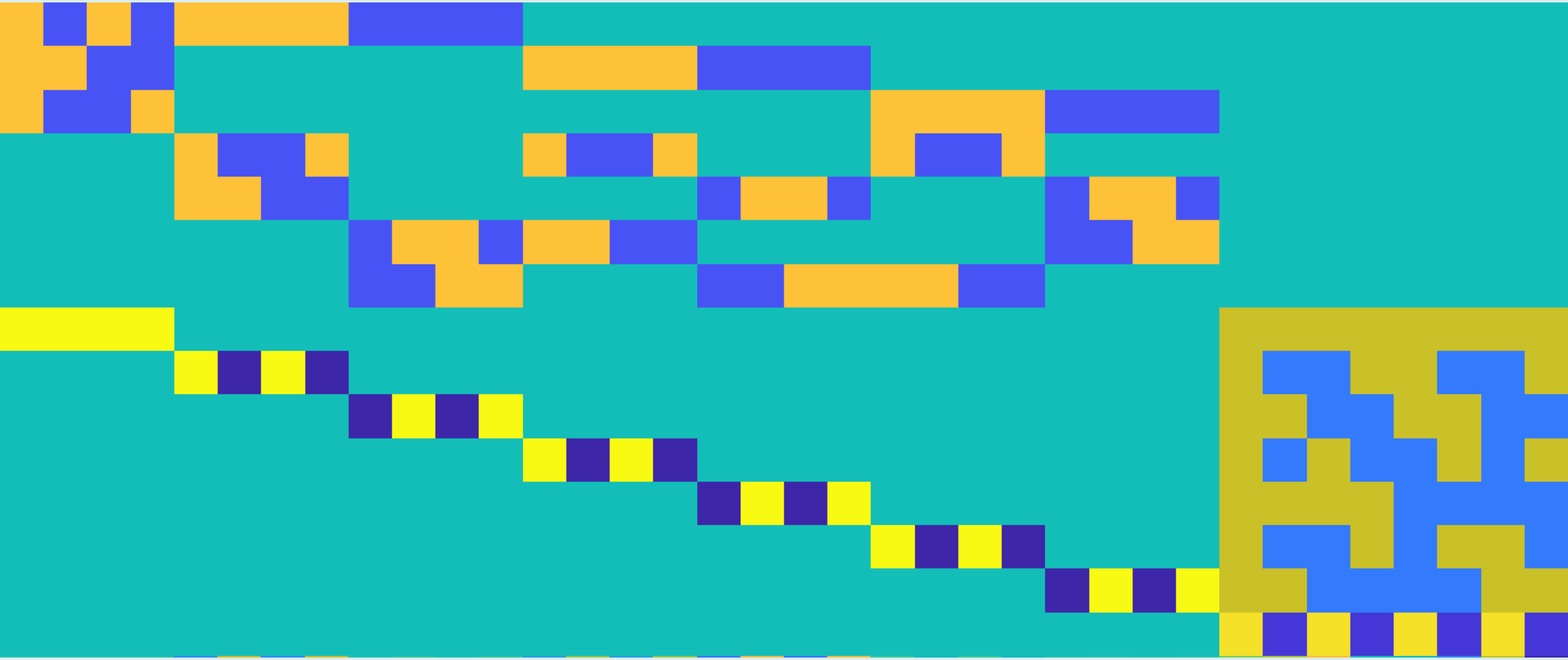

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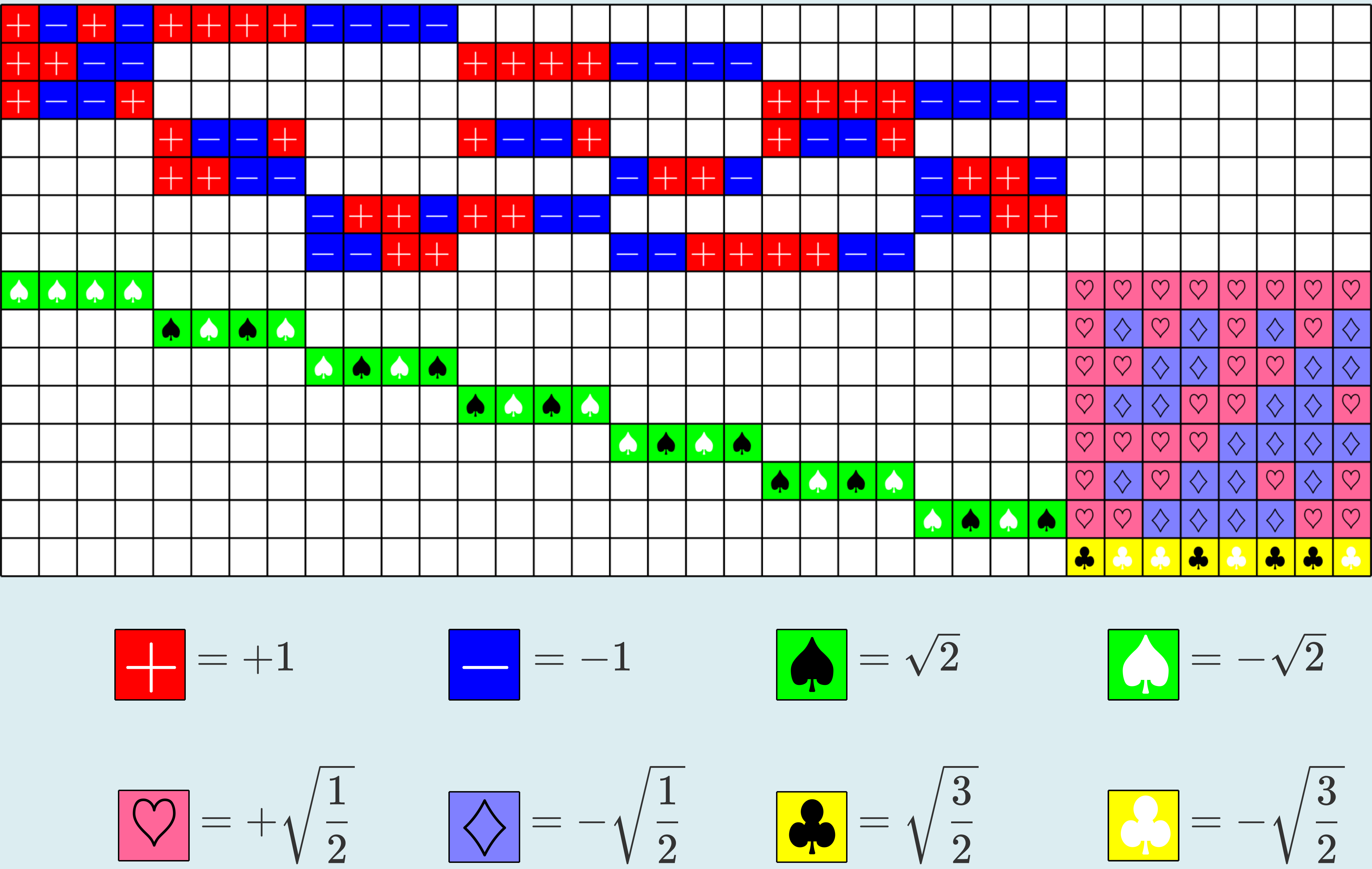

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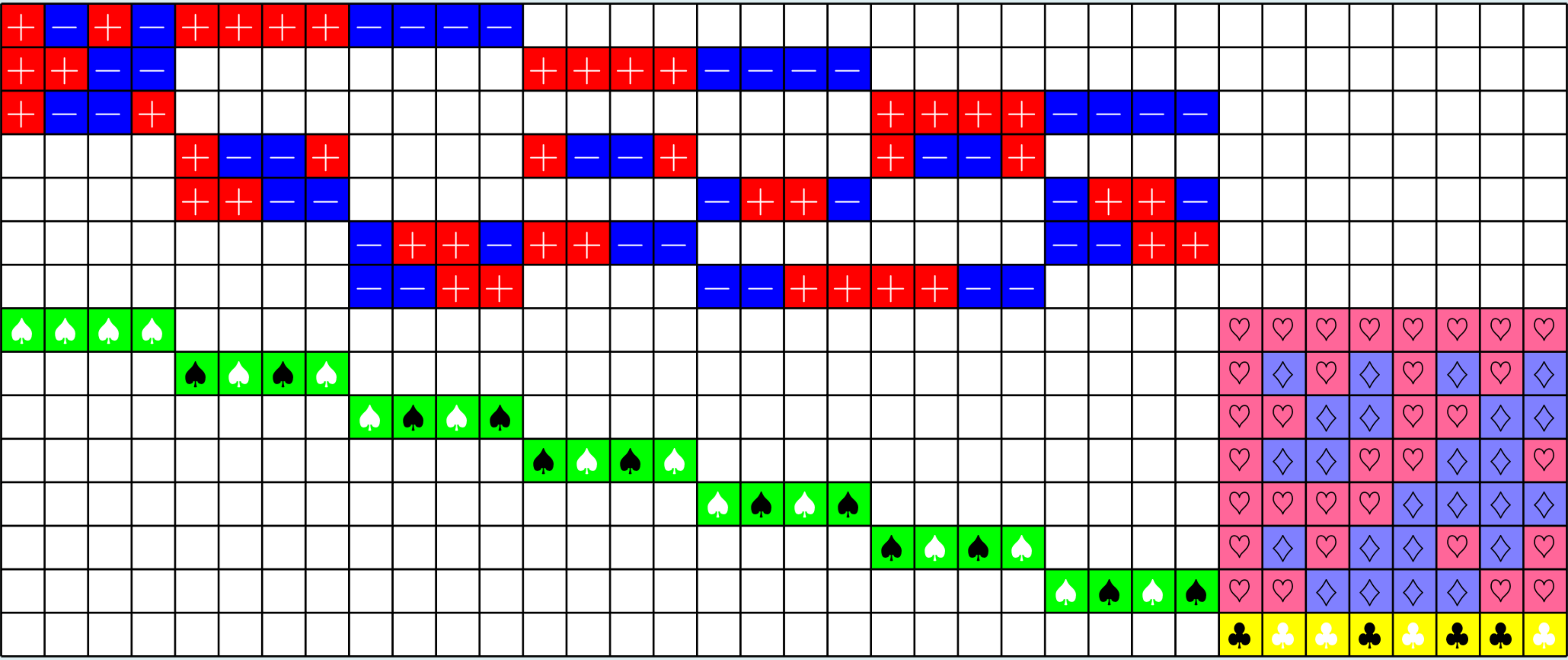

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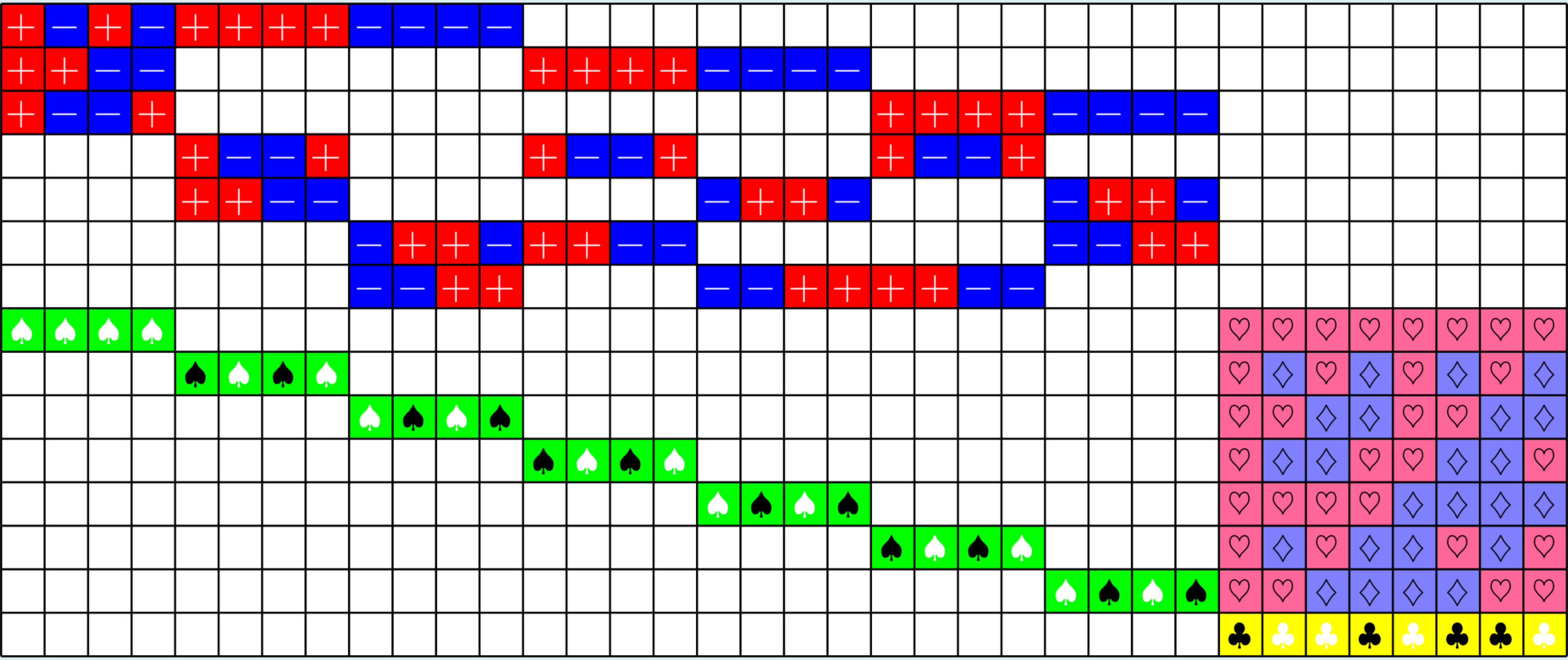
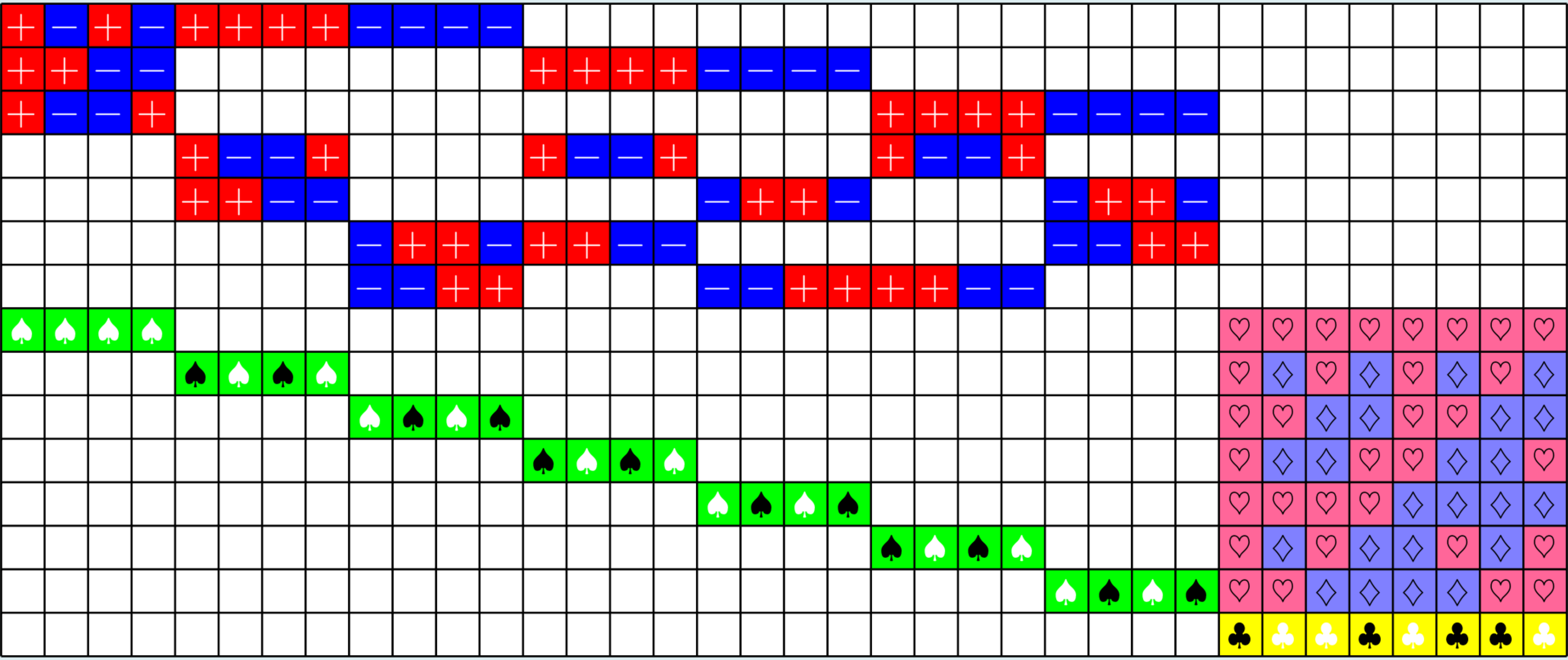


- $\mathbf{1} = (1, 1, \ldots, 1)$ is in the kernel

$NO_6^-(2)$



- $\mathbf{1} = (1, 1, \ldots, 1)$ is in the kernel
- works for all real Tremain ETFs!

Centered Tremain ETFs:

Theorem (J). If there exists an

$h \times h$ Hadamard matrix with $h \equiv 1$ or $2 \bmod 3$,

$$\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

then there exists a strongly regular graph with parameters

$$v = h(2h+1), \quad k = h^2 - 1, \quad \lambda = \frac{1}{2}(h^2 - 4), \quad \mu = \frac{1}{2}h(h-1)$$

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$\exists\, 20 \times 20$ Hadamard matrix $\Rightarrow$ SRG(820,399,198,190)

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Centered Tremain ETFs:

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+	+	+	+
+	−	+	−
+	+	−	−
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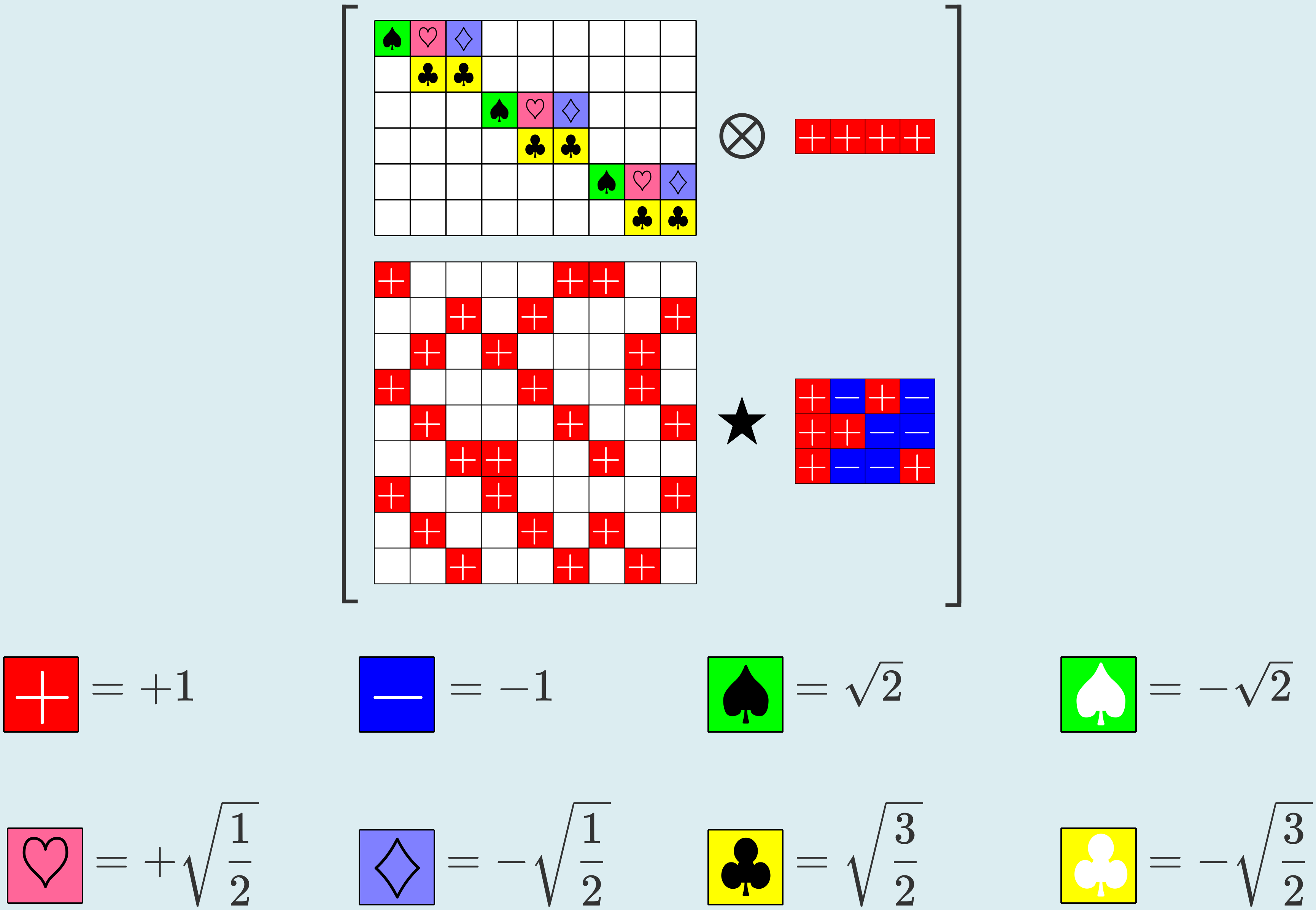
Group Divisible Design



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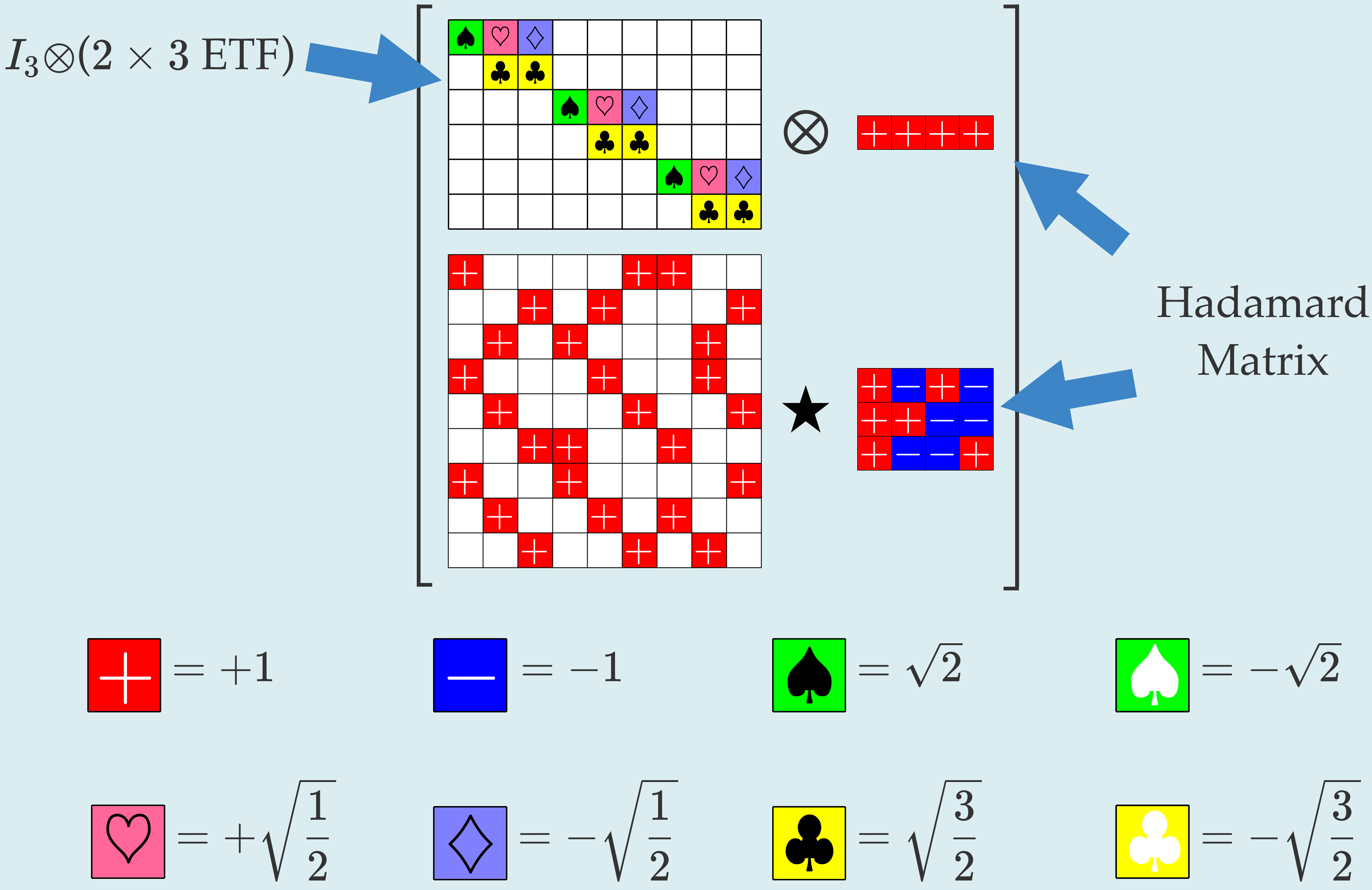
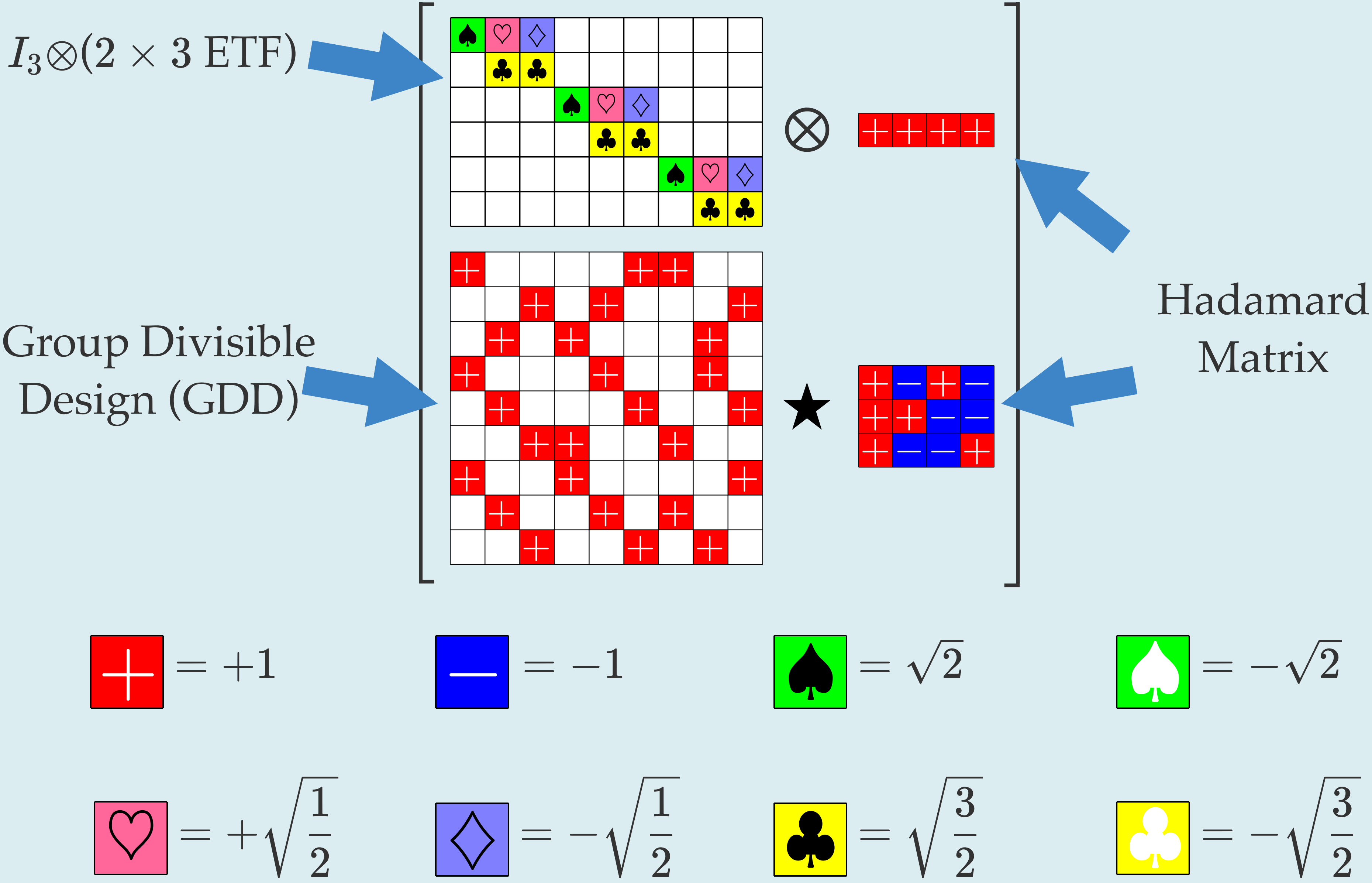
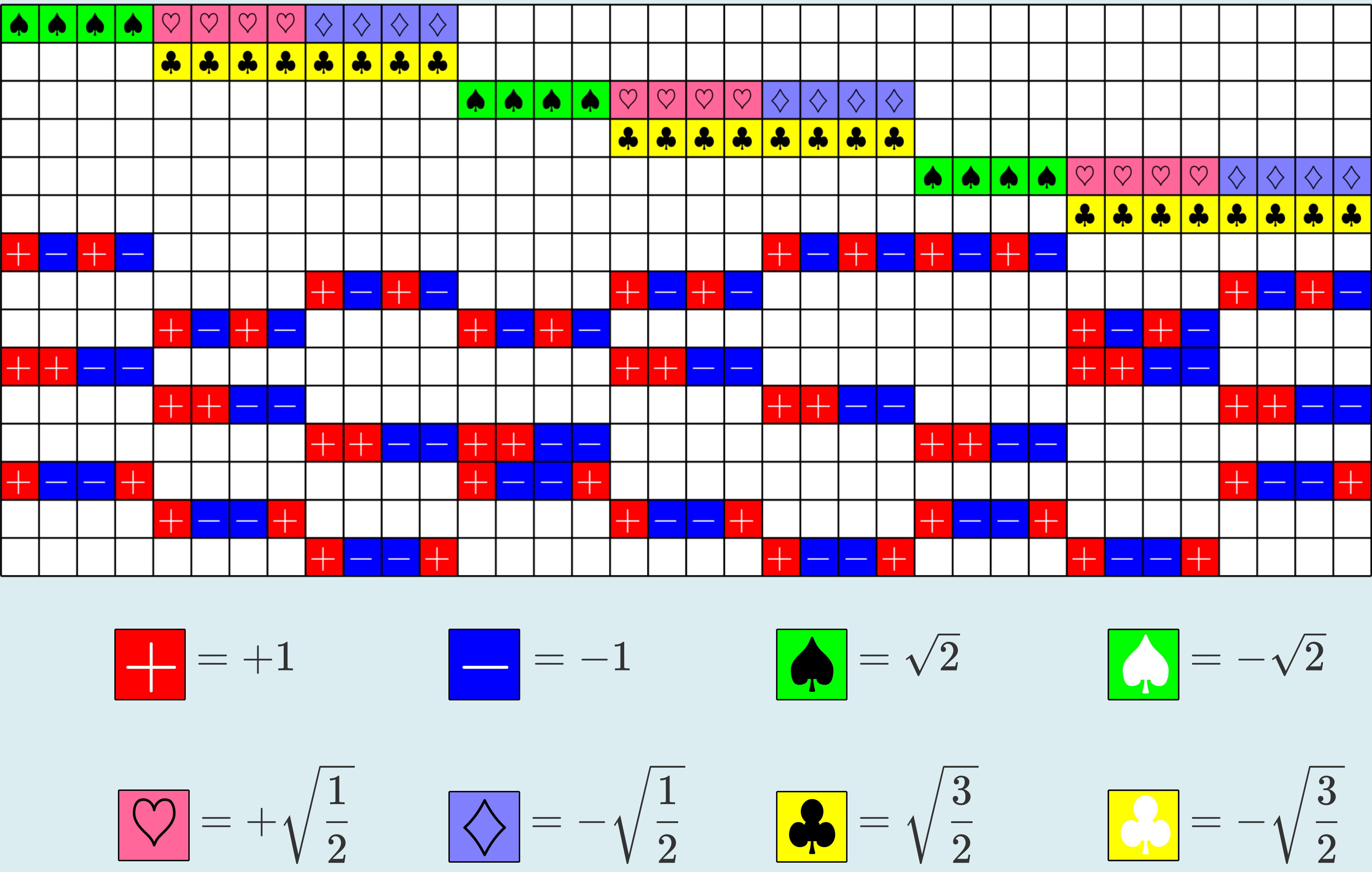


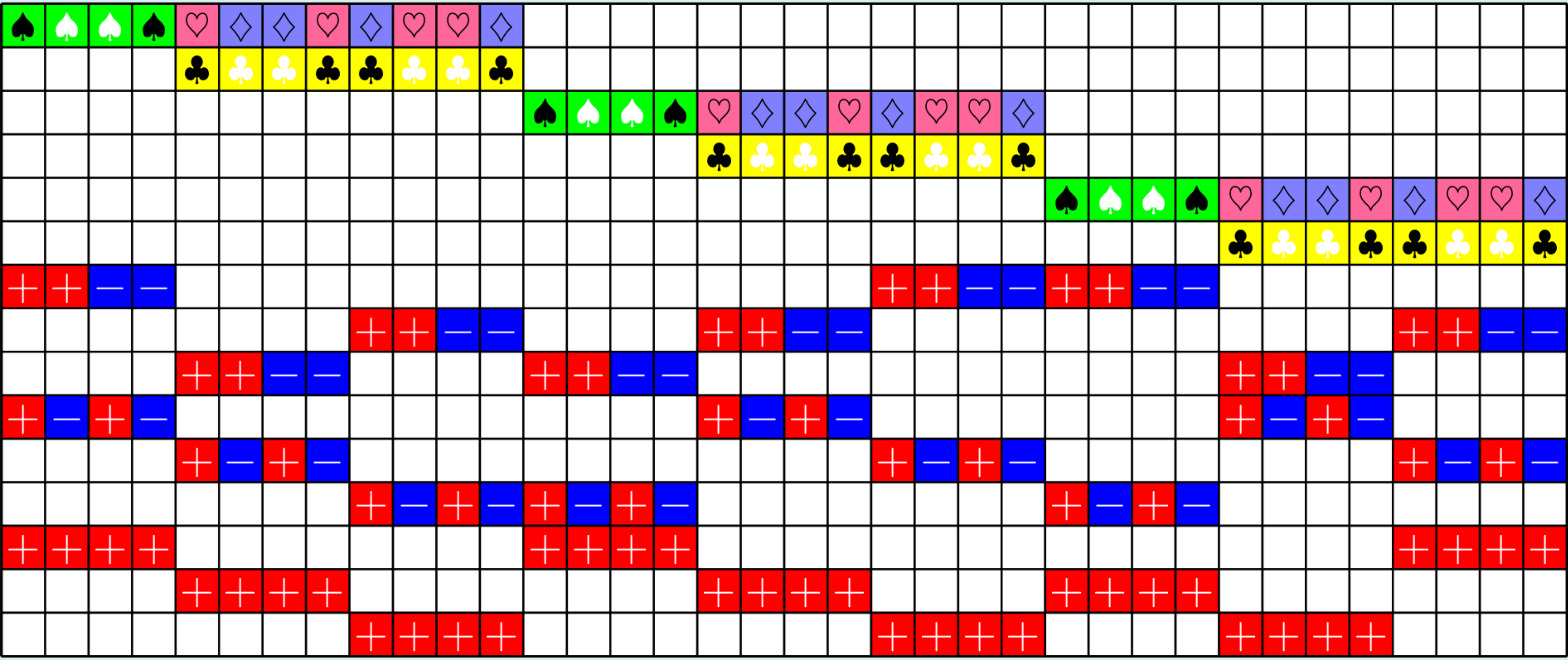
Group Divisible Design



Group Divisible Design



Group Divisible Design



+ $= +1$

− $= -1$

♠ (black spade) $= \sqrt{2}$

♤ (white spade) $= -\sqrt{2}$

♡ $= +\sqrt{\frac{1}{2}}$

♢ $= -\sqrt{\frac{1}{2}}$

♣ (black club) $= \sqrt{\frac{3}{2}}$

♧ (white club) $= -\sqrt{\frac{3}{2}}$

Axial GDD ETFs:

Theorem (J). If there exists an

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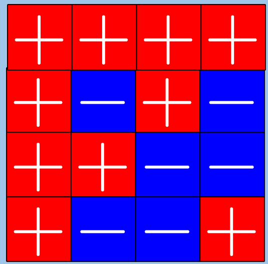

then there exists a strongly regular graph with parameters

$$v = h(2h+1), \quad k = \frac{(h+2)(2h-1)}{2}, \ \lambda = \frac{(h-1)(h+4)}{2}, \ \mu = \frac{h(h+2)}{2}$$

Axial GDD ETFs:

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$h \times h$ Hadamard matrix with $h \equiv 1 \bmod 3$,

$$\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

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$\exists\, 16 \times 16$ Hadamard matrix $\Rightarrow \exists$ SRG(528,279,150,144)

From Brouwer's table online:

?	528	248	112	120	8^{341}	-16^{186}	2-graph?
		279	150	144	15^{186}	-9^{341}	2-graph?
+	528	255	126	120	15^{187}	-9^{340}	$NO^{-}(10,2)$; Muzychuk S2 (r=4); 2-graph
		272	136	144	8^{340}	-16^{187}	2-graph

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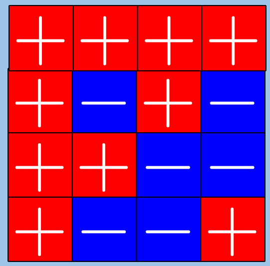

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		272	136	144	8^{340}	-16^{187}	2-graph

Compatible Orthobiangular Tight Frames

Ex:

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

	(0,0,0,0)	(0,0,1,0)	(0,0,2,0)	(0,0,0,1)	(0,0,1,1)	(0,0,2,1)	(0,0,0,2)	(0,0,1,2)	(0,0,2,2)	(1,0,0,0)	(1,0,1,0)	(1,0,2,0)	(1,0,0,1)	(1,0,1,1)	(1,0,2,1)	(1,0,0,2)	(1,0,1,2)	(1,0,2,2)	(0,1,0,0)	(0,1,1,0)	(0,1,2,0)	(0,1,0,1)	(0,1,1,1)	(0,1,2,1)	(0,1,0,2)	(0,1,1,2)	(0,1,2,2)	(1,1,0,0)	(1,1,1,0)	(1,1,2,0)	(1,1,0,1)	(1,1,1,1)	(1,1,2,1)	(1,1,0,2)	(1,1,1,2)	(1,1,2,2)
(0,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(0,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(0,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(0,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(0,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(0,0,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0
(0,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(0,0,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0
(0,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2
(1,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(1,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(1,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(1,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5
(1,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1
(1,0,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3
(1,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1
(1,0,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3
(1,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5
(0,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(0,1,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(0,1,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(0,1,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5
(0,1,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1
(0,1,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3
(0,1,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1
(0,1,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3
(0,1,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5
(1,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(1,1,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(1,1,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(1,1,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(1,1,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(1,1,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0
(1,1,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(1,1,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0
(1,1,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2

Ex:

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

	(0,0,0,0)	(0,0,1,0)	(0,0,2,0)	(0,0,0,1)	(0,0,1,1)	(0,0,2,1)	(0,0,0,2)	(0,0,1,2)	(0,0,2,2)	(1,0,0,0)	(1,0,1,0)	(1,0,2,0)	(1,0,0,1)	(1,0,1,1)	(1,0,2,1)	(1,0,0,2)	(1,0,1,2)	(1,0,2,2)	(0,1,0,0)	(0,1,1,0)	(0,1,2,0)	(0,1,0,1)	(0,1,1,1)	(0,1,2,1)	(0,1,0,2)	(0,1,1,2)	(0,1,2,2)	(1,1,0,0)	(1,1,1,0)	(1,1,2,0)	(1,1,0,1)	(1,1,1,1)	(1,1,2,1)	(1,1,0,2)	(1,1,1,2)	(1,1,2,2)
(0,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(0,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(0,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(0,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(0,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(0,0,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0
(0,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(0,0,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0
(0,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2
(1,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(1,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(1,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(1,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5
(1,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1
(1,0,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3
(1,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1
(1,0,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3
(1,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5
(0,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(0,1,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(0,1,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(0,1,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5
(0,1,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1
(0,1,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3
(0,1,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1
(0,1,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3
(0,1,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5
(1,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(1,1,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(1,1,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(1,1,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(1,1,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(1,1,2,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^3	w^5	w^1	w^1	w^3	w^5	w^5	w^1	w^3	w^0	w^2	w^4	w^4	w^0	w^2	w^2	w^4	w^0
(1,1,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(1,1,1,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^3	w^1	w^5	w^5	w^3	w^1	w^1	w^5	w^3	w^0	w^4	w^2	w^2	w^0	w^4	w^4	w^2	w^0
(1,1,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2

Ex:

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

	(0,0,0,0)	(0,0,1,0)	(0,0,2,0)	(0,0,0,1)	(0,0,1,1)	(0,0,2,1)	(0,0,0,2)	(0,0,1,2)	(0,0,2,2)	(1,0,0,0)	(1,0,1,0)	(1,0,2,0)	(1,0,0,1)	(1,0,1,1)	(1,0,2,1)	(1,0,0,2)	(1,0,1,2)	(1,0,2,2)
(0,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(0,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(0,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(0,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(0,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(0,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2
(1,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(1,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(1,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(0,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(0,1,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(0,1,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2
(1,1,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(1,1,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5
(1,1,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1

	(0,1,0,0)	(0,1,1,0)	(0,1,2,0)	(0,1,0,1)	(0,1,1,1)	(0,1,2,1)	(0,1,0,2)	(0,1,1,2)	(0,1,2,2)	(1,1,0,0)	(1,1,1,0)	(1,1,2,0)	(1,1,0,1)	(1,1,1,1)	(1,1,2,1)	(1,1,0,2)	(1,1,1,2)	(1,1,2,2)
(0,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2
(0,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4
(0,0,0,1)	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(0,0,1,1)	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4	w^0	w^4	w^2	w^4	w^2	w^0	w^2	w^0	w^4
(0,0,0,2)	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4
(0,0,2,2)	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2	w^0	w^2	w^4	w^2	w^4	w^0	w^4	w^0	w^2
(1,0,0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(1,0,1,0)	w^0	w^4	w^2	w^0	w^4	w^2	w^0	w^4	w^2	w^3	w^1	w^5	w^3	w^1	w^5	w^3	w^1	w^5
(1,0,2,0)	w^0	w^2	w^4	w^0	w^2	w^4	w^0	w^2	w^4	w^3	w^5	w^1	w^3	w^5	w^1	w^3	w^5	w^1
(0,1,0,0)	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3
(0,1,1,1)	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1	w^3	w^1	w^5	w^1	w^5	w^3	w^5	w^3	w^1
(0,1,2,2)	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5	w^3	w^5	w^1	w^5	w^1	w^3	w^1	w^3	w^5
(1,1,0,0)	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^3	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(1,1,0,1)	w^3	w^3	w^3	w^1	w^1	w^1	w^5	w^5	w^5	w^0	w^0	w^0	w^4	w^4	w^4	w^2	w^2	w^2
(1,1,0,2)	w^3	w^3	w^3	w^5	w^5	w^5	w^1	w^1	w^1	w^0	w^0	w^0	w^2	w^2	w^2	w^4	w^4	w^4

Ex:

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

	(0,0,0,0)	(0,1,0,0)	(1,0,0,0)	(1,1,0,0)	(0,0,1,1)	(0,1,1,1)	(1,0,1,1)	(1,1,1,1)	(0,0,2,2)	(0,1,2,2)	(1,0,2,2)	(1,1,2,2)
(0,0,2,2)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,0,1)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,1,0)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,2,0)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,0,2)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,1,1)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,1,0,0)	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3
(0,1,0,2)	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1
(0,1,0,1)	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5
(1,0,0,0)	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3
(1,0,1,1)	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5	w^4	w^4	w^1	w^1
(1,0,2,2)	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1	w^2	w^2	w^5	w^5
(1,1,0,0)	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0
(1,1,1,0)	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2
(1,1,2,0)	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4

	(0,0,1,0)	(0,1,1,0)	(1,0,1,0)	(1,1,1,0)	(0,0,2,1)	(0,1,2,1)	(1,0,2,1)	(1,1,2,1)	(0,0,0,2)	(0,1,0,2)	(1,0,0,2)	(1,1,0,2)
(0,0,2,2)	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4
(0,0,0,1)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,1,0)	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0
(0,0,2,0)	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0
(0,0,0,2)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,1,1)	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2
(0,1,0,0)	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3
(0,1,0,2)	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1
(0,1,0,1)	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5
(1,0,0,0)	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3
(1,0,1,1)	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5
(1,0,2,2)	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1
(1,1,0,0)	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0
(1,1,1,0)	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^0	w^3	w^3	w^0
(1,1,2,0)	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^0	w^3	w^3	w^0

	(0,0,0,1)	(0,1,0,1)	(1,0,0,1)	(1,1,0,1)	(0,0,1,2)	(0,1,1,2)	(1,0,1,2)	(1,1,1,2)	(0,0,2,0)	(0,1,2,0)	(1,0,2,0)	(1,1,2,0)
(0,0,2,2)	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4
(0,0,0,1)	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0
(0,0,1,0)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,2,0)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,0,2)	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0
(0,0,1,1)	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2
(0,1,0,0)	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3
(0,1,0,2)	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^0	w^3	w^0	w^3
(0,1,0,1)	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^0	w^3	w^0	w^3
(1,0,0,0)	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3
(1,0,1,1)	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5
(1,0,2,2)	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1
(1,1,0,0)	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0
(1,1,1,0)	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2
(1,1,2,0)	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4

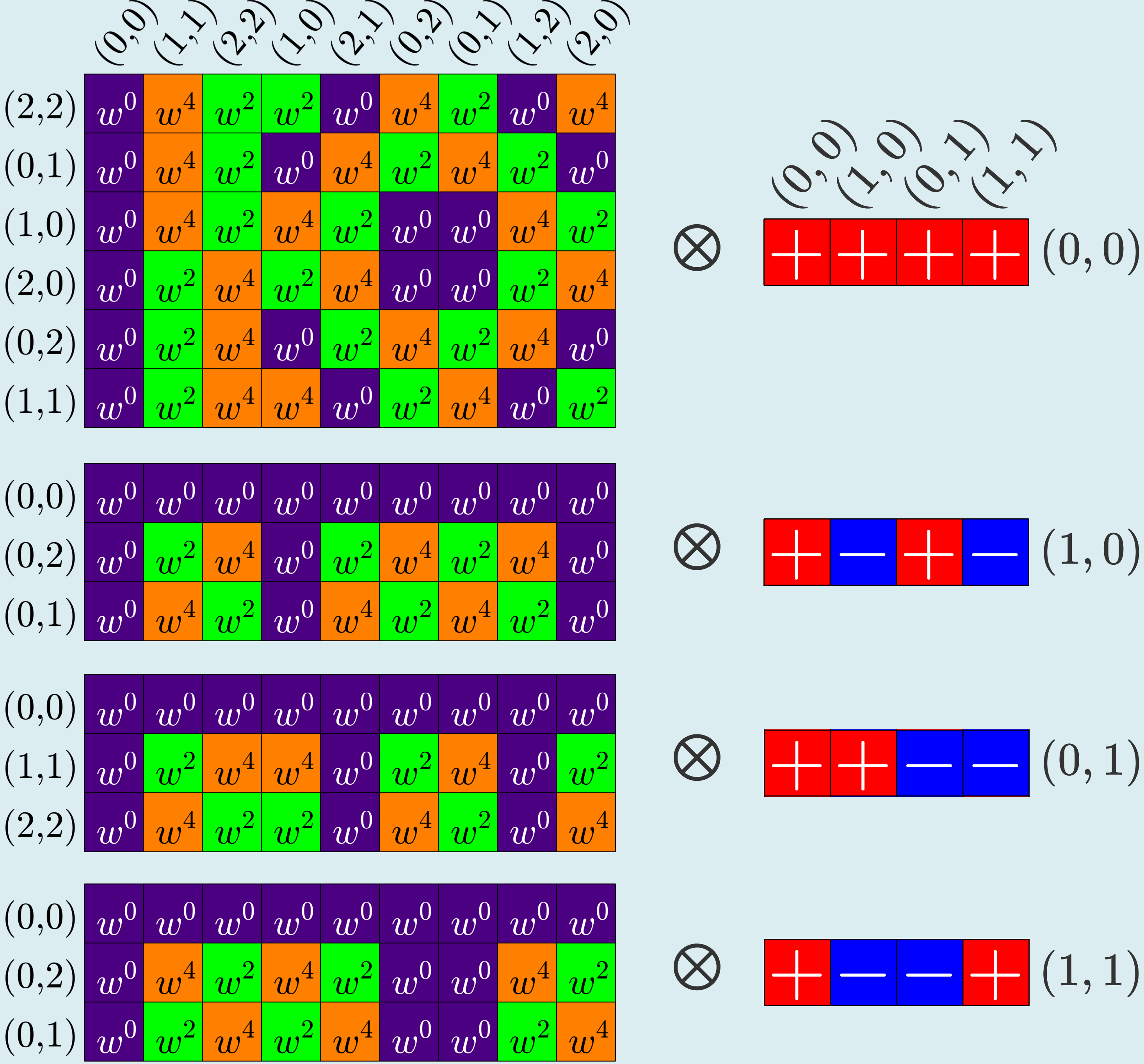
	(0,0,0,0)	(0,1,0,0)	(1,0,0,0)	(1,1,0,0)	(0,0,1,1)	(0,1,1,1)	(1,0,1,1)	(1,1,1,1)	(0,0,2,2)	(0,1,2,2)	(1,0,2,2)	(1,1,2,2)	(0,0,1,0)	(0,1,1,0)	(1,0,1,0)	(1,1,1,0)	(0,0,2,1)	(0,1,2,1)	(1,0,2,1)	(1,1,2,1)	(0,0,0,2)	(0,1,0,2)	(1,0,0,2)	(1,1,0,2)	(0,0,0,1)	(0,1,0,1)	(1,0,0,1)	(1,1,0,1)	(0,0,1,2)	(0,1,1,2)	(1,0,1,2)	(1,1,1,2)	(0,0,2,0)	(0,1,2,0)	(1,0,2,0)	(1,1,2,0)
(0,0,2,2)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4
(0,0,0,1)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0
(0,0,1,0)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,2,0)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,0,2)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0
(0,0,1,1)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2
(0,1,0,0)	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3
(0,1,0,2)	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^0	w^3	w^0	w^3
(0,1,0,1)	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^0	w^3	w^0	w^3
(1,0,0,0)	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3
(1,0,1,1)	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5	w^4	w^4	w^1	w^1	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5
(1,0,2,2)	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1	w^2	w^2	w^5	w^5	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1
(1,1,0,0)	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0
(1,1,1,0)	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2
(1,1,2,0)	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4

	(0,0,0,0)	(0,1,0,0)	(1,0,0,0)	(1,1,0,0)	(0,0,1,1)	(0,1,1,1)	(1,0,1,1)	(1,1,1,1)	(0,0,2,2)	(0,1,2,2)	(1,0,2,2)	(1,1,2,2)	(0,0,1,0)	(0,1,1,0)	(1,0,1,0)	(1,1,1,0)	(0,0,2,1)	(0,1,2,1)	(1,0,2,1)	(1,1,2,1)	(0,0,0,2)	(0,1,0,2)	(1,0,0,2)	(1,1,0,2)	(0,0,0,1)	(0,1,0,1)	(1,0,0,1)	(1,1,0,1)	(0,0,1,2)	(0,1,1,2)	(1,0,1,2)	(1,1,1,2)	(0,0,2,0)	(0,1,2,0)	(1,0,2,0)	(1,1,2,0)
(0,0,2,2)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4
(0,0,0,1)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0
(0,0,1,0)	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2
(0,0,2,0)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4
(0,0,0,2)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0
(0,0,1,1)	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2	w^4	w^4	w^4	w^4	w^0	w^0	w^0	w^0	w^2	w^2	w^2	w^2

(0,1,0,0)	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3	w^0	w^3
(0,1,0,2)	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^0	w^3	w^0	w^3	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^0	w^3	w^0	w^3
(0,1,0,1)	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^0	w^3	w^0	w^3	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^4	w^1	w^4	w^1	w^2	w^5	w^2	w^5	w^0	w^3	w^0	w^3

(1,0,0,0)	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3
(1,0,1,1)	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5	w^4	w^4	w^1	w^1	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5	w^4	w^4	w^1	w^1	w^0	w^0	w^3	w^3	w^2	w^2	w^5	w^5
(1,0,2,2)	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1	w^2	w^2	w^5	w^5	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1	w^2	w^2	w^5	w^5	w^0	w^0	w^3	w^3	w^4	w^4	w^1	w^1

(1,1,0,0)	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0
(1,1,1,0)	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2
(1,1,2,0)	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4	w^0	w^3	w^3	w^0	w^0	w^3	w^3	w^0	w^2	w^5	w^5	w^2	w^4	w^1	w^1	w^4


(0,0) (1,1) (2,2) (1,0) (2,1) (0,2) (0,1) (1,2) (2,0)
(2,2) w0 w4 w2 w2 w0 w4 w2 w0 w4
(0,1) w0 w4 w2 w0 w4 w2 w4 w2 w0
(1,0) w0 w4 w2 w4 w2 w0 w0 w4 w2
(2,0) w0 w2 w4 w2 w4 w0 w0 w2 w4
(0,2) w0 w2 w4 w0 w2 w4 w2 w4 w0
(1,1) w0 w2 w4 w4 w0 w2 w4 w0 w2
⊗
(0, 0) (1, 0) (0, 1) (1, 1)
+ + + + (0, 0)
(0,0) w0 w0 w0 w0 w0 w0 w0 w0 w0
(0,2) w0 w2 w4 w0 w2 w4 w2 w4 w0
(0,1) w0 w4 w2 w0 w4 w2 w4 w2 w0
⊗
+ — + — (1, 0)
(0,0) w0 w0 w0 w0 w0 w0 w0 w0 w0
(1,1) w0 w2 w4 w4 w0 w2 w4 w0 w2
(2,2) w0 w4 w2 w2 w0 w4 w2 w0 w4
⊗
+ + — — (0, 1)
(0,0) w0 w0 w0 w0 w0 w0 w0 w0 w0
(0,2) w0 w4 w2 w4 w2 w0 w0 w4 w2
(0,1) w0 w2 w4 w2 w4 w0 w0 w2 w4
⊗
+ — — + (1, 1)

$$\Phi_0 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline w^0 & w^4 & w^2 & w^2 & w^0 & w^4 & w^2 & w^0 & w^4 \\ \hline w^0 & w^4 & w^2 & w^0 & w^4 & w^2 & w^4 & w^2 & w^0 \\ \hline w^0 & w^4 & w^2 & w^4 & w^2 & w^0 & w^0 & w^4 & w^2 \\ \hline w^0 & w^2 & w^4 & w^2 & w^4 & w^0 & w^0 & w^2 & w^4 \\ \hline w^0 & w^2 & w^4 & w^0 & w^2 & w^4 & w^2 & w^4 & w^0 \\ \hline w^0 & w^2 & w^4 & w^4 & w^0 & w^2 & w^4 & w^0 & w^2 \\ \hline \end{array}$$

$$\Phi_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ \hline w^0 & w^2 & w^4 & w^0 & w^2 & w^4 & w^2 & w^4 & w^0 \\ \hline w^0 & w^4 & w^2 & w^0 & w^4 & w^2 & w^4 & w^2 & w^0 \\ \hline \end{array}$$

$$\Phi_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ \hline w^0 & w^2 & w^4 & w^4 & w^0 & w^2 & w^4 & w^0 & w^2 \\ \hline w^0 & w^4 & w^2 & w^2 & w^0 & w^4 & w^2 & w^0 & w^4 \\ \hline \end{array}$$

$$\Phi_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ \hline w^0 & w^4 & w^2 & w^4 & w^2 & w^0 & w^0 & w^4 & w^2 \\ \hline w^0 & w^2 & w^4 & w^2 & w^4 & w^0 & w^0 & w^2 & w^4 \\ \hline \end{array}$$

$$
\Phi_0 = \begin{pmatrix}
w^0 & w^4 & w^2 & w^2 & w^0 & w^4 & w^2 & w^0 & w^4 \\
w^0 & w^4 & w^2 & w^0 & w^4 & w^2 & w^4 & w^2 & w^0 \\
w^0 & w^4 & w^2 & w^4 & w^2 & w^0 & w^0 & w^4 & w^2 \\
w^0 & w^2 & w^4 & w^2 & w^4 & w^0 & w^0 & w^2 & w^4 \\
w^0 & w^2 & w^4 & w^0 & w^2 & w^4 & w^2 & w^4 & w^0 \\
w^0 & w^2 & w^4 & w^4 & w^0 & w^2 & w^4 & w^0 & w^2
\end{pmatrix}
\qquad
\Phi_0^*\Phi_0 = \frac{1}{3}\begin{pmatrix}
2 & - & - & & & & & & \\
- & 2 & - & & & & & & \\
- & - & 2 & & & & & & \\
 & & & 2 & - & - & & & \\
 & & & - & 2 & - & & & \\
 & & & - & - & 2 & & & \\
 & & & & & & 2 & - & - \\
 & & & & & & - & 2 & - \\
 & & & & & & - & - & 2
\end{pmatrix}
$$

$$
\Phi_1 = \begin{pmatrix}
w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\
w^0 & w^2 & w^4 & w^0 & w^2 & w^4 & w^2 & w^4 & w^0 \\
w^0 & w^4 & w^2 & w^0 & w^4 & w^2 & w^4 & w^2 & w^0
\end{pmatrix}
\qquad
\Phi_1^*\Phi_1 = \frac{1}{3}\begin{pmatrix}
+ & & & + & & & & & + \\
 & + & & & + & & + & & \\
 & & + & & & + & & + & \\
+ & & & + & & & & & + \\
 & + & & & + & & + & & \\
 & & + & & & + & & + & \\
 & + & & & + & & + & & \\
 & & + & & & + & & + & \\
+ & & & + & & & & & +
\end{pmatrix}
$$

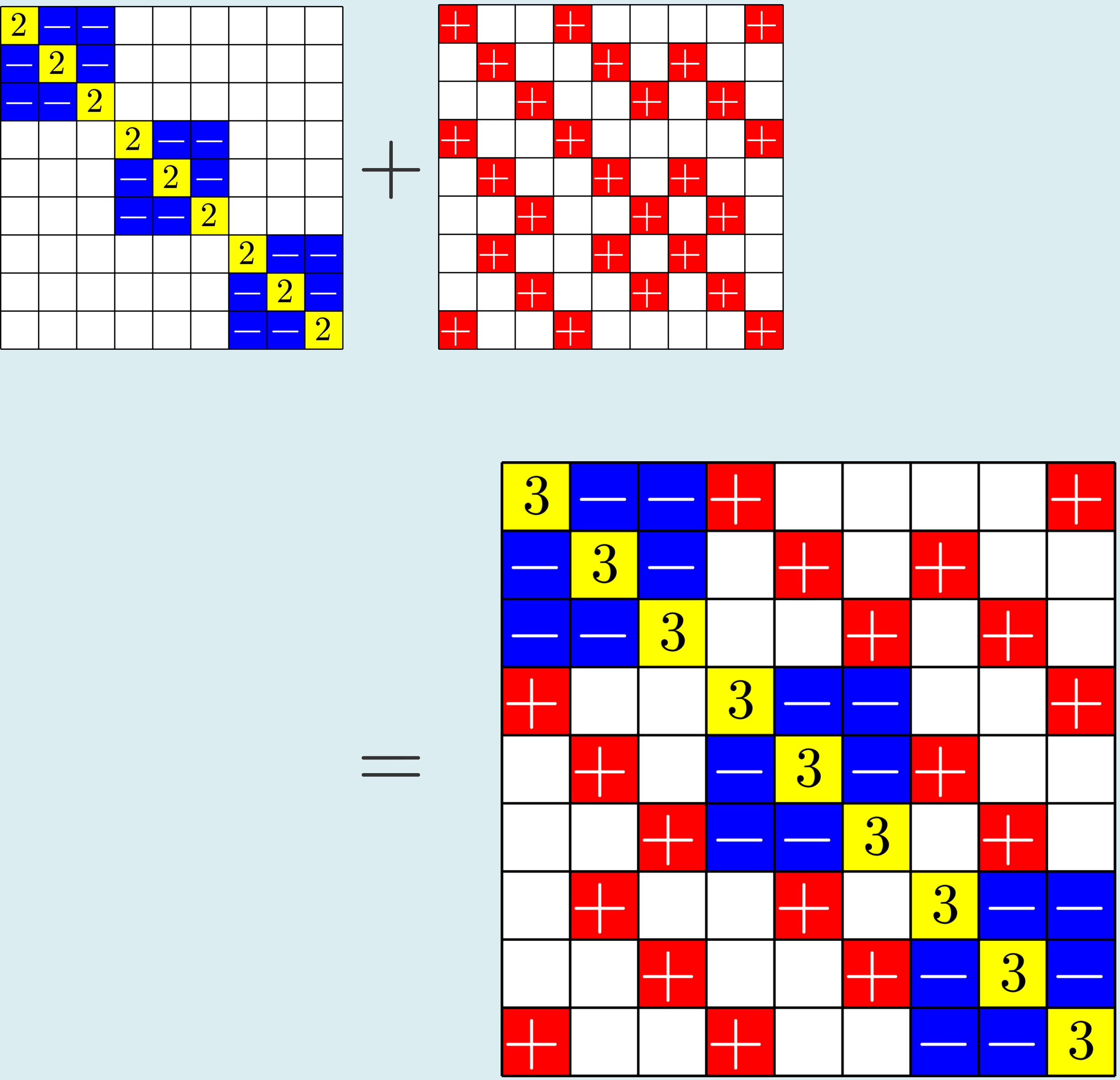
$$
\Phi_2 = \begin{pmatrix}
w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\
w^0 & w^2 & w^4 & w^4 & w^0 & w^2 & w^4 & w^0 & w^2 \\
w^0 & w^4 & w^2 & w^2 & w^0 & w^4 & w^2 & w^0 & w^4
\end{pmatrix}
\qquad
\Phi_2^*\Phi_2 = \frac{1}{3}\begin{pmatrix}
+ & & & & + & & & + & \\
 & + & & & & + & & & + \\
 & & + & + & & & + & & \\
 & & + & + & & & + & & \\
+ & & & & + & & & + & \\
 & + & & & & + & & & + \\
 & & + & + & & & + & & \\
+ & & & & + & & & + & \\
 & + & & & & + & & & +
\end{pmatrix}
$$

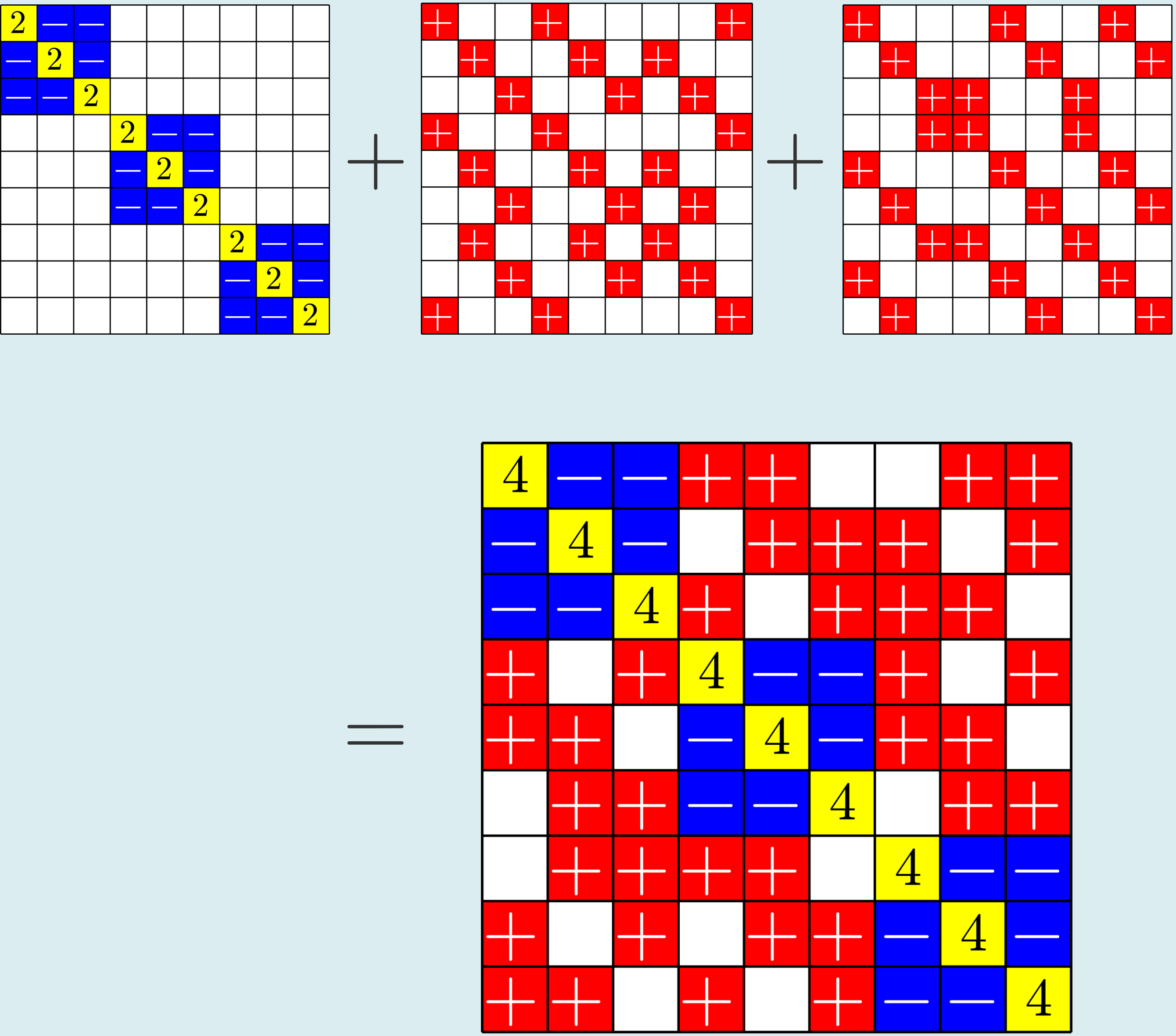
$$
\Phi_3 = \begin{pmatrix}
w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\
w^0 & w^4 & w^2 & w^4 & w^2 & w^0 & w^0 & w^4 & w^2 \\
w^0 & w^2 & w^4 & w^2 & w^4 & w^0 & w^0 & w^2 & w^4
\end{pmatrix}
\qquad
\Phi_3^*\Phi_3 = \frac{1}{3}\begin{pmatrix}
+ & & & & & + & + & & \\
 & + & & + & & & & + & \\
 & & + & & + & & & & + \\
 & + & & + & & & & + & \\
 & & + & & + & & & & + \\
+ & & & & & + & + & & \\
+ & & & & & + & + & & \\
 & + & & + & & & & + & \\
 & & + & & + & & & & +
\end{pmatrix}
$$

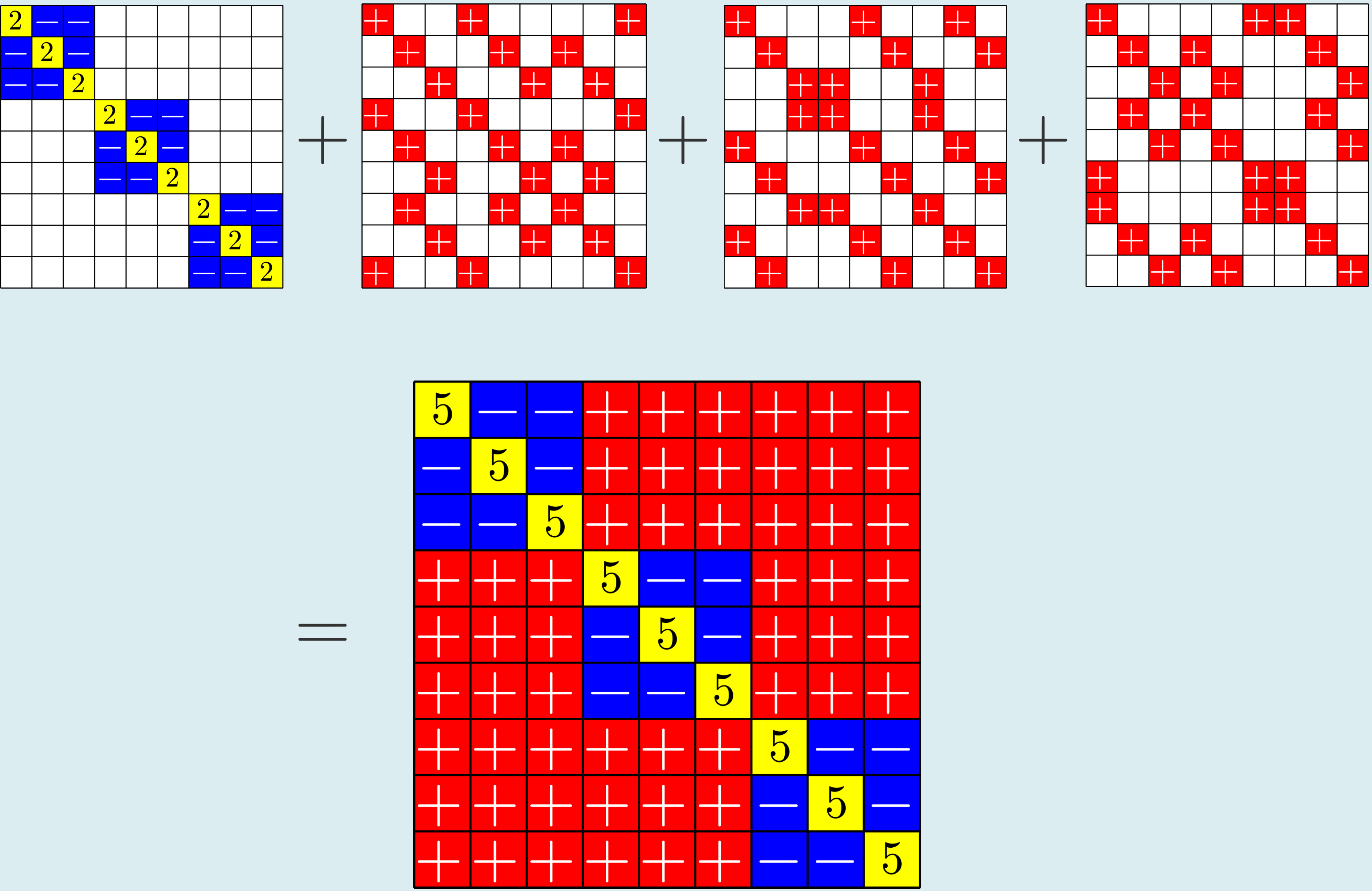
2	—	—						
—	2	—						
—	—	2						
			2	—	—			
			—	2	—			
			—	—	2			
						2	—	—
						—	2	—
						—	—	2

=

2	—	—						
—	2	—						
—	—	2						
			2	—	—			
			—	2	—			
			—	—	2			
						2	—	—
						—	2	—
						—	—	2







Theorem (Fickus, J, Myers '24). A sequence of matrices with M columns each

$$\Phi_0, \Phi_1, \ldots, \Phi_{L-1}$$

is a *compatible orthobiangular tight frame (COBTF)* if:

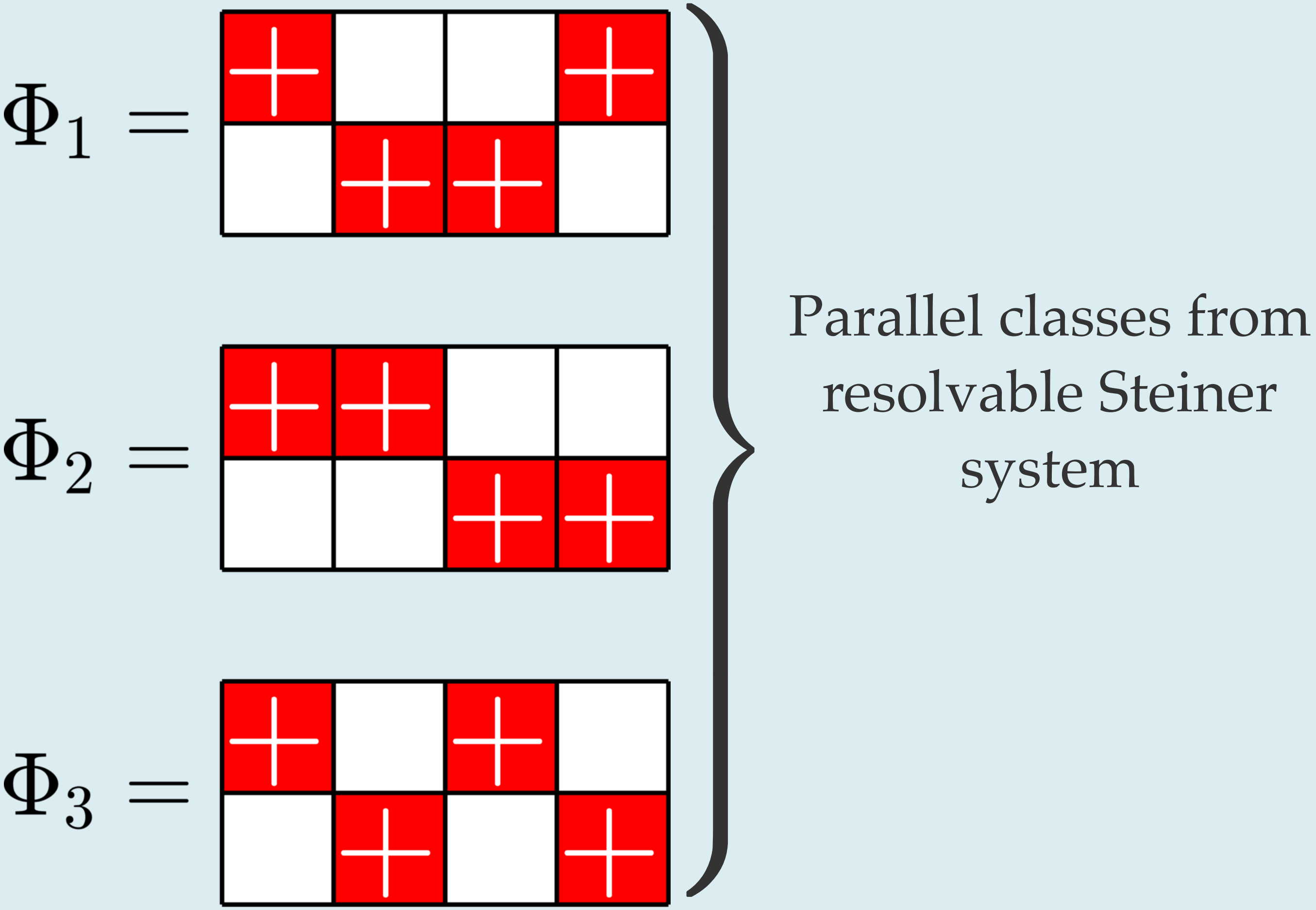
- $(\Phi_j^* \Phi_j)^2 = A \Phi_j^* \Phi_j$ (tight, same A)
- Off-diagonal $\Phi_j^* \Phi_j$ has modulus in $\{0, 1\}$ (orthobiangular)
- Off-diagonals supports $\Phi_j^* \Phi_j$ partition
- "Compatibility condition"

If $(h_i)_{i=0}^{L-1}$ are the rows of a (possibly complex) Hadamard matrix, then

$$\Psi = \begin{bmatrix} \Phi_0 \otimes h_0 \\ \Phi_1 \otimes h_1 \\ \vdots \\ \Phi_{L-1} \otimes h_{L-1} \end{bmatrix}$$

is an ETF.

COBTF Example 1 (Resolvable Steiner)



COBTF Example 1 (Resolvable Steiner)

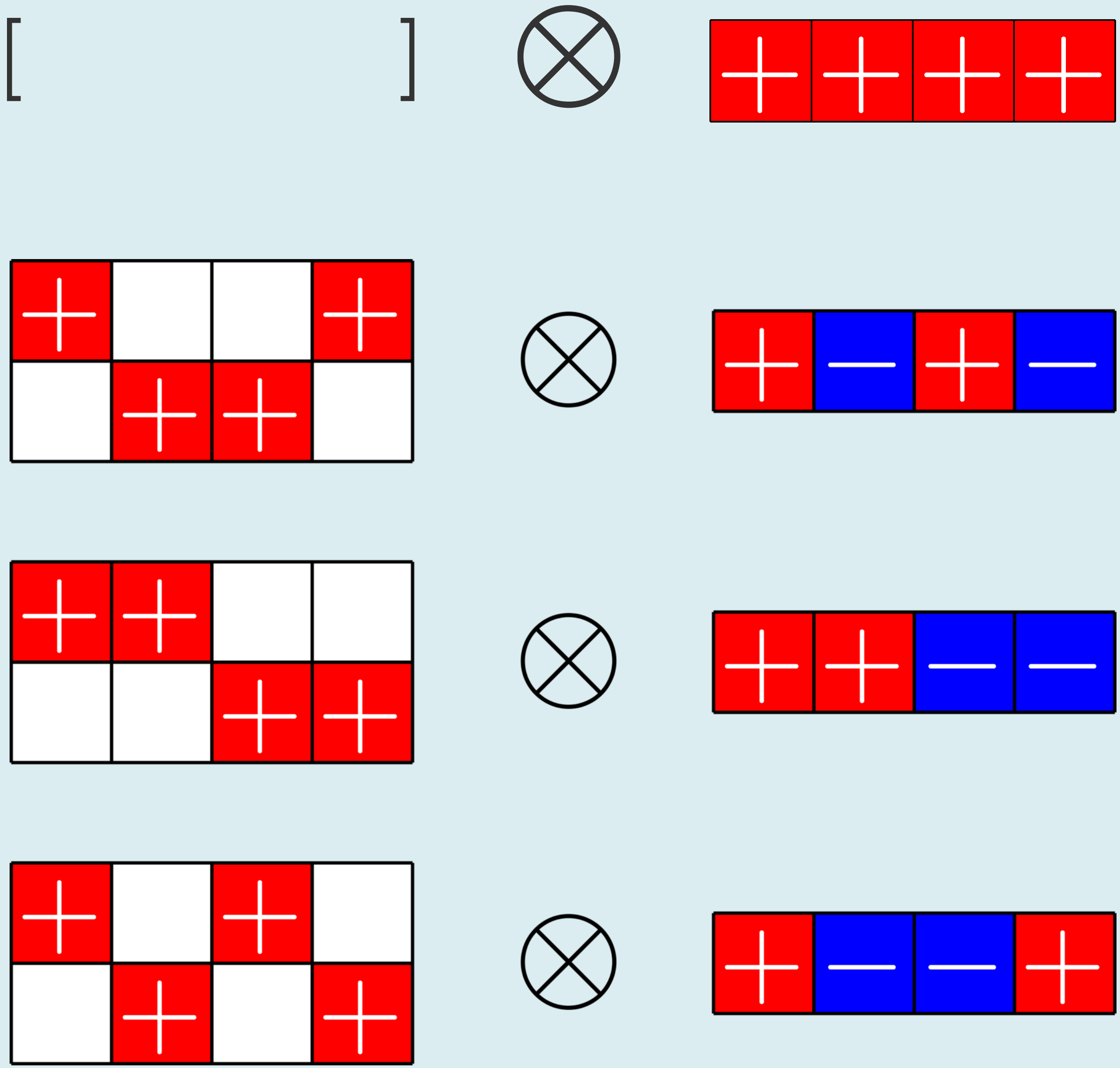
$$\Phi_0 = [\quad]$$

$$\Phi_1 = \begin{bmatrix} + & & & + \\ & + & + & \end{bmatrix}$$

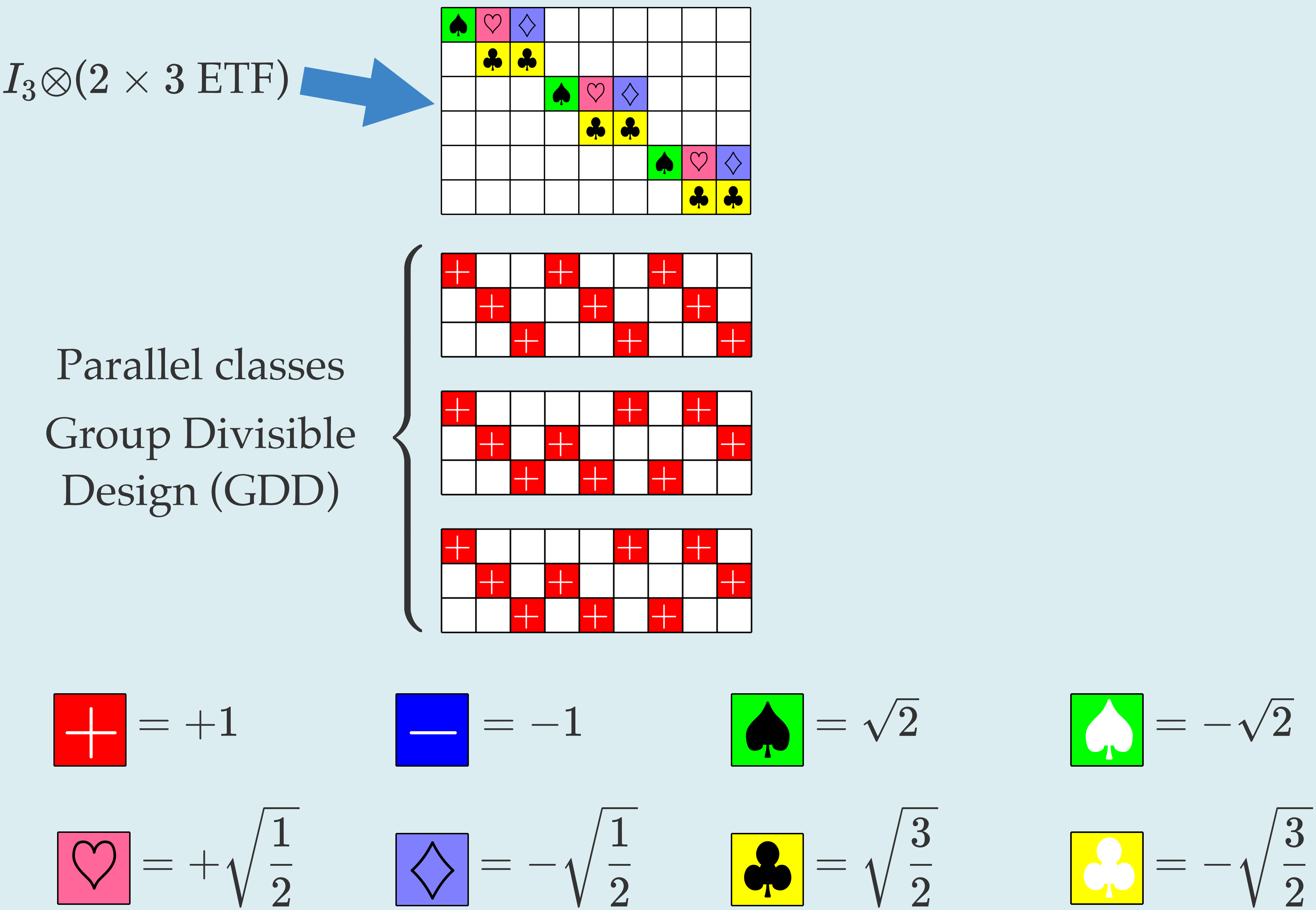
$$\Phi_2 = \begin{bmatrix} + & + & & \\ & & + & + \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} + & & + & \\ & + & & + \end{bmatrix}$$

COBTF Example 1 (Resolvable Steiner)



COBTF Example II (Group Divisible Design)



COBTF Example II (Group Divisible Design)

$\Phi_0 =$

♠	♡	♢						
	♣	♣						
			♠	♡	♢			
				♣	♣			
						♠	♡	♢
							♣	♣

$\Phi_1 =$

+			+			+		
	+			+			+	
		+			+			+

$\Phi_2 =$

+					+		+	
	+		+					+
		+		+		+		

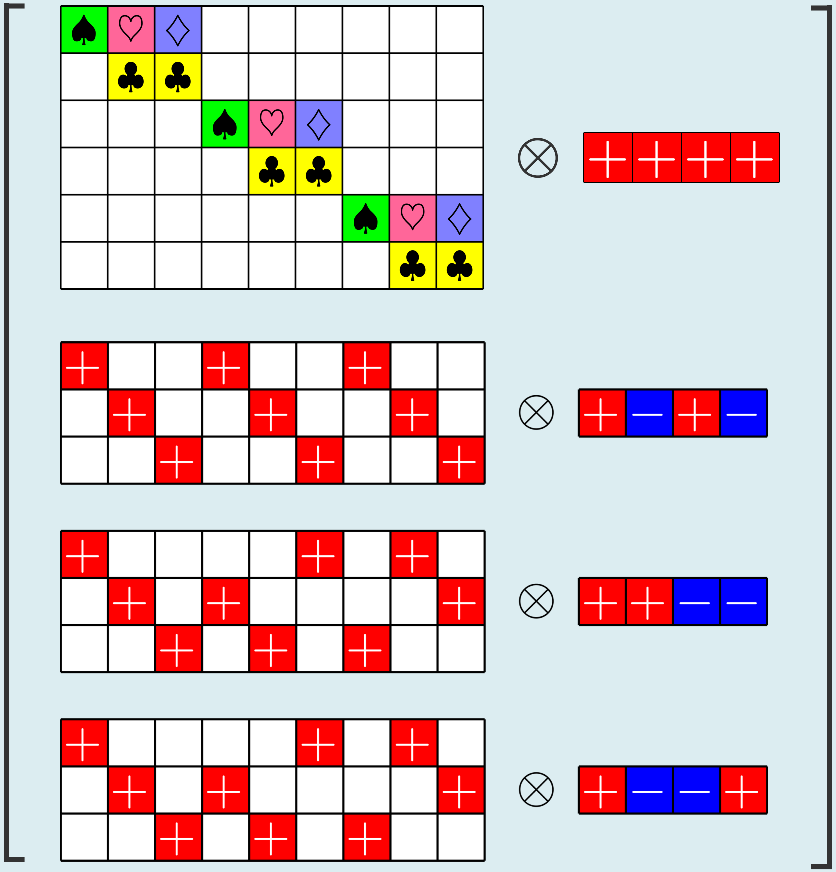
$\Phi_3 =$

+					+		+	
	+		+					+
		+		+		+		

+ (red) $= +1$ — (blue) $= -1$ ♠ (black on green) $= \sqrt{2}$ ♠ (white on green) $= -\sqrt{2}$

♡ $= +\sqrt{\frac{1}{2}}$ ♢ $= -\sqrt{\frac{1}{2}}$ ♣ (black on yellow) $= \sqrt{\frac{3}{2}}$ ♣ (white on yellow) $= -\sqrt{\frac{3}{2}}$

COBTF Example II (Group Divisible Design)



+ $= +1$

− $= -1$

♠ $= \sqrt{2}$

♤ $= -\sqrt{2}$

♡ $= +\sqrt{\frac{1}{2}}$

♢ $= -\sqrt{\frac{1}{2}}$

♣ $= \sqrt{\frac{3}{2}}$

♧ $= -\sqrt{\frac{3}{2}}$

COBTF Example II (Group Divisible Design)

♠	♠	♠	♠	♡	♡	♡	♡	♢	♢	♢	♢																								
				♣	♣	♣	♣	♣	♣	♣	♣																								
												♠	♠	♠	♠	♡	♡	♡	♡	♢	♢	♢	♢												
																♣	♣	♣	♣	♣	♣	♣	♣												
																								♠	♠	♠	♠	♡	♡	♡	♡	♢	♢	♢	♢
																												♣	♣	♣	♣	♣	♣	♣	♣
+	−	+	−																	+	−	+	−	+	−	+	−								
								+	−	+	−					+	−	+	−													+	−	+	−
				+	−	+	−					+	−	+	−													+	−	+	−				
+	+	−	−													+	+	−	−									+	+	−	−				
				+	+	−	−													+	+	−	−									+	+	−	−
								+	+	−	−	+	+	−	−									+	+	−	−								
+	−	−	+									+	−	−	+																	+	−	−	+
				+	−	−	+									+	−	−	+					+	−	−	+								
								+	−	−	+									+	−	−	+					+	−	−	+				

+ $= +1$

− $= -1$

♠ $= \sqrt{2}$

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♡ $= +\sqrt{\frac{1}{2}}$

♢ $= -\sqrt{\frac{1}{2}}$

♣ $= \sqrt{\frac{3}{2}}$

♧ $= -\sqrt{\frac{3}{2}}$

Chen's Construction

- Y. Q. Chen (1997): New difference sets!
- Used special sets $U_0, \ldots, U_{L-1}$ from a group G
- Φ_j by pulling rows U_j from the DFT over G
- Φ_0 by pulling rows $G \setminus U_0$.

$\Phi_0 =$

	(0,0)	(1,1)	(2,2)	(1,0)	(2,1)	(0,2)	(0,1)	(1,2)	(2,0)
(2,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^2	w^0	w^4
(0,1)	w^0	w^4	w^2	w^0	w^4	w^2	w^4	w^2	w^0
(1,0)	w^0	w^4	w^2	w^4	w^2	w^0	w^0	w^4	w^2
(2,0)	w^0	w^2	w^4	w^2	w^4	w^0	w^0	w^2	w^4
(0,2)	w^0	w^2	w^4	w^0	w^2	w^4	w^2	w^4	w^0
(1,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^4	w^0	w^2

$$U_0 = \{(0,0),(1,2),(2,1)\}$$

$\Phi_1 =$

(0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(0,2)	w^0	w^2	w^4	w^0	w^2	w^4	w^2	w^4	w^0
(0,1)	w^0	w^4	w^2	w^0	w^4	w^2	w^4	w^2	w^0

$$U_1 = \{(0,0),(0,2),(0,1)\}$$

$\Phi_2 =$

(0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(1,1)	w^0	w^2	w^4	w^4	w^0	w^2	w^4	w^0	w^2
(2,2)	w^0	w^4	w^2	w^2	w^0	w^4	w^2	w^0	w^4

$$U_2 = \{(0,0),(1,1),(2,2)\}$$

$\Phi_3 =$

(0,0)	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0	w^0
(0,2)	w^0	w^4	w^2	w^4	w^2	w^0	w^0	w^4	w^2
(0,1)	w^0	w^2	w^4	w^2	w^4	w^0	w^0	w^2	w^4

$$U_3 = \{(0,0),(0,2),(0,1)\}$$

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Example. Chen constructs

$$U_0, U_1, \ldots, U_{323} \subset \mathbb{Z}_3^8$$

This gives us a COBTF $(\Phi_j)_{j=0}^{323}$.

$\exists\ 324 \times 324$ real Hadamard matrix.

$\exists$ real 957177×2152008 ETF.

$\exists$ SRGs 2152008,

and 2152008 vertices.

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Off the charts!

?	1300	441	154	147	21^{507}	-14^{792}	
		858	563	572	13^{792}	-22^{507}	
?	1300	516	254	172	86^{52}	-4^{1247}	
		783	438	522	3^{1247}	-87^{52}	pg(9,86,6)?

Finding an axial 77×210

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From Brouwer's table online:

?	210	95	40	45	5^{133}	-10^{76}	2-graph?
		114	63	60	9^{76}	-6^{133}	2-graph?
+	210	99	48	45	9^{77}	-6^{132}	Sym(7) - Klin, cf. Klin_et_al; 2-graph
		110	55	60	5^{132}	-10^{77}	pg(11,9,6)?; 2-graph

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Thanks!

- This work was partially supported by NSF #1830066
- Webpage: https://tinyurl.com/jasperAFIT
- Slides: https://slides.com/johnjasper