RATIONAL HOMOTOPY THEORY IN GEOMETRY – ASSIGNMENT 1

(1) Compute directly the simplicial homology of a figure 8, and give a cellular decomposition for this space.

(2) Show that $S^n \subset \mathbb{R}^{n+1}$ is a deformation retract of $\mathbb{R}^{n+1} - \{0\}$.

(3) Consider the simplicial degree 2 map $S^1 \to S^1$ we discussed in class. Calculate its induced map on homology and on homotopy groups.

(4) (a) Consider the inclusion $S^n \xrightarrow{i} D^{n+1}$ of the n-sphere as the boundary of the $(n+1)$-disk $D^{n+1} = \{x \in \mathbb{R}^{n+1}, |x| \leq 1\}$. Prove that the disk does not retract onto the sphere, i.e. that there is no (continuous) map $D^{n+1} \to S^n$ such that the composition ri equals the identity map $S^n \to S^n$.

(b) Prove the Brouwer fixed theorem for disks: Every continuous map $D^n \xrightarrow{f} D^n$ has a fixed point, i.e. a point $x \in D^n$ such that $f(x) = x$. Hint: suppose there is no fixed point, then x and $f(x)$ are always distinct and so determine a line. Using this build a retract to the boundary. You do not have to carefully argue that your constructed map is continuous.

(c) Prove that the Brouwer fixed point theorem holds for any space homeomorphic to a disk D^n.

(d) Show that the Brouwer fixed point theorem need not hold on a space that is only homotopy equivalent to a disk. That is, find an example of a space X homotopy equivalent to a disk, together with a continuous map $X \xrightarrow{f} X$, which has no fixed points. Moral of the story: homeomorphism type knows about the individual points in a space, while the homotopy type does not.

The Brouwer fixed point theorem can be used to recover many disparate results: the fundamental theorem of algebra, the Nash equilibrium theorem, and more.

(e) Here is another basic application: prove that a matrix with non-negative real entries has at least one non-negative eigenvalue. Hint: if zero is an eigenvalue, then we're done; otherwise every non-zero vector with non-negative entries is sent to a non-zero vector with non-negative entries. You can take for granted that the intersection of the unit sphere in $\mathbb{R}^n$ with the set of vectors with non-negative entries is homeomorphic to a disk. (E.g. in $\mathbb{R}^2$ this intersection is a quarter of a circle, from 0 to $\frac{\pi}{2}$.)

(5) Consider a finite chain complex of finite-dimensional vector spaces C_i,

$$0 \xrightarrow{\partial} C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

(A finite chain complex is a chain complex with finitely many non-zero terms.) Define its **Euler characteristic** to be the alternating sum of dimensions of homology

$$\dim H_0 - \dim H_1 + \dim H_2 - \cdots + (-1)^k \dim H_k.$$

Now consider another chain complex, with the same vector spaces but another differential ∂':

$$0 \xrightarrow{\partial'} C_k \xrightarrow{\partial'} C_{k-1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} C_0 \xrightarrow{\partial'} 0$$

- Prove that the Euler characteristics of both chain complexes coincide.

That is, the Euler characteristic depends only on the vector spaces in the chain complex and not on the differential. In particular, taking the trivial differential $\partial'' = 0$, we conclude that the Euler characteristic is equal to the alternating sum

$$\dim C_0 - \dim C_1 + \dim C_2 - \cdots + (-1)^k \dim C_k.$$

We can take C_i to be the simplicial or cellular chains of some finite simplicial complex or finite CW complex. Since simplicial and cellular homology are isomorphic, the Euler characteristics of the resulting complexes are equal; furthermore, by the above, we can compute the Euler characteristic by taking the alternating sum of the number of i-simplices or i-cells. We denote the Euler characteristic of a topological space X (computed in any of these equivalent ways) by $\chi(X)$.

- Calculate the Euler characteristic of S^2.
- Calculate the Euler characteristic of S^n, for n a non-negative integer.
- Calculate the Euler characteristic of $S^n \times S^n$.
- Extra credit: Make a guess for a general formula for the Euler characteristic of a product $\chi(X \times Y)$. Use the Künneth theorem to prove it.

(6) Prove that the Hopf fiber bundle $S^1 \to S^3 \xrightarrow{p} S^2$ does not have a section, i.e. there is no map $S^2 \xrightarrow{s} S^3$ such that the composition ps is the identity map $S^2 \to S^2$.

(7) Show that $H_2(\mathbb{RP}^2 \times S^3; \mathbb{Z}) = 0$.