C&O 781 Topics in Quantum Information
Quantum Information Theory, Error-correction, and Cryptography
University of Waterloo
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Assignment 1, May 16, 2008
Due: May 30, 2008

In the following questions, $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces. A rectangular matrix $Y \in \mathrm{M}_{m,n}(F)$ over a ring F is said to be diagonal if all entries except Y_{ii}, $i \leq \min m, n$, are 0. For any positive semi-definite operator $M \in \mathrm{L}(\mathcal{H})$, $\sqrt{M}$ denotes the positive semi-definite operator N such that $N^2 = M$.

Question 1. [5 marks] Let $M \in \mathrm{L}(\mathcal{H} \otimes \mathcal{K})$ be any linear operator on $\mathcal{H} \otimes \mathcal{K}$. Let $\{|e\rangle\}$ be any orthonormal basis for $\mathcal{K}$. Recall that the partial trace operation over $\mathcal{K}$ is a linear transformation from $\mathrm{L}(\mathcal{H} \otimes \mathcal{K})$ onto $\mathrm{L}(\mathcal{H})$, and is defined as:

$$\mathrm{Tr}_{\mathcal{K}}(M) \quad = \quad \sum_e (I \otimes \langle e|) M (I \otimes |e\rangle).$$

Prove that the partial trace operation is well-defined. I.e., it is independent of the choice of basis $\{|e\rangle\}$.

Question 2. [5 marks] The singular value decomposition theorem states that any matrix $X \in \mathrm{M}_{m,n}(\mathbb{C})$ may be expressed as $M = UDV$, where $U \in \mathrm{M}_m(\mathbb{C}), V \in \mathrm{M}_n(\mathbb{C})$ are unitary, and $D \in \mathrm{M}_{m,n}(\mathbb{R})$ is diagonal, with non-negative entries. Show that this theorem is equivalent to the Schmidt decomposition theorem.

Question 3. [5 marks] Consider the trine states $|\psi_j\rangle = \cos\frac{2j\pi}{3}|0\rangle + \sin\frac{2j\pi}{3}|1\rangle$, for $j = 0, 1, 2$. Suppose that you are given one of these three states uniformly at random. Show that with any single qubit projective measurement, the maximum probability of correctly guessing which state you are given is $\frac{2}{3}\cos^2\frac{\pi}{12}$.

Question 4. [5 marks] Recall that a POVM on $\mathcal{H}$ is specified by a sequence $\{E_i\}$ of positive semi-definite operators such that $\sum_i E_i = I$. Verify that the sequence $A_i = \sqrt{E_i}$ defines a quantum operation (i.e., a completely positive trace preserving map). Using this, prove that there is an extension $\mathcal{H} \otimes \mathcal{K}$ with $\dim(\mathcal{K})$ at most the number of outcomes i, and a sequence of orthogonal projection operators $\Pi_i \in \mathrm{L}(\mathcal{H} \otimes \mathcal{K})$ such that $\mathrm{Tr}(E_i \rho) = \mathrm{Tr}(\Pi_i(\rho \otimes |\bar{0}\rangle\langle\bar{0}|))$ for all $\rho \in \mathrm{L}(\mathcal{H})$.

Question 5. [5 marks] Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be a bipartite pure state such that $\mathrm{Tr}_{\mathcal{H}}(\psi) = \rho$. If $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ is any mixed state over $\mathcal{K}$ with the same density matrix ρ, show that there is a measurement on $\mathcal{H}$ which when performed on $|\psi\rangle$, results in the mixed state $\mathcal{E}$ in $\mathcal{K}$.