Discrete Mathematics U. Waterloo ECE 103, Spring 2010
Ashwin Nayak June 19, 2010

Some applications of graph theory

In these notes, we visit some important applications of elementary Graph Theory.

Gray Codes

An n-bit Gray code, also called the *reflected binary code*, is an ordering of the 2^n strings of length n over $\{0, 1\}$ such that every pair of successive strings differ in exactly one position. For example, an 2-bit Gray code is $00, 01, 11, 10$, and a 3-bit Gray code is $000, 001, 101, 111, 011, 010, 110, 100$.

If we use successive strings in a Gray code to represent the integers from 0 up to $2^n - 1$, instead of the usual binary representation, we see that incrementing a number by one involves flipping only one bit. In the usual binary representation, incrementing by 1 could lead to a sequence of bits that are carried over, which may change several consecutive bits at once. For example, $7 + 1 = 8$ in the usual binary representation is $111_2 + 1_2 = 1000_2$, and we see that incrementing 111 by 1 causes four bits to be flipped. This simple property makes Gray codes very useful in practice.

Although they had appeared in mathematical puzzles earlier, Gray codes were proposed in 1947 by Frank Gray, a physicist and researcher at Bell Labs, to prevent spurious output from electromechanical switches. (As you may have guessed, the codes are named after him.) Today, Gray codes are widely used to facilitate error correction in digital communications such as digital terrestrial television and some cable TV systems.

We have seen Gray codes with 2 and 3 bits. Do they exist for all n? Below, we see that this question is closely related to a basic property of the n-dimensional hypercube H_n.

Recall that the vertices of H_n are precisely the set of all n-bit strings, and two vertices x, y form an edge iff they differ in exactly in one position. So, the question of existence of an n-bit Gray code may be reformulated as: Is there a path in H_n that includes all the 2^n vertices? The answer is, as you may have guessed, yes! In fact, the hypercube satisfies a stronger property, that there is a *cycle* that contains all 2^n of its vertices (when $n \geq 2$). Disregarding any one of the edges in such a cycle gives us the path we seek. Such a cycle is shown in bold for H_2 and H_3 below; H_1 cannot have such a cycle, but has the kind of path we seek.

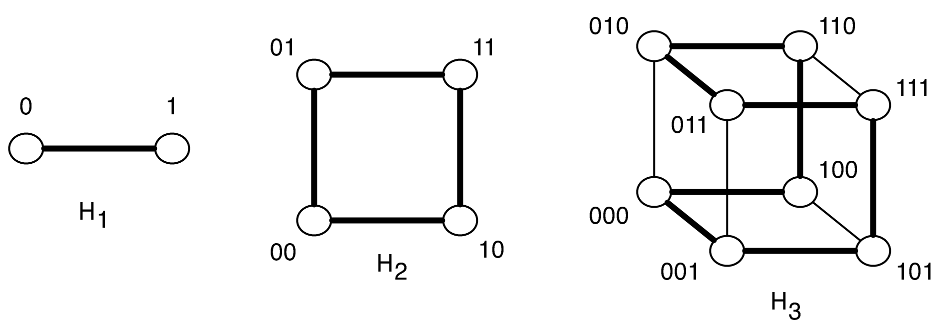

Figure 1: Hamilton paths and cycles in H_1, H_2, H_3, depicted in bold.

Definition 1 *A cycle in a graph G that contains all the vertices in it is called a* Hamilton *cycle (sometimes a* Hamiltonian *cycle).*

In the last home work, we saw that the grid graph G_n has a Hamilton cycle for even n. We now prove that the n-hypercube also does, for all $n \geq 2$. In proving this, we make use of the fact that we can construct the $(n+1)$-hypercube by connecting two copies of the n-hypercube with a suitable set of edges. So we can construct a cycle in the larger hypercube by "cutting and pasting" cycles in the two copies. For example, we obtained the Hamilton cycle in H_3 shown above, by taking two cycles in the front and back faces (which are isomorphic to H_2), removing the edges $\{000, 010\}$ and $\{001, 011\}$ from the two cycles, and pasting them with the edges $\{010, 011\}$ and $\{000, 001\}$.

Theorem 2 *For every integer $n \geq 2$, the n-hypercube H_n has a Hamilton cycle.*

Proof : The proof is by induction over n.

The base case is $n = 2$, and the statement is true, as H_2 is isomorphic to C_4, the cycle on 4 vertices.

Assume, as our induction hypothesis, that the k-hypercube has a Hamilton cycle for some $k \geq 2$. We prove that the $(k+1)$-hypercube also has a Hamilton cycle.

We partition the vertex set of H_{k+1} into two sets V_0 and V_1 of equal size. The set V_0 consists of all vertices (i.e., $(k+1)$-bit strings) beginning with a 0, and V_1 consists of all vertices beginning with a 1:

$$\begin{aligned} V_0 &= \left\{ 0x \;:\; x \in \{0,1\}^k \right\} \\ V_1 &= \left\{ 1x \;:\; x \in \{0,1\}^k \right\} \ . \end{aligned}$$

Observe that the subgraph induced by either set V_0 or V_1 is isomorphic to H_k. By the induction hypothesis, there is a cycle $C_0 = 0u_1, 0u_2, \ldots, 0u_{2^k}, 0u_1$ that contains all the 2^k vertices in V_0. Note that $C_1 = 1u_1, 1u_2, \ldots, 1u_{2^k}, 1u_1$ is also a cycle, and it contains all the vertices in V_1.

We cut and paste C_0 and C_1 as follows: we delete the edges $\{0u_{2^k}, 0u_1\}$ and $\{1u_{2^k}, 1u_1\}$, and insert edges $\{0u_{2^k}, 1u_{2^k}\}$ and $\{1u_1, 0u_1\}$ to get the Hamilton cycle in H_{k+1}:

$$0u_1, 0u_2, \ldots, 0u_{2^k}, 1u_{2^k}, 1u_{2^k-1}, \ldots, 1u_1, 0u_1 \ .$$

(Here, we have traversed the edges of the cycle C_1 in reserve order.)

By mathematical induction, the statement is true for all $n \geq 2$. ∎

Euler tours

You may have come across this puzzle in your childhood: can you draw the pattern depicted in Figure 2 in one stroke, without lifting your pencil off the paper, and without retracing any line? Of course, after trying

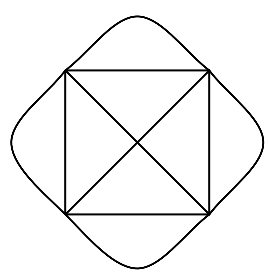

Figure 2: Can you draw this pattern without lifting your pencil off the paper, and without retracing any line?

all different ways of tracing this pattern, we realize it is impossible. This puzzle is closely related to the Seven Bridges of Königsberg problem, a historical problem in mathematics.

Much like the greater Montréal region formed on the St. Lawrence river, the city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River. It included two large islands which were connected to each other and the mainland by seven bridges (shown schematically in Figure 3).

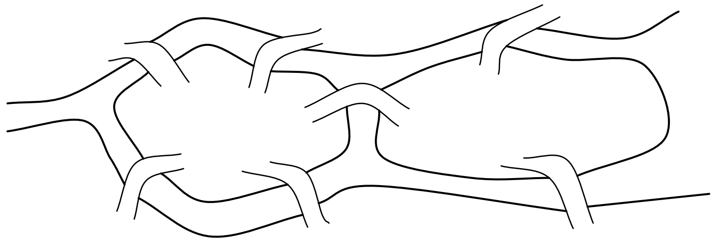

Figure 3: Schematic map of Königsberg in the 1700s showing the Pregel river, its two islands, and the seven bridges.

The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge was to be crossed completely every time. One could not walk half way onto the bridge and then turn around and later cross the other half from the other side.

Again, trying out all possibilities, we find that this task is impossible. In 1735, Leonhard Euler, the preeminent Swiss mathematician and physicist of the 18th century, explained this more generally by casting it in terms of graphs, thereby founding a new field of study, graph theory. Euler's solution was to represent the two banks of the river Pregel, and the two islands by vertices, and the bridges by edges between these vertices. Notice that this gives rise to multiple edges between pairs of vertices. The resulting object is therefore called a *multigraph*, see Figure 4. Euler proved that if there were a solution to the seven bridges problem, exactly two of the vertices would have odd degree, or all of them would have even degree. Since this is not the case, there is no solution. Similarly, the puzzle in Figure 2 has no solution, as there are four odd degree vertices.

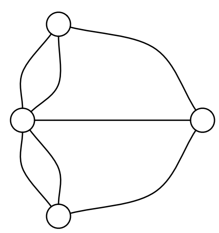

Figure 4: The multigraph corresponding to the Königsberg bridges problem.

Below we study this kind of problem for general graphs. The results we prove also apply to multigraphs. However, to avoid verifying which properties of graphs extend to multigraphs, and which do not, we restrict ourselves to graphs. The puzzles above may be recast as graph problems by introducing new vertices on the "extra" edges (see Figure 5), as can any other multigraph.

Definition 3 *An* Euler walk *in a connected graph G (sometimes also called an* Eulerian *walk) is a walk in G that traverses each edge exactly once. An* Euler tour *in a connected graph is a closed Euler walk, i.e., it starts and ends at the same vertex in the graph.*

The puzzles above ask us if there is an Euler walk in the (multi)graphs shown in Figure 5. Figure 6 shows a graph with an Euler tour.

Theorem 4 *Let G be a connected graph with at least two vertices. The graph G admits an Euler tour if and only if all its vertices have even degree. The graph G admits an Euler walk if and only if it has exactly zero or two odd degree vertices.*

Proof : We show the theorem in three parts.

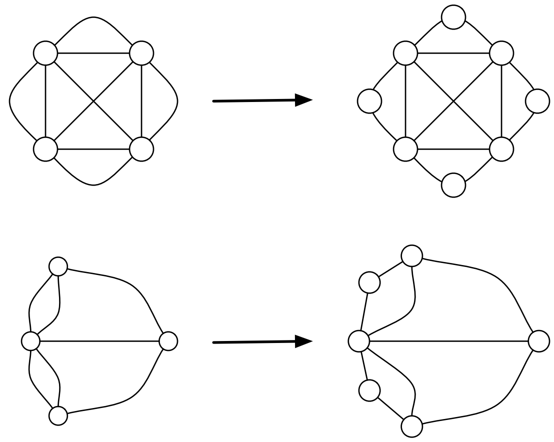

Figure 5: The multigraphs corresponding to the two puzzles. Multigraphs may be turned into graphs by introduction of new vertices that "split" edges.

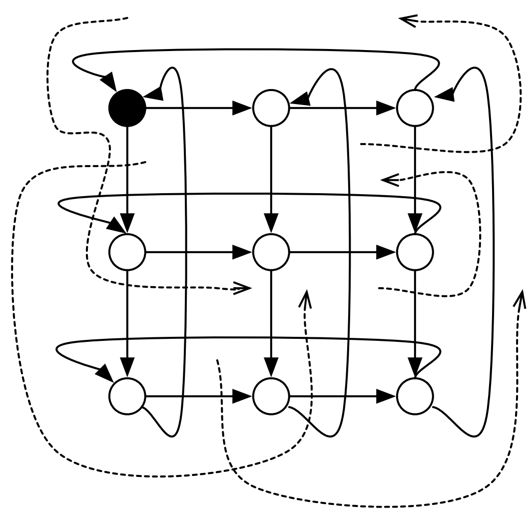

Figure 6: An Euler tour in the 2-dimensional Torus T_3. The solid black vertex is the start and the end of the tour. The arrows on the edges show the direction in which the tour traverses them. The dashed lines show which edge the tour follows, when there is more than one choice.

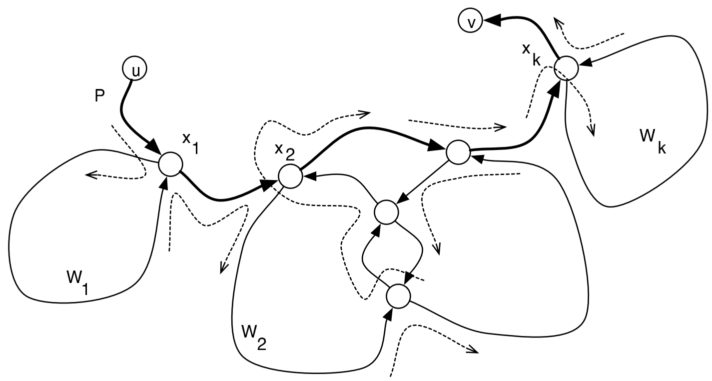

Figure 7: Constructing an Euler walk by splicing the path P (shown in bold) and a collection of tours $W_1, W_2, \ldots, W_k$. The dashed arrows show how the path and the tours are pasted together.

(a) Suppose the graph G has an Euler walk W. If the start and end vertices u, v of W are different, we show that all vertices *except* u, v have even degree. If $u = v$ (i.e., the walk is an Euler tour), then all vertices have even degree.

Consider any vertex x other than u, v. Since the graph is connected and has more than one vertex, there is some edge incident on x. Since W traverses all edges, it visits x as well. Each time W visits x, it traverses two edges incident on it. Since W does not traverse any edge more than once, and traverses all edges, the degree of x is even.

Similarly, every time W visits u or v after the start and before the end, it traverses two edges. At the start, or at the end, it traverses only one edge. Therefore, if $u \neq v$, they have odd degree. Otherwise, they too have even degree.

(b) Suppose the graph G has exactly two vertices u, v with odd degree. Then we show that it has an Euler walk that starts at u and ends at v.

By the property shown in Quiz 9.2, there is a path P from u to v in G. If P is an Euler walk in G, we are done. Otherwise, we remove the edges in P from G and any resulting vertices with degree 0. This leaves us with a graph G' with k connected components $G_1, G_2, \ldots, G_k$, for some $k \geq 1$. Each subgraph G_i is connected, and has only even degree vertices (by Part (a)). Therefore, by the property in Part (c) we prove below, it has an Euler tour W_i. We construct an Euler walk W by splicing P and W_i as follows.

Since G is connected, P has at least one (distinct) vertex x_i in common with each G_i. Without loss of generality, let the vertices x_i occur in the order $x_1, x_2, \ldots, x_k$ as we go from u to v in P. The walk W starts at u and follows P up to x_1, then follows the Euler tour W_1 of G_1, returns to x_1, then follows P to x_2, then follows the Euler tour W_2 of G_2, returns to x_2, and so on until all the subgraphs G_i have been toured, and we are at x_k. Then, we follow P to v, and we have our Euler walk. Note that since P is a path, and W_i are Euler tours of disjoint components (that do not contain edges of P), every edge is traversed exactly once.

This splicing is depicted in Figure 7. As shown in this figure, the path P may intersect a component in more than one vertex.

(c) If all its vertices have even degree, then we show that G has an Euler tour.

The proof is by induction on the number of edges m in G. The base case is $m = 3$, since we are concerned with simple graphs (that do not allow multiple edges). The graph then is isomorphic to C_3, the cycle on three vertices, which has an Euler tour.

Assume, as induction hypothesis, that every connected graph with at most m edges, $m \geq 3$, and only even degree vertices has an Euler tour.

Consider a connected graph with $m + 1 \geq 4$ edges with only even degree vertices. The graph G has at least two vertices, as the number of edges is at least 4. By the property shown in Quiz 9.1, there is a cycle C in G. If C is an Euler tour of G, we are done. Otherwise, as in the proof in Part (b), we remove the edges in cycle C from G, along with any vertices that have degree 0 as a result. This leaves us with components $G_1, G_2, \ldots, G_k$, for some $k \geq 1$, that have at least 2 vertices each, and only even degree vertices. Moreover, each component has at most m edges as we removed all the edges of C, and C has at least three edges. By the induction hypothesis, there is an Euler tour W_i in G_i. We splice together C with an arbitrary start and end vertex u, and the tours W_i as in Part (b) above, to obtain an Euler tour of G.

This completes the proof of the theorem. ■

The Euler tour in Figure 6 was obtained by precisely the process which we have described in the above proof.