BINARY SUBMATROIDS

CanaDAM 2019

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(Binary) matroid $M = (E, G)$

- $G \cong PG(n-1, \mathbb{F}_2) \cong \mathbb{F}_2^n \setminus \{0\}$
- $E \subseteq G$

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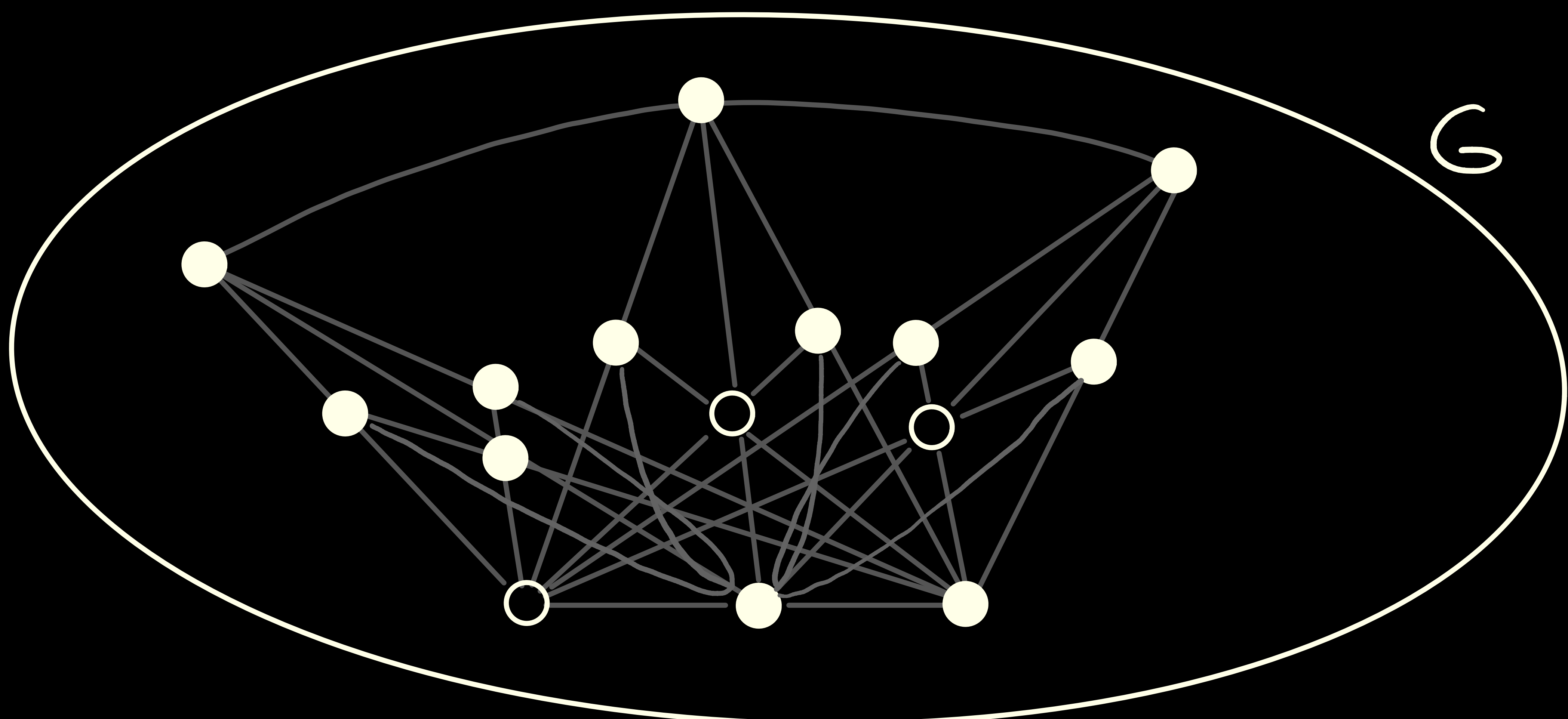
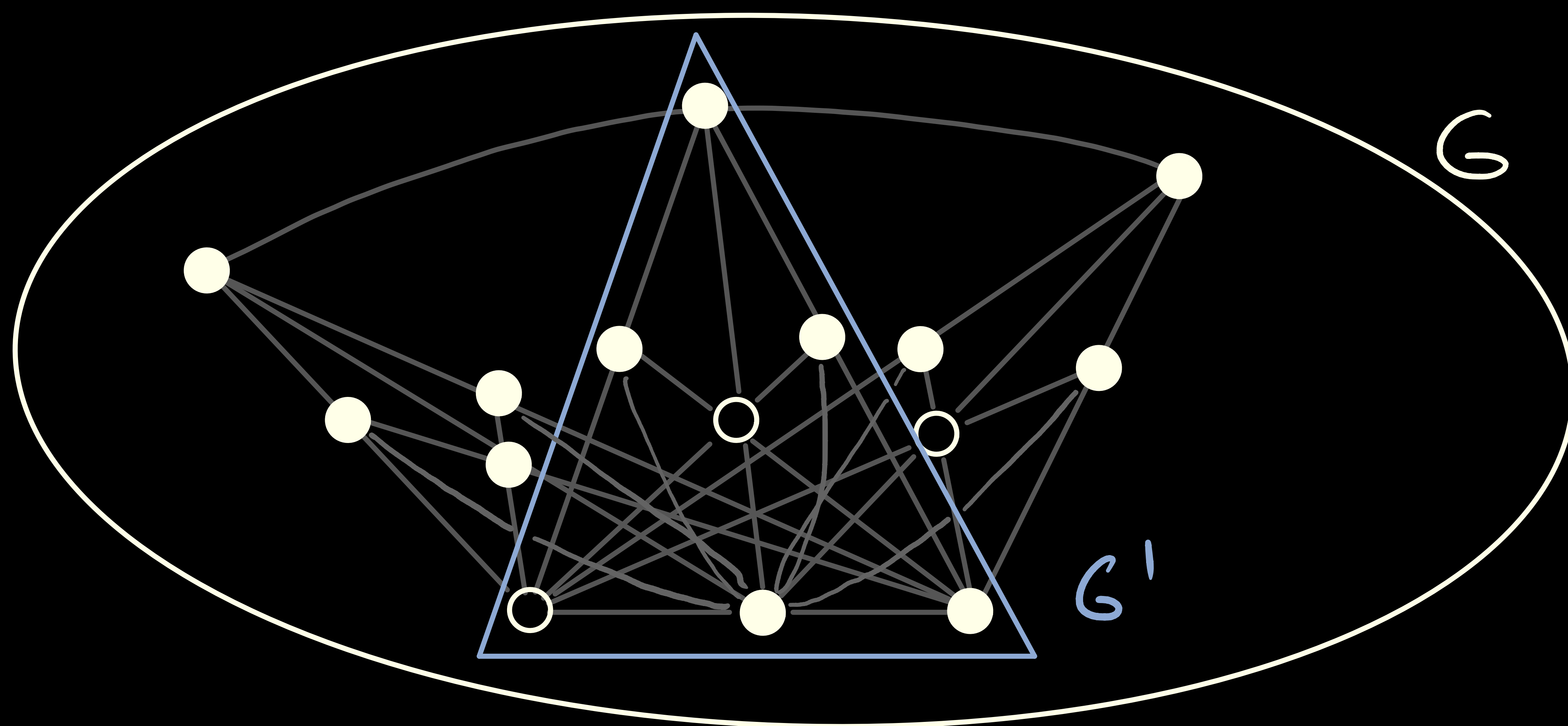


$\dim(M) := n$

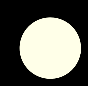


Submatroid $M' = (E', G')$

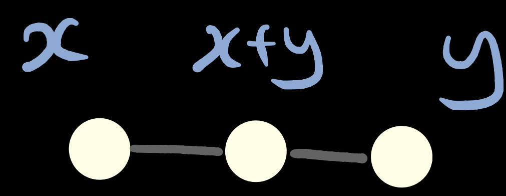
where $E' \subseteq E \cap G'$

Induced submatroid $M' = (E \cap G', G')$

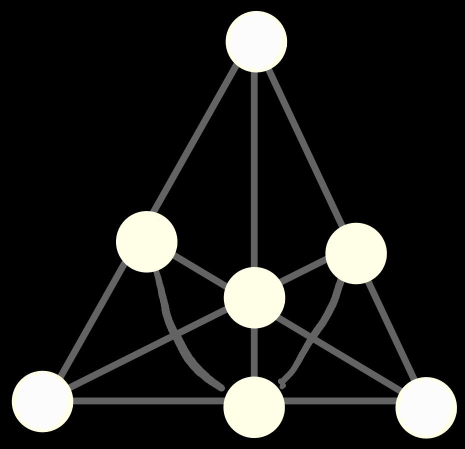
Projective Geometries



PG_1

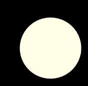


PG_2

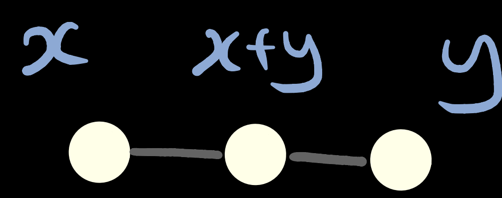


PG_3

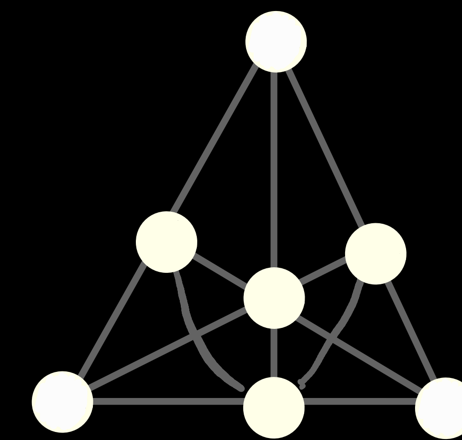
Projective Geometries



PG_1

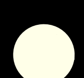


PG_2

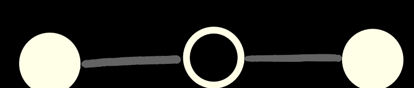


PG_3

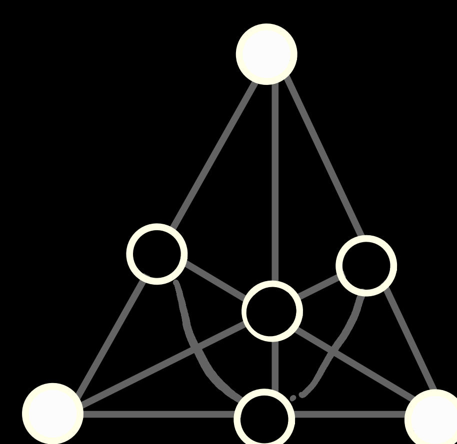
'Trees'



I_1



I_2



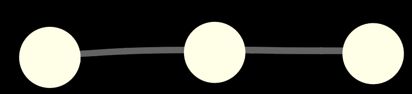
I_3

(CLAW)

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

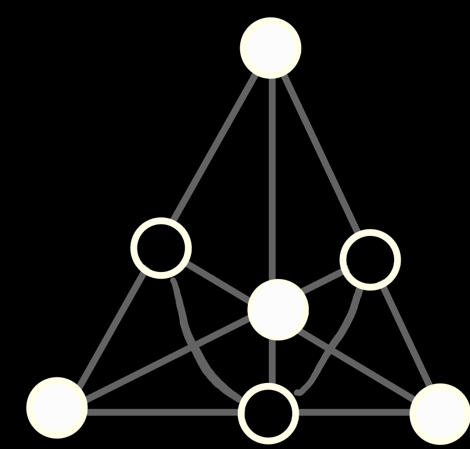
I_4

Circuits



C_3

(TRIANGLE)



C_4

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

C_5

girth(M) = size of smallest circuit in M

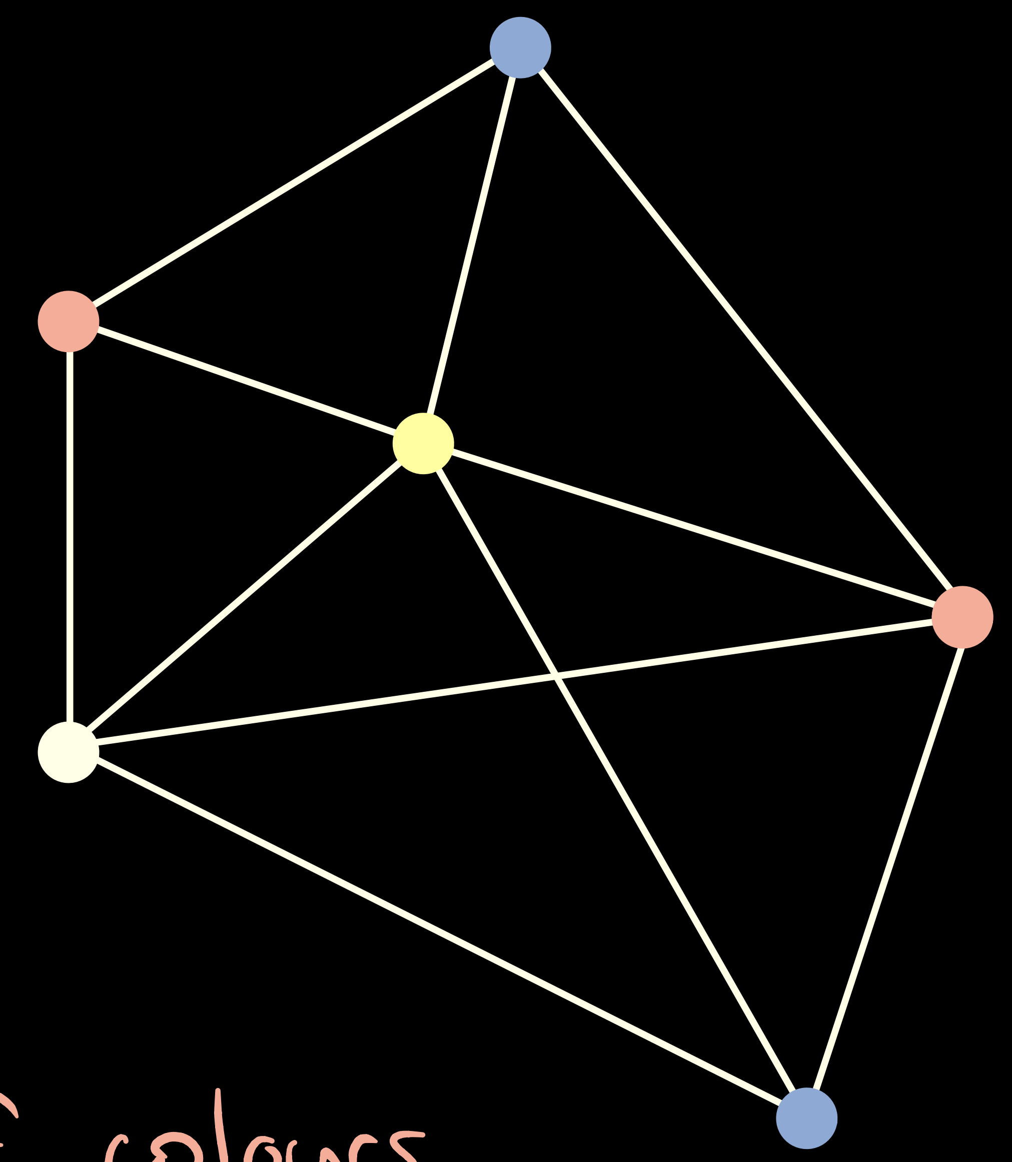
Connections

- Additive combinatorics
  - Triangle-free : 'sum-free', 'capset'
  - Submatroid : 'System of linear forms'
  - Regularity, Pseudorandomness

Connections

- Additive combinatorics
  - Triangle-free : 'sum-free', 'capset'
  - Submatroid : 'System of linear forms'
  - Regularity, Pseudorandomness
- Coding Theory
  - $E \leftrightarrow$ set of columns of parity-check matrix
  - girth : 'minimum distance'

GRAPH PARAMETERS



$\omega(G)$ = size of largest complete subgraph

$\alpha(G)$ = size of largest stable set

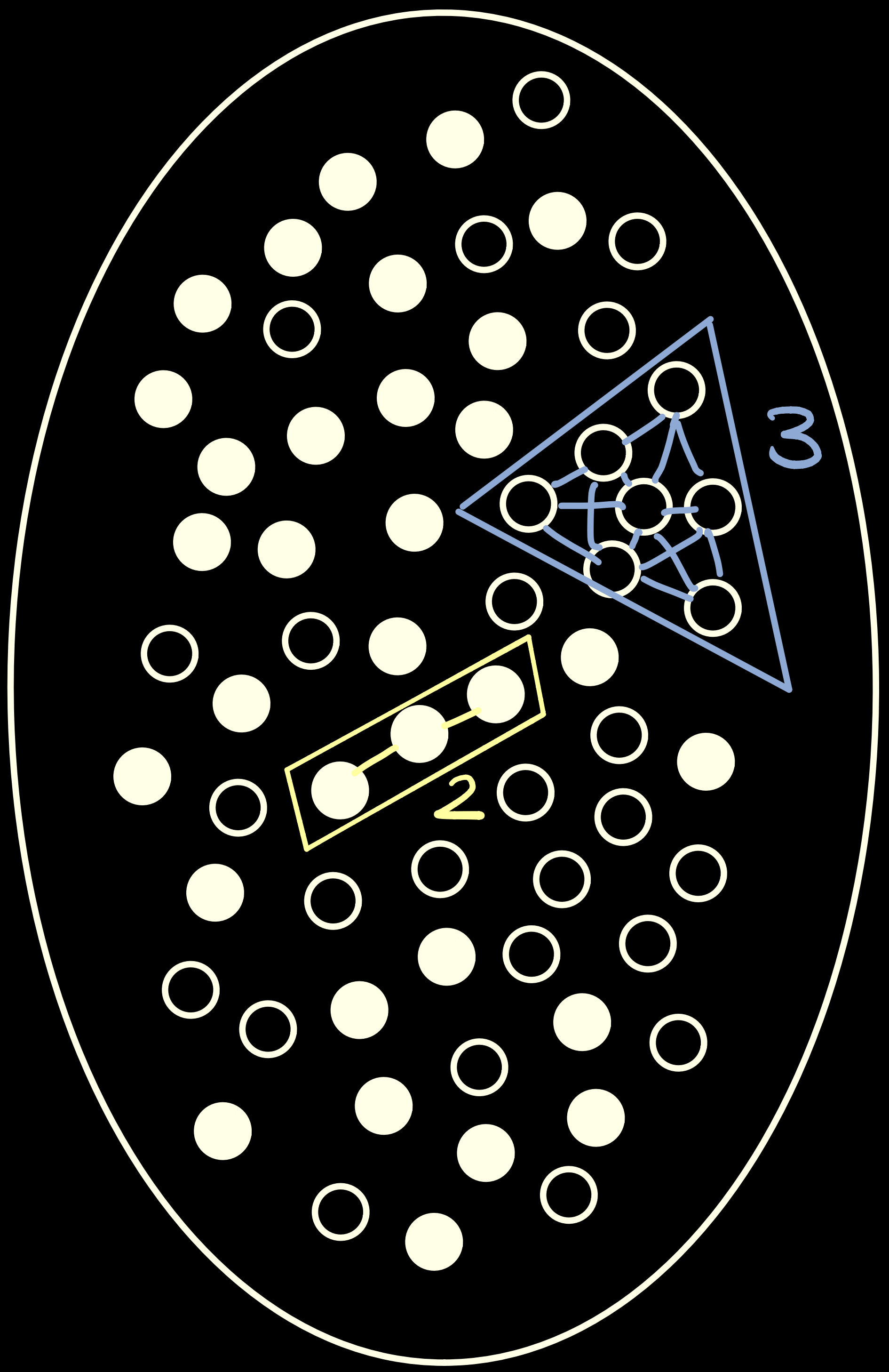
$\chi(G)$ = minimum number of colours in a proper vertex-colouring

$$\chi(G) \geq \omega(G) = \alpha(G^c)$$

MATROID PARAMETERS

$\omega(M)$ = dimension of largest PG-submatroid.

$\alpha(M) = \omega(M^c)$



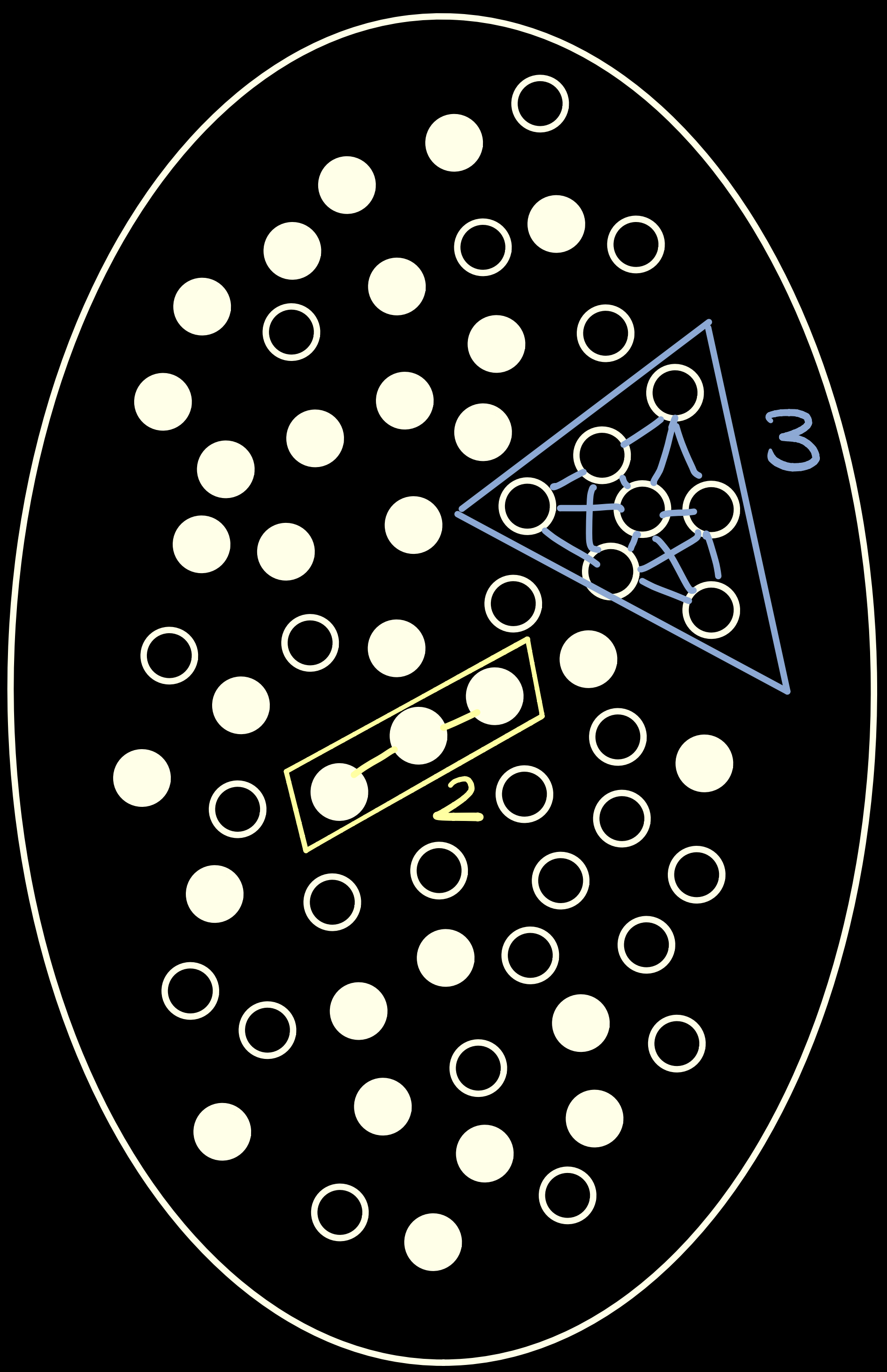
MATROID PARAMETERS

$\omega(M)$ = dimension of largest PG-submatroid.

$\alpha(M) = \omega(M^c)$

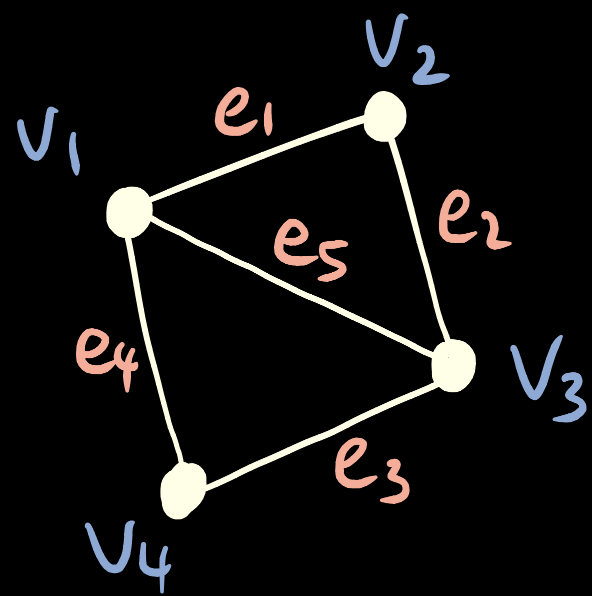
$\chi(M) = \dim(M) - \alpha(M)$

CRITICAL NUMBER



$\chi(M) \geq \omega(M)$

Graphs



G

$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix}1\\1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\\1\end{pmatrix} & \begin{pmatrix}1\\0\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \end{array} \subseteq \mathbb{F}_2^{V}$$

$M(G)$

- circuits $\longleftrightarrow$ circuits
  trees $\longleftrightarrow$ I_t
- $\omega(M(G)) \leq 2$
- $\chi(M(G)) = \lceil \log_2 \chi(G) \rceil$

Extremal Functions

<u>Theorem</u> (Erdős-Stone-Simonovits '46) If $G=(V,E)$ has no H-subgraph, then $|E| \leq \left(1-\frac{1}{\chi(H)-1}+o(1)\right)\binom{|V|}{2}$.

<u>Theorem</u> (Geelen, N. '12) If $M=(E,G)$ has no N-submatroid, then $|E| \leq \left(1-\frac{1}{2^{\chi(N)-1}}+o(1)\right)|G|$.

Brooks' Theorem

Theorem (Brooks '41) For connected G, $\chi(G) \leq \Delta(G)+1$ and $\chi(G) \leq \Delta(G)$ unless $G \cong C_{2n+1}$ or K_n.

Theorem (Oxley '16) For every connected matroid M, $\chi(M) \leq \lceil \log_2(\Delta(M)+1) \rceil$, and $\chi(M) \leq \lceil \log_2(\Delta(M)) \rceil$ unless $M \cong C_{2n+1}$ or PG_n.

RAMSEY THEORY

Ramsey Theorem (GR): If $n \geq R(k,\ell)$, then every red-blue colouring of PG_n contains a red PG_k or a blue PG_ℓ.

How does $R(k,\ell)$ behave?

RAMSEY THEORY

<u>Ramsey Theorem (GR)</u> : If $n \geq R(k, \ell)$, then every red-blue colouring of PG_n contains a red PG_k or a blue PG_ℓ.

How does $R(k, \ell)$ behave?

<u>Conjecture</u> : If $E \subseteq \mathbb{F}_2^n$ and $x+y+z \neq 0$ for all $x, y, z \in E$, then $\mathbb{F}_2^n \setminus E$ contains a subspace of dimension $\frac{n}{2}$.

(ie. $R(k, 2) \leq 2k$)

LARGE χ

<u>Theorem</u> (Rodl '77): If $\chi(G) \geq f(t)$, then G has a triangle-free subgraph H with $\chi(H) \geq t$.

Large χ

Theorem (Rodl '77): If $\chi(G) \geq f(t)$, then G has a triangle-free subgraph H with $\chi(H) \geq t$.

Conjecture (Geelen '19): If $\chi(M) \geq f(t)$, then M has a triangle-free submatroid N with $\chi(N) \geq t$.

Bounding χ?

Recall that $\chi(M) \geq \omega(M)$

Can we bound χ above by a function of ω?

<u>No</u>! There exist M with $\omega(M) = 1$ and χ large.

Bounding χ ?

Recall that $\chi(M) \geq \omega(M)$

Can we bound χ above by a function of ω ?

No! There exist M with $\omega(M) = 1$ and χ large.

Conjecture (Gyarfas-Sumner) : For every tree T, if G has no induced T-subgraph, then

$$\chi(G) \leq f_T(\omega(G))$$

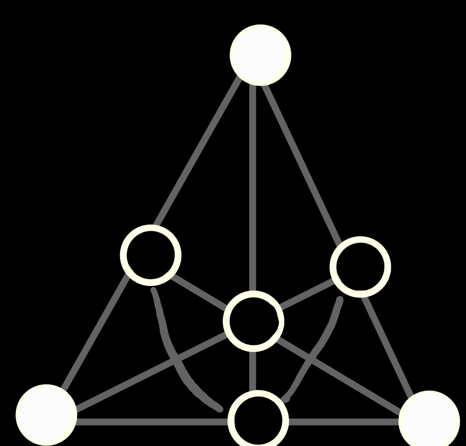
Matroid Gyarfas - Sumner : For every 'tree' I_t, if M has no induced I_t-submatroid, then

$$\chi(M) \leq f_t(\omega(M)).$$

~~Matroid Gyarfas - Sumner: For every 'tree' I_t, if M has no induced I_t-submatroid, then~~

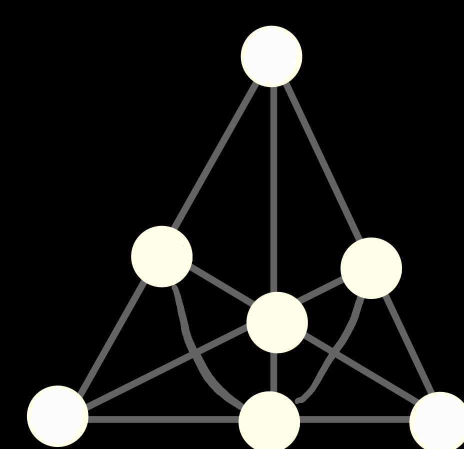
$$\cancel{\chi(M) \leq f_t(\omega(M))}.$$

False! There exist M with no induced



I_3

or



PG_3

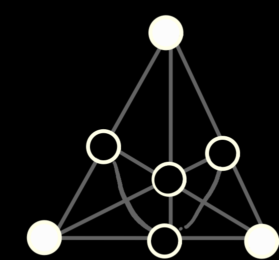
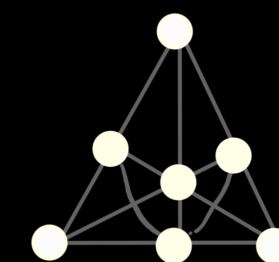
-submatroid,

but with $\chi(M)$ arbitrarily large.

(Bonamy, Kardos, Kelly, N., Postle)

Even Plane Matroids

$\mathcal{E}_3 = \{ M :$ every 3-dimensional induced restriction of M has even size $\}$

Matroids in $\mathcal{E}_3$ are  -free,  -free

I_3 $\qquad$ PG_3

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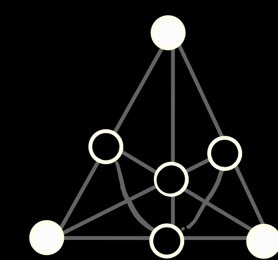
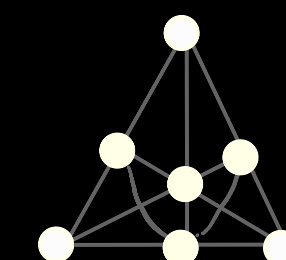
Matroids in $\mathcal{E}_3$ are  -free,  -free

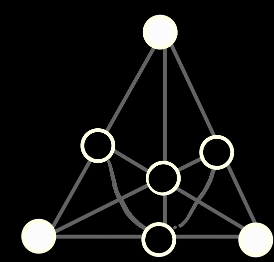
I_3 PG_3

Prop: $M \in \mathcal{E}_3$ iff there is a polynomial $p : \mathbb{F}_2^n \to \mathbb{F}_2$ of degree ≤ 2 so that

$$E = \{x \in \mathbb{F}_2^n : p(x) = 1\}$$

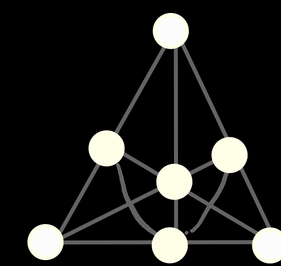
Prop: (Norine) If $p(x) = x^T Q x$, then $\chi(M) \geq \frac{1}{2} \mathrm{rank}(Q)$

Thm (Bonamy, Kardos, Kelly, N., Postle '18): If M is induced-



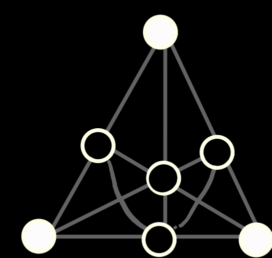
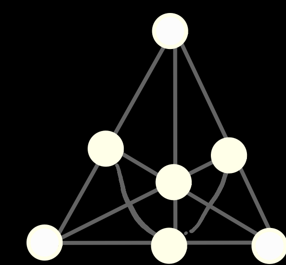
I_3

-free,



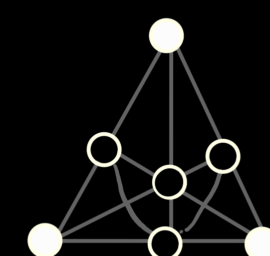
PG_3

-free and $\chi(M) > 2$, then $M \in \mathcal{E}_3$.

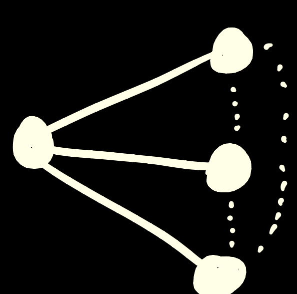
Thm (Bonamy, Kardos, Kelly, N., Postle '18): If M is induced -  -free,  -free and

I_3 PG_3

$\chi(M) > 2$, then $M \in \mathcal{E}_3$.

What is the structure of  -free matroids?

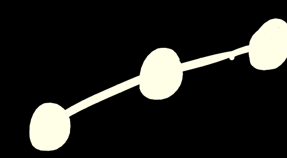
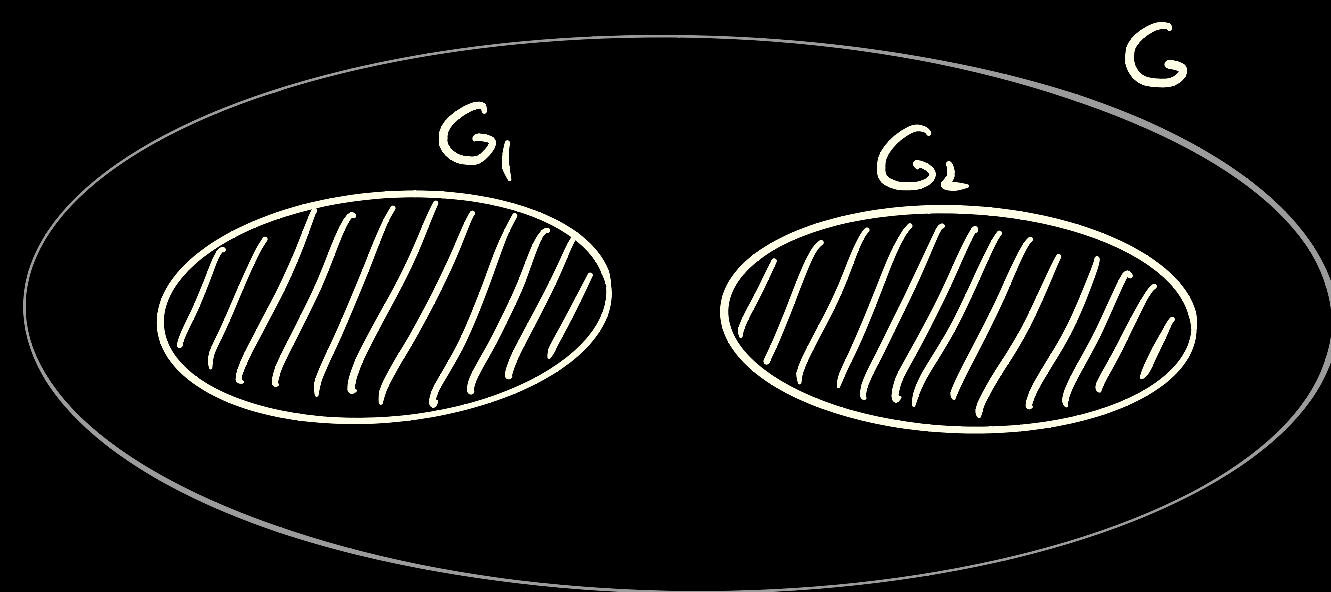
CLAW

 -free graphs have a precise structure

CLAW

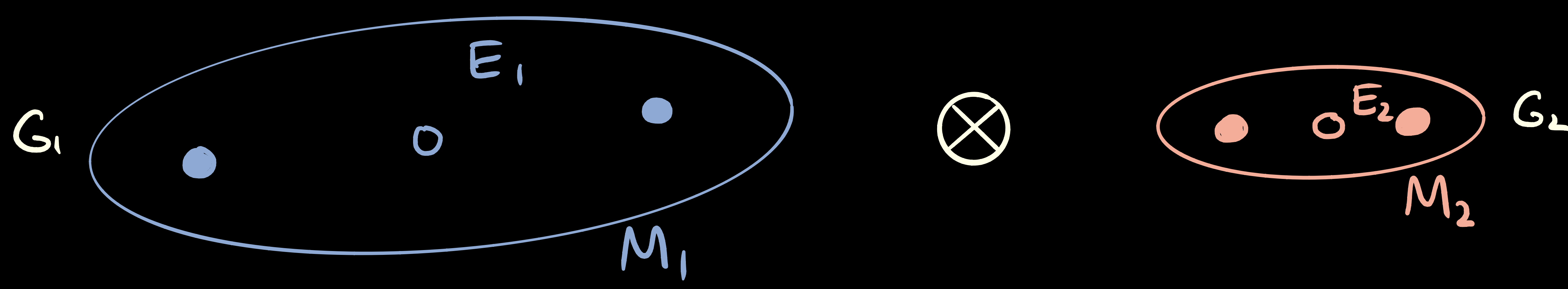
(Chudnovsky, Seymour)

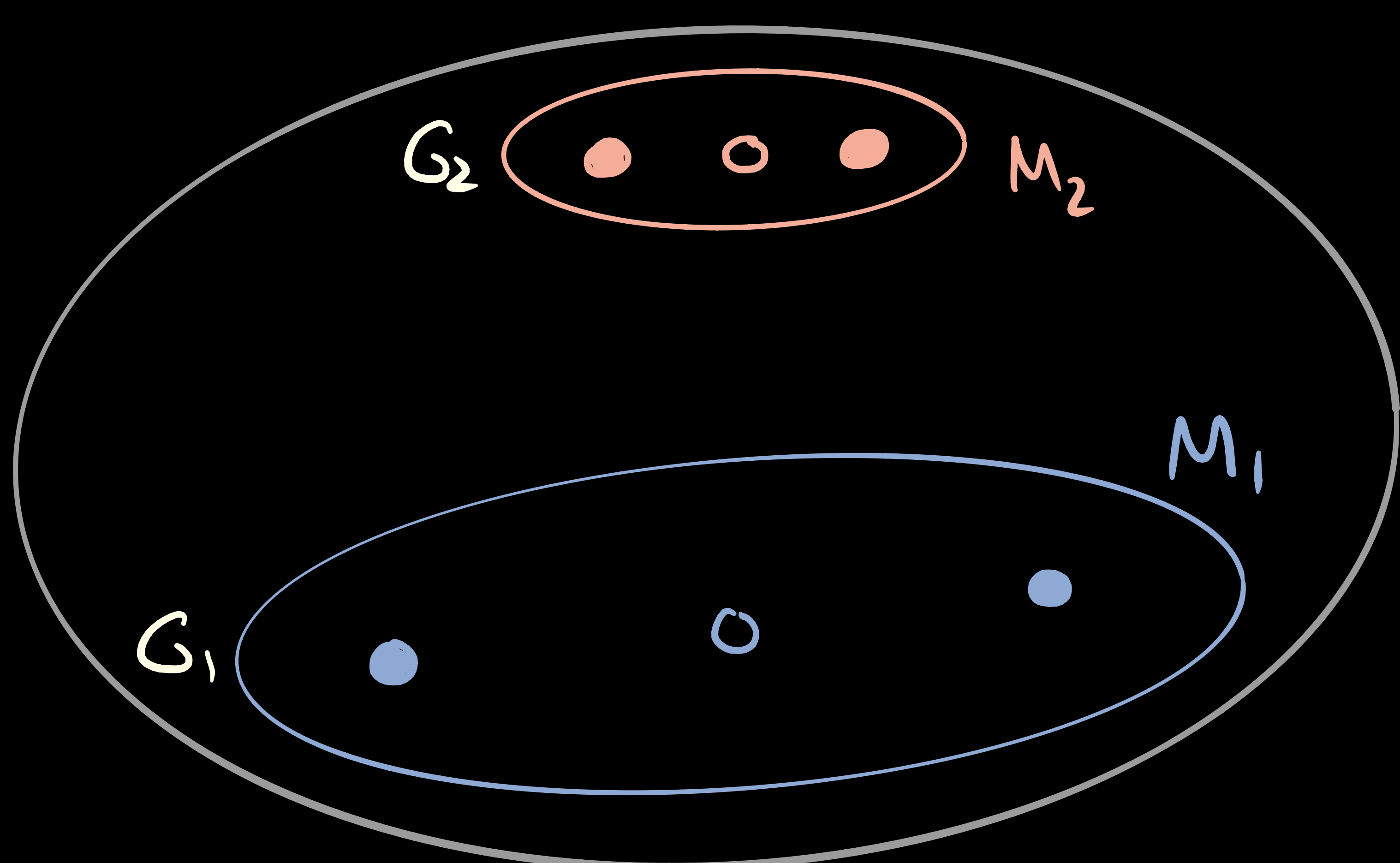
Classes of claw-free matroids

- Even plane matroids
- Complements of  -free matroids
- PG-sums 

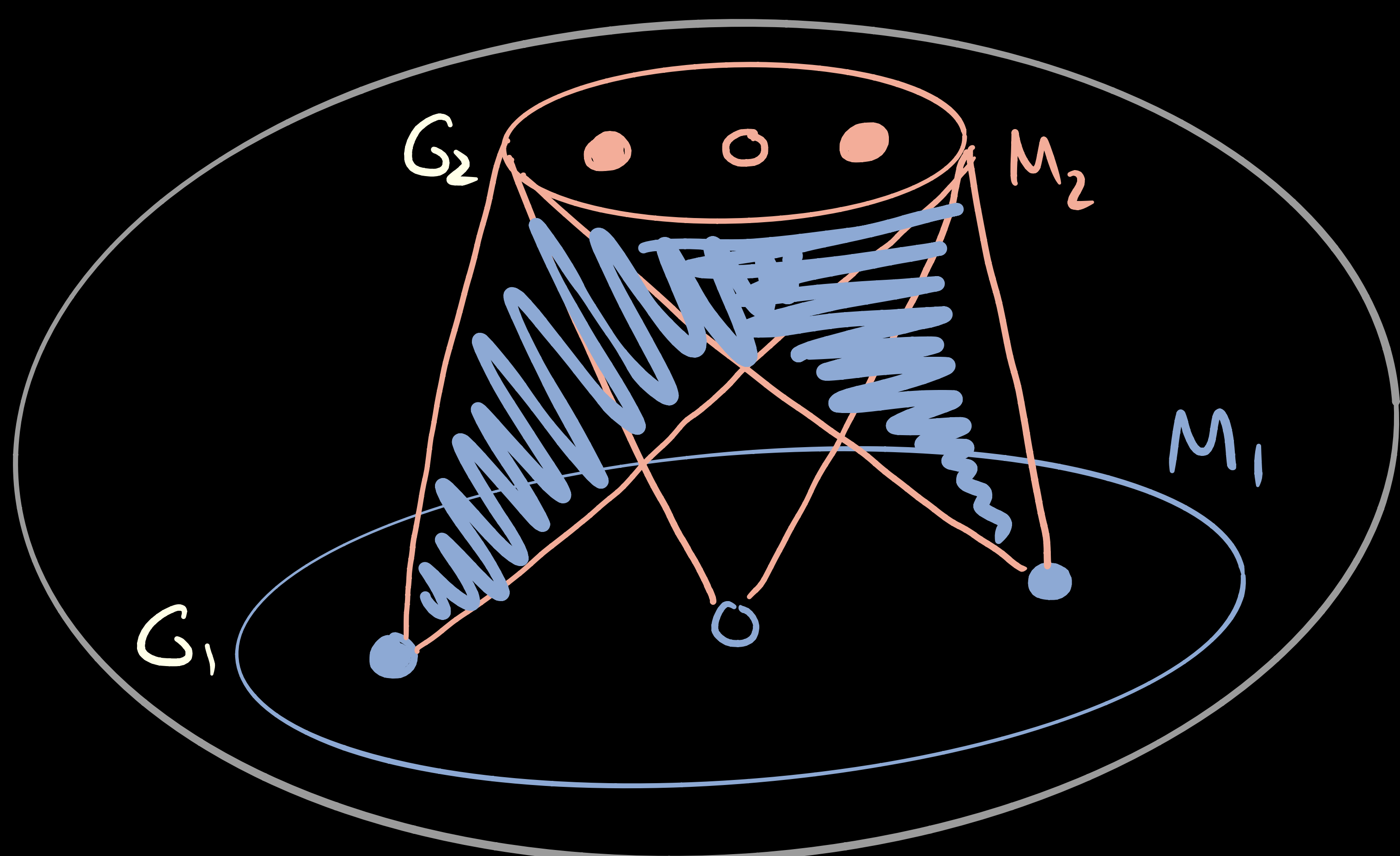
LIFT-JOINS

LIFT-JOINS





Lift-Joins



M_1, M_2 claw-free

$\Rightarrow$ $M_1 \otimes M_2$ claw-free.

Theorem (N., Nomoto): A matroid M is claw-free if and only if M can be obtained via lift-joins from matroids that are either

- even-plane,
- the complement of triangle-free,

or
- PG-sums.

Theorem (N., Nomoto): A matroid M is claw-free if and only if M can be obtained via lift-joins from matroids that are either

- even-plane,
- the complement of triangle-free,
- or PG-sums.

Pf: A minimum counterexample must have an induced submatroid N so that $\dim(N) = \dim(M) - 1$ and N decomposes as a lift-join.

Show that this lift-join extends to decompose M.

Recall: (I_3, PG_3)-free matroids can have arbitrarily large χ.

Prop: $(I_3, \text{triangle})$-free matroids have $\chi \leq 1$

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Prop: $(I_3, \text{triangle})$-free matroids have $\chi \le 1$

Theorem (N., Nomoto): $(I_4, \text{triangle})$-free matroids have $\chi \le 2$.

Conjecture: $(I_t, \text{triangle})$-free matroids have $\chi \le f(t)$.

Recall: I_3-free matroids with $\omega \leq 2$ can have χ arbitrarily large.

$\chi(M)$ = min t so that there exist linear functionals $p_1, \cdots, p_t$ for which

$$E \subseteq \bigcup_{i=1}^{t} \{x : p_t(x) = 1\}$$

Recall: I_3-free matroids with $\omega \leq 2$ can have χ arbitrarily large.

$\chi(M)$ = min t so that there exist linear functionals $p_1, \cdots, p_t$ for which

$$E \subseteq \bigcup_{i=1}^{t} \{x : p_t(x) = 1\}$$

Prop: If M is claw-free and $\omega(M) \leq t$, then there are quadratic polynomials $p_1, \cdots, p_{f(t)}$ so that $p_i(0) = 0$ while

$$E \subseteq \bigcup_{i=1}^{f(t)} \{x : p_t(x) = 1\}$$

Conjecture (N., Norine): If M has no induced I_t-restriction and $\omega(M) \leq s$, then there is a polynomial p of degree at most $f(s,t)$ so that $p(0) = 0$ but $p(x) = 1$ for all $x \in E$.

THANK YOU!