On Representations Subordinate to Topologically Introverted Spaces

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Joint Work with M. Filali and M. Neufang

• Given a Banach algebra A, the dual space A^* can be viewed as a Banach A-bimodule with the canonical operations:

$$\langle \lambda \cdot a, b \rangle = \langle \lambda, ab \rangle, \quad \langle a \cdot \lambda, b \rangle = \langle \lambda, ba \rangle,$$

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where $\lambda \in A^*$ and $a, b \in A$.

• Let X be a norm closed A-submodule of A^*. For $\Psi \in X^*$ and $\lambda \in X$, define $\Psi \cdot \lambda \in A^*$ by

$$\langle \Psi \cdot \lambda, a \rangle = \langle \Psi, \lambda \cdot a \rangle.$$

If $\Psi \cdot \lambda \in X$ for all choices of $\Psi \in X^*$ and $\lambda \in X$, then X is called an introverted subspace of A^*. Lau and Loy (1997); Dales and Lau (2005) .

• The dual of an introverted subspace of A^* is a Banach algebra: if $\Phi, \Psi \in X^*$ we define $\Phi \Box \Psi \in X^*$ by

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If G is a locally compact group, then $LUC(L^1(G)) = LUC(G)$.

(iii) $X = WAP(A) = \{\ \lambda \in A^* :$ the linear map

$$a \mapsto \lambda \cdot a,\ A \longrightarrow A^*,\ \text{is weakly compact}\}.$$

• For more examples see: Granirer (1987), Dales (2000), Dales–Lau (2005).

Subordination

• Let E be a dual Banach space and $\mathscr{L}(E)$ be equipped with the W*OT. Given a continuous representation

$$\pi : A \longrightarrow \mathscr{L}(E),$$

and given $y \in E, \lambda \in E_*$, we define $\pi_{y,\lambda} \in A^*$ by

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- If $X \subset A^*$ is introverted, we say π is subordinate to X if

$$\pi_{y,\lambda} \in X \qquad (y \in E, \lambda \in E_*).$$

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(ii) If $T \in \mathscr{L}(H)$, $T_* = \sum_{i=1}^{\infty} x_i \otimes y_i \in H\widehat{\otimes}_\gamma H = \mathscr{L}(H)_*$, then for almost all $t \in G$:

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(iii) $\pi_{T,T^*} \in LUC(G)$, hence π subordinate to $LUC(L^1(G))$.

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A Little Digression...

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• Question: Is every $\lambda \in WAP(A)$ a coordinate function of some continuous representation of A on a reflexive Banach space?

Yes, if A has a bounded approximate identity.

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In other words, there exists a norm $\|\cdot\|_\mu$ on

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such that the completion of $A \cdot \mu$ is a reflexive space E_μ, and the left module action of A on E_μ induces a w^*-continuous representation

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Moreover, there exists $x \in E_\mu$ and $\lambda \in E_\mu^*$ such that

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• In 2003, Megrelishvili proved a representation theorem for WAP-functions associated to a semitopological flow (S, X).

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in which $\Psi \in X^*$, $y \in E$, $f \in E^*$, is a w^*-continuous representation of X^*.

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(v) If $\widetilde{\pi}$ is irreducible then so is π, the converse holds if X is faithful.

π-invarience

- Let a representation

$$\pi : A \longrightarrow \mathscr{L}(H)$$

be subordinate to an introverted space $X \subset A^*$. Suppose that in a suitable orthonormal basis $(e_i)_{i \in I}$ of H, we have

$$C_j = \sum_{i \in I} \|\pi_{ij}\|_{A^*} < \infty \quad \text{and} \quad C = \sup_{j \in I} C_j < \infty. \qquad (*)$$

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Then for an $\overline{\Psi} \in \ell^\infty(I, X^*)$ the following are equivalent:

(i) $a \cdot \overline{\Psi} = {}^t\pi(a)\overline{\Psi}$, for all $a \in A$;

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- We call such an element $\overline{\Psi} \in \ell^\infty(I, X^*)$ to be π-invariant.

Application to ideal theory

- Theorem: Using the preceeding notation, if

$$\overline{\Psi} \in \ell^{\infty}(I, X^{*})$$

is a non-zero, π-invariant element, then

$$M := \overline{\mathrm{lin}\,\{\overline{\Psi}(i) \colon i \in I\}}^{\|\cdot\|}$$

is a closed left ideal of $(X^{*}, \Box)$.

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- The converse of the above theorem holds for finite dimensional left ideals of $(X^*, \Box)$

A Cohomological Property

- Kaniuth, Lau, and Pym (2008) have shown that if

$$\varphi\colon A \longrightarrow \mathbb{C}$$

is a non-zero character, then existence of a φ-mean in A^{**}, that is, an element $\Psi \in A^{**}$ such that

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- Since a representation $\pi\colon A \longrightarrow \mathscr{L}(H)$ can be interpreted as a generalized character, a question arises of whether an analogous connection exists if the character φ is replaced by a representation π.

• Let $\pi : A \longrightarrow \mathscr{L}(H)$ be a representation satisfying $(*)$. Let E is a Banach right A-module, and I be a set with $|I| = \dim(H)$.

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Given $\overline{x} \in \ell^1(I, E)$, we may define:

$$(a \cdot \overline{x})(i) = (\pi(a)\overline{x})(i), \tag{1}$$

$$(\overline{x} \cdot a)(i) = \overline{x}(i) \cdot a, \tag{2}$$

where $a \in A,\ i \in I$. This turns $\ell^1(I, E)$ into a Banach A-bimodule.

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• We can link the existence of π-invariant elements in $\ell^\infty(I, A^{**})$ to the study of derivations

$$d \colon A \longrightarrow \ell^1(I, E)^*.$$

• Theorem: Let $\pi\colon A \longrightarrow \mathscr{L}(H)$ be a continuous representation satisfying the condition $(*)$ as well as the strong Hahn–Banach separation property on a column j, for some $j \in I$.

Suppose that for every Banach right A-module E, every continuous derivation $d\colon A \longrightarrow \ell^1(I, E)^*$ is inner. In that case, there exists a π-invariant element $\overline{\Phi}_j \in \ell^\infty(I, A^{**})$ such that:

$$\langle \overline{\Phi}_j(i), \pi_{kj} \rangle = \delta_{ik} \quad (i, k \in I).$$

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• We can show the following converse result as well.

• Theorem: Suppose that $\pi \colon A \longrightarrow \mathscr{L}(H)$ satisfies $(*)$, and for each $j \in I$, there exists a π-invariant element $\overline{\Phi}_j \in \ell^\infty(I, A^{**})$ such that

(i) $\sup_j \|\overline{\Phi}_j\|_\infty < \infty$;

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If E is any Banach right A-module and $\ell^1(I, E)$ is equipped with the Banach A-bimodule structure defined in (1)–(2), then every continuous derivation $d = (d_i)_{i \in I} \colon A \longrightarrow \ell^1(I, E)^*$ is inner, provided that

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• In the special case that $\pi\colon A \longrightarrow \mathbb{C}$ is a character, the above theorem and its converse recover with the main result of Kaniuth, Lau, and Pym (2008).

Thank you for your attention.