Proof of the Lebesgue Differentiation Theorem

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Lebesgue's Differentiation Theorem:

Theorem [Lebesgue's Differentiation Theorem]: Let $f : [a, b] \to \mathbb{R}$ be increasing. Then f is differentiable almost everywhere on $[a, b]$, f' is measureable, integrable and

$$\int_{[a,b]} f' dm \leq f(b) - f(a).$$

Corollary: If f is of bounded variation on $[a, b]$, then it is differentiable almost everywhere. In particular, if f is absolutely continuous on $[a, b]$, then it is differentiable almost everywhere.

Lebesgue's Differentiation Theorem:

Proof: Consider $E = \{x \in [a,b] \,|\, D^+f(x) > D_-f(x)\}$. Let

$$E_{r,s} = \{x \in [a,b] \,|\, D^+f(x) > r > s > D_-f(x)\},$$

for $r, s \in \mathbb{Q}$.
Then

$$E = \bigcup_{r,s \in \mathbb{Q}} E_{r,s}.$$

Let $\alpha = m^*(E_{r,s})$ and $\epsilon > 0$. Choose U open such that $m(U) < \alpha + \epsilon$ and $E_{r,s} \subseteq U$. For each $x \in E_{r,s}$, there is an arbitrarily small $h > 0$ such that $[x-h, x] \subseteq U$ and

$$\frac{f(x) - f(x-h)}{h} < s.$$

By Vitali's Lemma, there are finitely may disjoint intervals $I_1, \ldots, I_N$ of this type such that the interiors of the I_n's cover a subset A of $E_{r,s}$ with $m^*(A) > \alpha - \epsilon$.

Differentiation of Monotone Functions:

Proof Cont'd:
If $I_n = [x_n - h_n, x_n]$ for each $n = 1, 2, \cdots, N$, then

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] \leq s \sum_{n=1}^{N} h_n < s \cdot m(U) < s(\alpha + \epsilon)$$

Now for each point $y \in A$, there is an arbitrarily small interval $[y, y + k]$ that is contained in some I_n and is such that

$$\frac{f(y + k) - f(y)}{k} > r.$$

Therefore

$$f(y + k) - f(y) > rk.$$

Appying Vitali's Lemma again, we get intervals $J_1, \ldots, J_M$ disjoint and of the type $[y, y + k] \subseteq I_n$ such that $\bigcup_{i=1}^{M} J_i$ contains a subset B of A with outer measure at least $\alpha - 2\epsilon$.

Proof Cont'd: Hence

$$\sum_{i=1}^{M}[f(y_i+k_i)-f(y_i)] > r\sum_{i=1}^{M}k_i \geq r(\alpha-2\epsilon).$$

Since $J_i \subseteq I_n$ and f is increasing,

$$\sum_{J_i\subseteq I_n}[f(y_i+k_i)-f(y_i)] \leq f(x_n)-f(x_n-h_n).$$

It follows that

$$r(\alpha-2\epsilon) \leq \sum_{i=1}^{M}[f(y_i+k_i)-f(y_i)] \leq \sum_{n=1}^{N}[f(x_n)-f(x_n-h_n)] \leq s(\alpha-\epsilon)$$

Since $\epsilon > 0$ is arbitrary, $r\alpha \leq s\alpha$. Since $r > s$, this implies that $\alpha = 0$. Therefore $m^*(E_{r,s}) = 0$, so $m(E_{r,s}) = 0$ and so $m(E) = 0$.

Differentiation of Monotone Functions:

Proof Cont'd: Similarly, if

$$E_1 = \{x \in [a, b] \mid D^- f(x) > D_+ f(x)\}$$

then $m(E_1) = 0$. From this we can deduce that

$$D^+ f(x) = D^- f(x) = D_+ f(x) = D_- f(x)$$

almost everywhere. Therefore

$$g(x) = \lim_{h \to 0} \frac{f(x) - f(x+h)}{h}$$

exists as an extended real number almost everywhere on $[a, b]$.

Differentiation of Monotone Functions:

Proof Cont'd: Let

$$g_n(x) = n[f(x + \frac{1}{n}) - f(x)],$$

where $f(x) = f(b)$ for $x \geq b$. Then $g_n \to g$ almost everywhere on $[a, b]$ and since f is increasing, $g_n \geq 0$. Hence $g \geq 0$ and g is measurable. By Fatou's Lemma,

$$\begin{aligned}
\int_{[a,b]} g\, dm &\leq \liminf_n \int_{[a,b]} g_n\, dm \\
&= \liminf_n\ n \int_{[a,b]} [f(x + \frac{1}{n}) - f(x)]\, dx \\
&= \liminf_n \left[n \int_{[b,b+\frac{1}{n}]} f\, dm - n \int_{[a,a+\frac{1}{n}]} f\, dm \right] \\
&= \liminf_n \left[f(b) - n \int_{[a,a+\frac{1}{n}]} f\, dm \right] \\
&\leq f(b) - f(a)
\end{aligned}$$

Therefore, g is integrable. Hence $g(x)$ is finite almost everywhere so $f(x)$ is differentiable almost everywhere with $f'(x) = g(x)$.

Finally,

$$\int_{[a,b]} f' \, dm = \int_{[a,b]} g \, dm \leq f(b) - f(a).$$