PACKING T-JOINS AND EDGE COLOURING IN PLANAR GRAPHS

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ABSTRACT. Let $\mathcal{G}$ be a planar graph and let T be a subset of vertices of $\mathcal{G}$ of even cardinality. Suppose that there exists a T-cut of $\mathcal{G}$ of cardinality at most five and that the parity of the cardinality of every T-cut is the same. We show that in that case the cardinality of the smallest T-cut is equal to the maximum number of pairwise disjoint T-joins. As a corollary we obtain that for $k \in \{4, 5\}$, a k-regular planar graph has chromatic index k if and only if for every subset of vertices X of odd cardinality there are at least k edges with exactly one end in X. The case where $k = 4$ was conjectured by Seymour in 1979.

1. INTRODUCTION

In this paper graphs will be allowed parallel edges but will be loopless. A *cut* of a graph $\mathcal{G}$ is a set of edges $\delta_{\mathcal{G}}(U) := \{uv \in E(\mathcal{G}) : u \in U, v \notin U\}$ where $U \neq \emptyset, U \neq V(\mathcal{G})$. The cardinality of $\delta_{\mathcal{G}}(U)$ is denoted $d_{\mathcal{G}}(U)$. Let v be a vertex of $\mathcal{G}$, we write $\delta_{\mathcal{G}}(v)$ and $d_{\mathcal{G}}(v)$ for $\delta_{\mathcal{G}}(\{v\})$ and $d_{\mathcal{G}}(\{v\})$ respectively. In cases when there is no ambiguity we shall omit the index $\mathcal{G}$. Thus $d(v)$ denotes the degree of v.

A *graft* is a pair $(\mathcal{G}, T)$ where $\mathcal{G}$ is a graph and T a subset of vertices of even cardinality. A T*-cut* is a cut $\delta(U)$ where $|U \cap T|$ is odd. A T*-join* is a set of edges B which has the property that T is the set of vertices of odd degree of $\mathcal{G}[B]$ (the graph induced by B). Note that we do not require T-cuts and T-joins to be inclusion-wise minimal. We say that $(\mathcal{G}, T)$ is a *postman set* if $E(\mathcal{G})$ is a T-join.

The cardinality of the smallest T-cut is denoted $\tau(\mathcal{G}, T)$. We call a collection of pairwise disjoint T-joins a *packing* (of T-joins). The cardinality of the largest packing is denoted $\nu(\mathcal{G}, T)$. The following observation is easy and well known,

Remark 1.1. Let $\delta(U)$ be a T-cut and let B be a T-join then $|\delta(U) \cap B|$ is odd.

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The last remark implies in particular that $\tau(\mathcal{G},T) \geq \nu(\mathcal{G},T)$. The graft $(\mathcal{G},T)$ *packs* when equality holds. The *parity* of a T-cut $\delta(U)$ is the parity of $d(U)$. Next we characterize grafts which have the property that all T-cuts have the same parity.

Proposition 1.2. *The following statements are equivalent for a graft $(\mathcal{G},T)$ where $T \neq \emptyset$.*

(1) *all T-cuts have the same parity,*
(2) *$\mathcal{G}$ is Eulerian or $(\mathcal{G},T)$ is a postman set.*

Proof. Suppose that (2) holds. If $\mathcal{G}$ is Eulerian then all cuts are even, in particular so are all T-cuts. If $(\mathcal{G},T)$ is a postman set then for every T-cut $\delta(U)$ we have $d(U) = |\delta(U) \cap E(\mathcal{G})|$ which is odd because of Remark 1.1 and the fact that $E(\mathcal{G})$ is a T-join. Suppose that (1) holds. Since $T \neq \emptyset$ there exists a T-cut, say $\delta(U)$. Consider a vertex $v \notin T$. Then $\delta(U \triangle \{v\})$ is a T-cut. Note that $\delta(v) = \delta(U \triangle U \triangle \{v\}) = \delta(U) \triangle \delta(U \triangle \{v\})$. Since $\delta(U), \delta(U \triangle \{v\})$ have the same parity, $d(v)$ is even. Thus vertices not in T have even degree. We may assume that $\mathcal{G}$ has vertex w of odd degree for otherwise $\mathcal{G}$ is Eulerian. Then $w \in T$. To complete the proof it suffices to show that if a vertex $v \in T$ then v has odd degree. Since $\delta(U)$ is a T-cut so is $\delta(U \triangle \{v\} \triangle \{w\})$. Note that $\delta(v) \triangle \delta(w) = \delta(\{v\} \triangle \{w\}) = \delta(U \triangle U \triangle \{v\} \triangle \{w\}) = \delta(U) \triangle \delta(U \triangle \{v\} \triangle \{w\})$. Because $\delta(U), \delta(U \triangle \{v\} \triangle \{w\})$ have the same parity and $d(w)$ is odd, so is $d(v)$. □

The following is the main result of the paper (which will be proved in sections 2, 3, and 4).

Theorem 1.3. *Let $(\mathcal{G},T)$ be a graft where all T-cuts have the same parity. Then $(\mathcal{G},T)$ packs if $\mathcal{G}$ is planar and $\tau(\mathcal{G},T) \leq 5$.*

The condition that all T-cuts have the same parity cannot be omitted in the hypothesis of the last theorem. Indeed consider the bipartite graph $K_{2,3}$ and let T consist of all vertices of $K_{2,3}$ with the exception of one of the vertices of degree 3. Then $2 = \tau(K_{2,3},T) > \nu(K_{2,3},T) = 1$ and $(K_{2,3},T)$ does not pack.

Edge colouring. We write $[n]$ for $\{1,\ldots,n\}$. We say that $\theta : E(\mathcal{G}) \to [k]$ is a *colouring* of $\mathcal{G}$ with k colours if for every pair of edges e, e' incident to the same vertex we have $\theta(e) \neq \theta(e')$. The minimum number of colours $\chi'(\mathcal{G})$ needed colour the edges of $\mathcal{G}$ is the *chromatic index* of $\mathcal{G}$.

The condition that $\mathcal{G}$ be planar is also required in Theorem 1.3. Indeed let $\mathcal{G}$ denote the Petersen graph. $\mathcal{G}$ is 3-regular and every cut of $\mathcal{G}$ contains at least 3-edges. Hence $\tau(\mathcal{G}, V(\mathcal{G})) = 3$. Suppose for a contradiction that $(\mathcal{G}, V(\mathcal{G}))$ packs. Then there exists 3 disjoint T-joins. Since $\delta(v)$ is a T-cut for every vertex v it follows that each of the T-joins is a perfect matching. But this implies that $\chi'(\mathcal{G}) = 3$, a contradiction as it is well known that $\chi'(\mathcal{G}) = 4$.

Corollary 1.4. *Let $\mathcal{G}$ be a k-regular planar graph where $k \leq 5$. Then $\chi'(\mathcal{G}) = k$ if and only if for all $U \subseteq V(\mathcal{G})$ where $|U|$ is odd, we have $d(U) \geq k$.*

Proof. Suppose that $\chi'(\mathcal{G}) = k$ and let $J_1, \ldots, J_k$ denote each of the colour classes. Note that each colour class is a perfect matching. It follows that for $i \in [k]$ and for all $U \subseteq V(\mathcal{G})$ where $|U|$ is odd, we have $J_i \cap \delta(U) \neq \emptyset$. Since $J_1, \ldots, J_k$ are pairwise disjoint it follows that $d(U) \geq k$. Conversely, suppose that for all $U \subseteq V(\mathcal{G})$ where $|U|$ is odd, we have $d(U) \geq k$. Then $|V(\mathcal{G})|$ is even and let $T := V(\mathcal{G})$. Let $\delta(U)$ be any T-cut. Then $|U \cap T| = |U|$ and $|U|$ is odd. Hence $d(U) \geq k$. It follows that $\tau(\mathcal{G}, T) \geq k$. If k is even then $\mathcal{G}$ is Eulerian. If k is odd then $(\mathcal{G}, T)$ is a postman set. Proposition 1.2 implies that all T-cuts have the same parity. Theorem 1.3 imply that there exists a packing of k T-joins, say $J_1, \ldots, J_k$. Since $T = V(\mathcal{G})$, $\delta(v)$ is a T-cut for every $v \in V(\mathcal{G})$. Since $\mathcal{G}$ is k-regular, $J_1, \ldots, J_k$ must be perfect matchings. Let each matching correspond to a colour class, then $\chi'(\mathcal{G}) = k$. □

Remark 1.5. The cases $k = 0, 1, 2$ are trivial. The case $k = 3$ states that every bridgeless cubic planar graph has chromatic index three. By a result of Tait [9] this is equivalent to the 4-colour theorem (which states that any map can be coloured using four colours so that adjacent countries get different colours).

The case $k = 4$ was conjectured by Seymour [7] (see also [4] problem 12.18). Seymour (personal communication) pointed out that it implies the following strengthening of the four colour theorem. (Note that we rely on the 4-colour theorem for the proof of Theorem 1.3.)

Corollary 1.6. *Let $\mathcal{G}$ be a bridgeless plane graph. Then we can colour the vertices and the faces of $\mathcal{G}$ using four colours such that for every edge uv:*

(1) *vertices u and v are assigned distinct colours,*

(2) *the two faces F_1, F_2 which share edge uv are assigned distinct colours,*

(3) *exactly three colours are used to colour $\{u, v, F_1, F_2\}$.*

Proof. We construct the medial graph $\mathcal{H}$ of $\mathcal{G}$, i.e. vertices of $\mathcal{H}$ correspond to the edges of $\mathcal{G}$ and two vertices of $\mathcal{H}$ are adjacent if and only if they correspond to consecutive edges in some face of $\mathcal{G}$. Since $\mathcal{G}$ is bridgeless, $\mathcal{H}$ is simple and 4-regular, moreover, it can be readily checked that for every $X \subseteq V(G)$ where $|X|$ is odd, $d(X) \geq 4$. It follows from corollary 1.4 that $\chi'(H) = 4$. Choose the colours to be the elements, say $\alpha, \beta, \gamma, \delta$, of the $Z_2 \times Z_2$ group. Then for every $v \in V(H)$ the sum of the colours over the edges of $\delta_H(v)$ is $\alpha + \beta + \gamma + \delta = (0, 0)$. It follows that for the dual $\mathcal{H}^*$ of $\mathcal{H}$ the sum of the colours over the edges of every facial circuit is $(0, 0)$. Since for plane graphs the cycle space is generated by the facial circuits, the sum of the colours over the edges of any circuit of $\mathcal{H}^*$ is $(0, 0)$. Therefore, the following assignment of colours to the vertices of $\mathcal{H}^*$ is well defined: give colour $(0, 0)$ to some initial vertex z_o and for all vertices z find a path P from z_o to z and assign to z the colour which is equal to the sum of all colours along the edges of the path P. The graph $\mathcal{H}$ can be drawn on the plane so that each vertex is placed in the middle of its corresponding edge of $\mathcal{G}$ and so that faces of $\mathcal{H}$ correspond to either vertices of $\mathcal{G}$ or to faces of $\mathcal{G}$. Hence, the vertex colouring of $\mathcal{H}^*$ correspond to a colouring of the faces and vertices of $\mathcal{G}$. Let $e = uv$ be an edge of $\mathcal{G}$, let F_1, F_2 be the faces of $\mathcal{G}$ containing e, and for $i = 1, 2$ let f_i, g_i be the edge of F_i incident to u and v respectively. Then $e, f_1, f_2, g_1, g_2 \in V(H)$ and $f_1e, f_2e, g_1e, g_2e \in E(H)$. We may assume that f_1e, g_1e, g_2e, f_2e are assigned colours $\alpha, \beta, \gamma, \delta$ respectively. Suppose face u of $\mathcal{H}$ is assigned colour x then faces F_1, v, F_2 of $\mathcal{H}$ are assigned colours $x + \alpha, x + \alpha + \beta, x + \alpha + \beta + \gamma$. Since

$x \neq x+\alpha+\beta$, (1) holds; and since $x+\alpha \neq x+\alpha+\beta+\gamma$ (2) holds. (1) and (2) imply that two colours are used at least to colour u, v, F_1, F_2. But since $x, x+\alpha, x+\alpha+\beta, x+\alpha+\beta+\gamma$ do not sum to $(0,0)$, exactly three colours are used, i.e. (3) holds. □

A general conjecture. Let $(\mathcal{G}, T)$ be a graft. Consider a partition of $V(\mathcal{G})$ into subsets $V_1, \ldots, V_r$ such that for each $i \in [r]$ the induced subgraph $\mathcal{G}[V_i]$ is connected and $|V_i \cap T|$ is odd. Let $\mathcal{G}'$ be the simple graph with vertex set $[r]$ and where $ij \in E(\mathcal{G}')$ if and only if there exists an edge of $\mathcal{G}$ with one end in V_i and one end in V_j. A subgraph of $\mathcal{G}'$ is called a *T-minor* of $\mathcal{G}$. Observe that if a graph is a T-minor of $\mathcal{G}$ then it is minor of $\mathcal{G}$ as well, however the converse is not true in general.

Conjecture 1.7. *Let $(\mathcal{G}, T)$ be a graft where all T-cuts have the same parity. Then the graft packs if $\mathcal{G}$ does not contain the Petersen graph as a T-minor.*

Seymour [8] introduced a property called *cycling* for multi-commodity flows in binary matroids. Characterizing which grafts (where all T-cuts have the same parity) pack can be viewed as a special case of the problem of characterizing which binary matroids are cycling. Proposition 1.2 implies that if $(\mathcal{G}, T)$ is a postman set then all T-cuts have the same parity. Hence the next conjecture is a special case of Conjecture 1.7.

Conjecture 1.8 (Conforti and Johnson [5, 1])**.** *Let $(\mathcal{G}, T)$ be a postman set where $\mathcal{G}$ has no Petersen minor. Then $(\mathcal{G}, T)$ packs.*

Since the Petersen graph is not planar, Conjecture 1.7 suggests that the condition that $\tau(\mathcal{G}, T) \leq 5$ is not required in Theorem 1.3. In other words,

Conjecture 1.9. *A graft $(\mathcal{G}, T)$ packs if $\mathcal{G}$ is planar and all T-cuts have the same parity.*

Following the argument in the proof of Corollary 1.4, this in turn would imply,

Conjecture 1.10 (Seymour (personal communication))**.** *Let $\mathcal{G}$ be a k-regular graph with no Petersen minor. Then $\chi'(\mathcal{G}) = k$ if and only if for all $U \subseteq V(\mathcal{G})$ where $|U|$ is odd, we have $d(U) \geq k$.*

Note that conjectures 1.8 and 1.10 would both imply the recently proved conjecture of Tutte [6, 2, 3] which states that cubic bridgeless graphs with no Petersen minor have chromatic index three. It can be easily shown that a graph has a *nowhere zero* 4*-flow* if and only if it contains three pairwise disjoint postman sets. Thus conjecture 1.7 would also imply,

Conjecture 1.11 (Tutte [10])**.** *Every bridgeless graph not containing the Petersen graph as a minor has a nowhere zero* 4*-flow.*

Organization of the paper. In Section 2 we derive properties that minimal counterexamples to Theorem 1.3 or Conjecture 1.9 must satisfy. Using the 4-colour theorem we deduce that any such minimal counterexample satisfies $\tau(\mathcal{G}, T) \geq 4$. We use discharging arguments to show that both cases $\tau(\mathcal{G}, T) = 4$ and $\tau(\mathcal{G}, T) = 5$ lead to a contradiction, this is done in Sections 3 and 4.

2. General properties

Let β be some fixed non-negative integer. Consider the following statement: "Let $(\mathcal{G}, T)$ be a graft where all T-cuts have the same parity, then $(\mathcal{G}, T)$ packs if $\mathcal{G}$ is planar and $\tau(\mathcal{G}, T) \leq \beta$". Observe that if $\beta = 5$ then the statement is equivalent to Theorem 1.3. Moreover, if the statement holds for every β then Conjecture 1.9 is true. Suppose that there is a graft which contradicts the above statement. Among all such grafts we choose one, say $(\mathcal{G}, T)$, which satisfies the following properties,

(1) it minimizes $|V(\mathcal{G})|$;

(2) it minimizes $\tau(\mathcal{G}, T)$ among all grafts satisfying (1);

(3) it minimizes $|E(G)|$ among all grafts satisfying (1) and (2).

Throughout the remainder of the paper, k denotes $\tau(\mathcal{G}, T)$.

Lemma 2.1. *Let* $\delta(X)$ *be a* T*-cut with* $d(X) = k$*. Then* $|X| = 1$ *or* $|\bar{X}| = 1$*.*

Proof. Suppose for a contradiction, $|X| > 1$ and $|\bar{X}| > 1$. Let $X_1 := X$ and $X_2 := \bar{X}$. Suppose for a contradiction $|X_1|, |X_2| > 1$. Let us first show that for $i = 1, 2$, $\mathcal{G}[X_i]$ (the graph induced by vertices X_i) is connected. If not we can partition X_i into X_i', X_i'' so that there are no edges between X_i' and X_i''. Since $|X_i \cap T|$ is odd, we may assume that $|X_i' \cap T|$ is odd. Note that we may assume that $\mathcal{G}$ is connected. It follows that there is at least one edge between X_{3-i} and X_i''. Hence the T-cut $\delta(X_i') \subset \delta(X_i)$, a contradiction.

For $i = 1, 2$, let $\mathcal{G}_i$ be obtained from $\mathcal{G}$ by identifying all vertices of $\mathcal{G}$ in X_{3-i} to a single new vertex z_i and deleting all loops. Also let $T_i = (T \cap X_i) \cup \{z_i\}$. By construction $|T_i|$ is even, and (G_i, T_i) is a graft. Since $\mathcal{G}[X_{3-i}]$ is connected it follows that $\mathcal{G}_i$ is a minor of $\mathcal{G}$. In particular $\mathcal{G}_i$ is a planar graph. Observe that T_i-cuts of $\mathcal{G}_i$ correspond to T-cuts of $\mathcal{G}$ which contain all vertices X_{3-i} on the same shore. It follows that, $\tau(G_i, T_i) \geq k$ and that all T_i-cuts of $\mathcal{G}_i$ have the same parity. Because $d(z_i) = k$ we have in fact $\tau(G_i, T_i) = k$. Since $|X_{3-i}| > 1$, $|V(G_i)| < |V(G)|$. It follows from the choice of $(\mathcal{G}, T)$ that (G_i, T_i) packs.

Thus for $i = 1, 2$ there exists packing of T_i-joins $B_1^i, \ldots, B_k^i$. As $d(z_i) = k$, each $B_1^i, \ldots, B_k^i$ uses exactly one edge incident to z_i. We may assume for each $l \in [k]$ that B_l^1 and B_l^2 contain the edge incident to z_1 and z_2 which corresponds to the same edge of $\mathcal{G}$. For $l \in [k]$ let $B_l := B_l^1 \cup B_l^2$. Since B_l^i is a T_i-join, $T \cap X_i$ is the set vertices of odd degree of $\mathcal{G}[B_l]$ in X_i. It follows that B_l is a T-join of $\mathcal{G}$ and that $(\mathcal{G}, T)$ packs, a contradiction. □

Lemma 2.2. *(1) $T = V(\mathcal{G})$ and (2) $\mathcal{G}$ is k-regular.*

Proof. Suppose for a contradiction that there exists $v \in V(\mathcal{G})$ where either $v \notin T$ or $d(v) > k$.

Claim: Not all edges incident to v are parallel.

Proof of claim: Suppose for a contradiction that all edges in $\delta(v)$ have both the same ends. Consider first the case $v \in T$. Let $e_1, e_2 \in \delta(v)$, and let $\mathcal{G}' := G \backslash \{e_1, e_2\}$. Since all T-cuts of $\mathcal{G}$ have the same parity, $d_{\mathcal{G}}(v) \geq k + 2$. As edges of $\delta_{\mathcal{G}}(v)$ are parallel this implies $\tau(G', T) \geq k$. Note that all T-cuts of $\mathcal{G}'$ have the same parity. By the choice of $(\mathcal{G}, T)$, $(\mathcal{G}', T)$ has a packing of k T-joins. A contradiction since T-joins of $\mathcal{G}'$ are T-joins of $\mathcal{G}$. Suppose now that $v \notin T$. Let $\mathcal{G}' := \mathcal{G} \backslash \delta(v)$.

Since edges of $\delta(v)$ are parallel, no minimal T-cut of $\mathcal{G}$ uses any edge of $\delta(v)$. Hence, $\tau(\mathcal{G}',T) \geq k$. By the choice of $(\mathcal{G},T)$, $(\mathcal{G}',T)$ packs, hence so does $(\mathcal{G},T)$, a contradiction. $\diamond$

Consider a planar embedding of $\mathcal{G}$ and visit edges of $\delta(v)$ in a clockwise fashion. It follows from the Claim that there are two consecutive edges of the form u_1v and u_2v where $u_1 \neq u_2$. Let $\mathcal{G}'$ be obtained by replacing edges u_1v, u_2v of $\mathcal{G}$ by an edge u_1u_2. Note that the planar embedding of $\mathcal{G}$ can be transformed into a planar embedding of $\mathcal{G}'$. Observe also that since $\mathcal{G}$ has no loops neither does $\mathcal{G}'$. Clearly, for all $w \in T - \{v\}$, $d_{\mathcal{G}'}(w) = d_{\mathcal{G}}(w) \geq k$. Let $\delta(X)$ be a T-cut of $\mathcal{G}$ where either $\{v\} = X$ or where both $|X| > 1$ and $|\bar{X}| > 1$. Then $d(X) > k$, in the former case by the choice of v and in the latter one by Lemma 2.1. Since all T-cuts have the same parity $d(X) \geq k+2$. It follows that $\tau(\mathcal{G}',T) = k$. Note that all T-cuts of $\mathcal{G}'$ have the same parity. By the choice of $(\mathcal{G},T)$ there exists a packing $B'_1,\dots,B'_k$ of T-joins of $\mathcal{G}'$. If none of $B'_1,\dots,B'_k$ use u_1u_2 then they are T-joins of $\mathcal{G}$ as well and $(\mathcal{G},T)$ packs, a contradiction. Otherwise there is an edge, say $u_1u_2 \in B'_1$, and $B'_1 - \{u_1u_2\} \cup \{u_1v, u_2v\}, B'_2, \dots, B'_k$ is a packing of T-joins of $\mathcal{G}$, and $(\mathcal{G},T)$ packs, a contradiction. $\square$

Lemma 2.3. $k \geq 4$.

Proof. Lemma 2.2 states that $V(\mathcal{G}) = T$ and $\mathcal{G}$ is k-regular. Let $X \subseteq V(\mathcal{G})$ where $|X|$ is odd. Then $|X| = |X \cap T|$ and since $\tau(\mathcal{G},T) \geq k$, we have $d(X) \geq k$. Suppose $k \leq 3$. It follows from Remark 1.5 that $\chi'(\mathcal{G}) = k$. Since $\mathcal{G}$ is k-regular and $T = V(\mathcal{G})$, every colour class is a T-join. But then $(\mathcal{G},T)$ packs, a contradiction. $\square$

We say that a collection of T-joins, $B_1,\dots,B_k$ is an *e-colouring* for some edge e, if they pairwise intersect at most in e (i.e. for all $i,j \in [k], i \neq j$, $B_i \cap B_j \subseteq \{e\}$) and if there does not exist a collection, $B'_1,\dots,B'_k$ of T-joins which pairwise intersect at most in e and for which $|\{B'_i : e \in B'_i, i \in [k]\}| < |\{B_i : e \in B_i, i \in [k]\}|$.

Lemma 2.4. *Suppose $B_1, \ldots, B_k$ is an e-colouring then (1) for every vertex v which is not an end of e every edge of $\delta(v)$ is in exactly one of $B_1, \ldots, B_k$. Moreover, we can assume that (2) for some odd $r \geq 3$, e is in all of $B_1, \ldots, B_r$ and in none of $B_{r+1}, \ldots, B_k$.*

Proof. Let v be a vertex of $\mathcal{G}$ which is not an end of e. Since $v \in T$, $\delta(v)$ is a T-cut. Now (1) follows from the fact that v has degree k and that $B_1, \ldots, B_k$ intersect at most in e. It follows from the fact that $\mathcal{G}$ is k-regular and $T = V(\mathcal{G})$ that $E(\mathcal{G})$ is a T-join if k is odd, and that $E(\mathcal{G})$ is an Eulerian subgraph if k is even. Let $B' := B_1 \triangle B_2 \triangle \ldots \triangle B_k$. Note that B' is a T-join if k is odd and is an Eulerian subgraph if k is even. Hence, in either cases, $B'' := B' \triangle E(\mathcal{G})$ is an Eulerian subgraph. (1) implies that $B'' \subseteq \{e\}$. Since $\mathcal{G}$ has no loops, $B'' = \emptyset$. It follows that there exists an odd number of T-joins among $B_1, \ldots, B_k$ which use the edge e. Hence, upon relabeling we can assume that for some odd r, e is in all of $B_1, \ldots, B_r$ and in none of $B_{r+1}, \ldots, B_k$. Moreover, $r \geq 2$, for otherwise $(\mathcal{G}, T)$ packs. This proves (2). □

Lemma 2.5. *There exists an e-colouring for every edge e.*

Proof. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by contracting edge e and deleting all loops. Let u, v denote the ends of e and let $T' := T - \{u, v\}$. Since $|T|$ is even so is $|T'|$. T'-cuts of $\mathcal{G}'$ correspond to T-cuts of $\mathcal{G}$ where both u and v are on the same shore. In particular, all T'-cuts of $\mathcal{G}'$ have the same parity and $\tau(\mathcal{G}', T') \geq k$. It follows by the choice of $(\mathcal{G}, T)$ that there exists a packing of k T'-joins of $\mathcal{G}'$. These T'-joins can be extended to T-joins of $\mathcal{G}$ by possibly adding e. Hence, there exists a collection of k T-joins, say $B_1, \ldots, B_k$, which pairwise intersect at most in e. Among such collections choose one with the fewest number of T-joins using e. □

Consider $e \in E(\mathcal{G})$ and an e-colouring $B_1, \ldots, B_k$ at e. We say that a T-cut $\delta(U)$ is a *mate* of B_i ($i \in [k]$) if $e \in \delta(U)$ and for all $j \in [k] - \{i\}$, $|\delta(U) \cap B_j| = 1$.

Lemma 2.6. *Let $B_1, \ldots, B_k$ be any e-colouring then all of $B_1, \ldots, B_k$ have mates.*

Proof. Let $r := |\{B_i : e \in B_i, i \in [k]\}|$. Let $\mathcal{G}'$ be obtained by adding $r - 3$ parallel edges to e. Lemma 2.4 implies that r is odd. It follows that since all T-cuts of $\mathcal{G}$ have the same parity so do all

T-cuts of $\mathcal{G}'$. The e-colouring $B_1, \dots, B_k$ of $\mathcal{G}$ implies that there exists an e-colouring $B'_1, \dots, B'_k$ of $\mathcal{G}'$ where $\{j : e \in B'_j, j \in [k]\} = \{1, 2, 3\}$.

Claim: Let $i \in [k]$ and let $\mathcal{G}'' := \mathcal{G}' \setminus B'_i$. Then $\tau(\mathcal{G}'', T) \leq k - 3$.

Proof of claim: If all T-cuts of $\mathcal{G}'$ are even (resp. odd) then Remark 1.1 implies that all T-cuts of $\mathcal{G}''$ are odd (resp. even). For every vertex v which is not an end of e, $d_{\mathcal{G}''}(v) < k$. Thus $\tau(\mathcal{G}'', T) < k$ and it follows from the choice of $(\mathcal{G}, T)$ that $(\mathcal{G}'', T)$ packs. Suppose for a contradiction that $\tau(\mathcal{G}'', T) \geq k - 1$. Then there exists a packing of $k - 1$ T-joins of $\mathcal{G}''$, say $\hat{B}_1, \dots, \hat{B}_{k-1}$. But then B'_i together with $\hat{B}_1, \dots, \hat{B}_{k-1}$ imply that there exists an e-colouring of $\mathcal{G}$ where at most $r - 2$ of the T-joins use edge e, a contradiction. Hence $\tau(\mathcal{G}'', T) < k - 1$. Since T-cuts of $\mathcal{G}'$ and T-cuts of $\mathcal{G}''$ have distinct parities, $\tau(\mathcal{G}'', T) \leq k - 3$. ◇

Let $\delta_{\mathcal{G}''}(U)$ be a minimum T-cut of $\mathcal{G}''$. Consider first the case where $i \in \{1, 2, 3\}$. Then for $j = 4, \dots, k$, $B'_i \cap B'_j = \emptyset$. Hence, for each $j = 4, \dots, k$, B'_j is a T-join of $\mathcal{G}''$ and in particular $\delta_{\mathcal{G}''}(U) \cap B'_j \neq \emptyset$. The Claim states that $|\delta_{\mathcal{G}''}(U)| \leq k - 3$. Hence, $|\delta_{\mathcal{G}''}(U) \cap B'_j| = 1$ for $j = 4, \dots, k$, and $\delta_{\mathcal{G}''}(U) \cap (B'_1 \cup B'_2 \cup B'_3) = \emptyset$. Since $\delta_{\mathcal{G}'}(U) \subseteq \delta_{\mathcal{G}''}(U) \cup B'_i$ is a T-join of $\mathcal{G}'$, it intersects each of B'_1, B'_2, B'_3. Hence $e \in \delta_{\mathcal{G}'}(U)$. But then $\delta_{\mathcal{G}}(U)$ is a mate of B_i. Consider now the case where $i > 3$. For $j \in [k] - \{i\}$, B'_j are T-joins of $\mathcal{G}''$, thus $\delta_{\mathcal{G}''}(U) \cap B'_j \neq \emptyset$. The Claim states that $|\delta_{\mathcal{G}''}(U)| \leq k - 3$. Hence, $|\delta_{\mathcal{G}''}(U) \cap B'_j| = 1$ for $j \in [k] - \{i\}$ and $e \in \delta_{\mathcal{G}''}(U)$. But then $\delta_{\mathcal{G}}(U)$ is a mate of B_i. □

Lemma 2.7. *Let $B_1, \dots, B_k$ be an e-colouring and let $\delta(U)$ be a mate of B_i for some $i \in [k]$. Suppose there is a circuit C such that $|C \cap \delta(U) \cap B_i| \geq 2$. Then $|C| \geq 3$. Moreover, if C is a triangle with edges, say f, g, h, then $f, g \in B_i, h \notin B_i$ and f, g are both incident to a same end of e.*

Proof. Let C be a circuit where $|C| \leq 3$ and $|C \cap \delta(U) \cap B_i| \geq 2$. Lemma 2.4(1) implies that the common endpoint of the two edges of C in B_i is an end of e. Suppose $|C| = 2$. Then Lemma 2.4(1) implies that both edges of C are parallel to e. It follows that all cuts using e contain two edges in B_i. Thus none of B_j where $j \in [k] - \{i\}$ has a mate, a contradiction to Lemma 2.6. □

We say that a triangle f, g, h as given in the previous proposition and where $e \notin \{f, g, h\}$ is a *bad triangle* for $\delta(U)$.

Lemma 2.8. *Let $B_1, \ldots, B_k$ where $k \in \{4, 5\}$ be any e-colouring. Then for all $i \in [k]$, B_i has a mate with no bad triangles.*

Proof. Consider a counterexample which minimizes $|B_i|$, i.e. every mate of B_i has a bad triangle but for every e-colouring $B'_1, \ldots, B'_k$ where $|B'_i| < |B_i|$, B'_i has a mate with no bad triangles. Let u, v be the ends of e. Since B_i has a least one mate, there is a triangle with edges f, g, h where $f, g \in B_i$, $h \in B_j$ ($j \neq i$) and both f, g are incident to u. Define, $B'_i = B_i \triangle C$, $B'_j := B_j \triangle C$ and let $B'_l := B_l$ for all $l \in [k] - \{i, j\}$. Clearly, $B'_1, \ldots, B'_k$ is an e-colouring. Since $|B'_i| < |B_i|$ there is a mate $\delta(U)$ of B'_i with no bad triangle. It can be readily checked that $\delta(U)$ is also a mate of B_i. Let f', g', h' be the edges of a bad triangle of $\delta(U)$ for B_i where $f', g' \in \delta(U) \cap B_i$. Since f', g', h' is no longer a bad triangle for B'_i, triangles f, g, h, and f', g', h' must have an edge in common, say $g = g'$.

Consider the case where $k = 4$. Then $i \notin [3]$ since otherwise $d(u) = 4$ implies that u is not incident to an edge of B_4, a contradiction as $u \in T$ and B_4 is a T-join. Hence, $i = 4$, but then $\delta(U) - \{e\} \subseteq B'_4$, a contradiction as $g' \in \delta(U) \cap B'_j$. Consider the case where $k = 5$. Then e is not included in all of $B_1, \ldots, B_5$, for otherwise, $\delta(U) - \{e\} \subseteq B'_i$, a contradiction as $g' \in \delta(U) \cap B'_j$. Thus we know from Lemma 2.4 that $e \in (B_1 \cap B_2 \cap B_3) - B_4 - B_5$. Then $i \notin [3]$ since otherwise $d(u) = 5$ implies that u is not incident to both an edge of B_4 and B_5, a contradiction as $u \in T$ and B_4, B_5 are T-joins. Hence, we can assume $i = 4$. Since $h \in \delta(U)$ we must have $h \in B_5$. By symmetry (interchange in the previous arguments the roles of triangles f, g, h and f', g', h'), we must also have $h' \in B_5$. It follows that the vertex common to both h and h' is v. But then every cut $\delta(U)$ which contains e intersects either B_4 or B_5 at least twice. It follows that none of B_1, B_2, B_3 have a mate, a contradiction. $\square$

It can be easily checked that every cut is either a T-cut or the symmetric difference of two T-cuts. Hence, Remark 1.1 implies the following observation.

Remark 2.9. For all cuts $\delta(U)$ and T-joins B, B', $|\delta(U) \cap B|$ and $|\delta(U) \cap B'|$ have the same parity.

Lemma 2.10. *$\mathcal{G}$ does not have a two edge cutset. In particular faces of $\mathcal{G}$ share at most one edge.*

Proof. Suppose there are edges e, e' such that $\mathcal{G} \backslash \{e', e'\}$ has two components $\mathcal{G}_1, \mathcal{G}_2$. Since $\mathcal{G}$ is k-regular, there exists an edge e_1 in $\mathcal{G}_1$ and an edge e_2 in $\mathcal{G}_2$. It follows from Remark 2.9 that for every e_1-colouring and for every e_2-colouring both e and e' are in the same T-join. Then we can combine the e_1-colouring and the e_2-colouring to find an edge colouring of $\mathcal{G}$, a contradiction. □

Lemma 2.11. *Let n, m, f denote respectively, the number of vertices, edges, and faces of $\mathcal{G}$. Then*

$$2m + (k-4)n < 4f$$

Proof. Since $\mathcal{G}$ is k-regular, $2m = kn$. Euler's formula states that $n - m + f = 2$, hence in particular $f > m - n$ or equivalently, $4f > 4m - 4n = 2m + (2m - 4n)$. Since $2m = kn$ we have $2m + (2m - 4n) = 2m + (kn - 4n) = 2m + (k-4)n$. □

3. DISCHARGING FOR $\tau(\mathcal{G}, T) = 4$

In this section we shall prove that no minimum counterexample satisfies $k = 4$. Let us assign every face F of $\mathcal{G}$ an initial charge equal to $|F|$. We say that we *move* one charge from a face H to a face F, if we decrease the number of charges of face H by one and increase the number of charges of face F by one. Note that the total number of charges remains unchanged. We say that two faces are *adjacent* if they share at least one edge. We apply, once for each face H, the following rule,

Discharging rule: if $|H| > 4$ and H has 4 distinct adjacent faces G_1, G_2, G_3, G_4 where $|G_i| \geq 4$ for $i \in [4]$, then for all adjacent faces F of H where $|F| < 4$ move one charge from H to F.

For each face F let α_F denote the resulting charge.

Remark 3.1. If F is a face where $|F| \geq 4$ then $\alpha_F \geq 4$.

Suppose that for every face F, $\alpha_F \geq 4$, then

$$2m = \sum_{\text{faces } F} |F| = \sum_{\text{faces } F} \alpha_F \geq 4f$$

which contradicts, Lemma 2.11. Hence the next lemma shows $k \neq 4$.

Lemma 3.2. *For every face F we have $\alpha_F \geq 4$.*

Proof. Because of Remark 3.1 we may assume that $|F| \leq 4$. Since $\mathcal{G}$ is loopless we may assume that $|F| \geq 2$. Hence, it suffices to consider the following two cases.

Case 1: $|F| = 2$.

Let e, e' be the two edges in F. Lemma 2.5 states that there is an e-colouring B_1, B_2, B_3, B_4. Lemma 2.4 implies that $e \in (B_1 \cap B_2 \cap B_3) - B_4$ and that $E(\mathcal{G}) = B_1 \cup B_2 \cup B_3 \cup B_4$. Lemma 2.8 implies that for each $i \in [4]$, B_i has a mate $\delta(U_i)$ with no bad triangle. Since $\delta(U_4) - \{e\} \subseteq B_4$, $e' \in B_4$. It follows that for $i \in [4]$, $\delta(U_i) - \{e, e'\} \subseteq B_i$. Let H (resp. H') be the face distinct from F with e (resp. e'). For all $i \in [4]$, let g_i be an edge, distinct from e, in $H \cap \delta(U_i) \subseteq B_i$ and let G_i be the face distinct from H with g_i. Since $|\delta(U_i) \cap G_i|$ is even, there is an edge $g'_i \in G_i \cap \delta(U_i)$ distinct from g_i. Clearly, $g'_i \neq e$. Because $\delta(U_i)$ have no bad triangles, $|G_i| \geq 4$ for all $i \in [4]$. Lemma 2.10 implies that the faces G_1, G_2, G_3, G_4 are distinct. It follows from the discharge rule that face F receives one charge from the face H. By symmetry, F also receives one charge from H'. It follows that $\alpha_F \geq 4$.

Case 2: $|F| = 3$.

Let e, f, f' be the three edges in F. As in the previous case there is an e-colouring B_1, B_2, B_3, B_4 and $e \in (B_1 \cap B_2 \cap B_3) - B_4$, $E(G) = B_1 \cup B_2 \cup B_3 \cup B_4$. Moreover, for each $i \in [4]$, B_i has mate $\delta(U_i)$ with no bad triangles. Since $\delta(U_4) - \{e\} \subseteq B_4$, one of f, f', say f, is in B_4. Let H be the face distinct from F which contains f. Lemma 2.4(1) implies that $f' \notin B_4$. After relabeling we may assume that $f' \in B_1$. It follows that for $i = 2,3,4$, each mate $\delta(U_i)$ contains f. For $i = 2,3,4$, we let g_i be an edge distinct from f in $H \cap \delta(U_i) \subseteq B_i$; we let G_i be the face distinct from H with g_i; and we let $g_i' \in G_i \cap \delta(U_i) \subseteq B_i$ be an edge distinct from g_i. Note $g_i' \neq e$ for otherwise $\delta(U_i) = \{e, f, g_i\}$, hence $k \leq 3$, a contradiction. Since $\delta(U_i)$ have no bad triangles, $|G_i| \geq 4$ ($i = 2,3,4$). Define $B_1' := B_4 \triangle F$ and $B_4' := B_1 \triangle F$. Note that B_1', B_2, B_3, B_4' form a e-colouring. By Lemma 2.8, B_4' has a mate $\delta(U)$ with no bad triangles. Since $f' \in B_1'$ and $f \in B_4'$, $\delta(U)$ contains f. Let $\tilde{g}$ be an edge distinct from f in $H \cap \delta(U) \subseteq B_4'$, let $\tilde{G}$ be the face distinct from H with $\tilde{g}$, and let $\tilde{g}' \in \tilde{G} \cap \delta(U) \subseteq B_4'$ be an edge distinct from g_i. Since $\delta(U)$ has no bad triangle, $|\tilde{G}| \geq 4$. Since $g_4 \in B_1'$ and $\tilde{g} \in B_4'$ faces $G_2, G_3, G_4, \tilde{G}$ are all distinct. This implies that F receives one charge from the face H and that $\alpha_F \geq 4$. □

4. Discharging for $\tau(\mathcal{G}, T) = 5$

In this section we shall prove that no minimum counterexample satisfies $k = 5$. Given a face F we write $N(F)$ for the set of faces adjacent to F (but distinct from F). Let us assign to every face F an initial charge equal to $|F|$.

Discharging rules for faces H:

(F1) if $|H| > 4$ and there exists faces $G_1, G_2, G_3, G_4 \in N(H)$ where $|G_i| \geq 4$ for $i \in [4]$ then move 1 charge from H to all faces $F \in N(H)$ where $|F| < 4$.

(F2) suppose $|H| > 4$ and there exists faces $G_1, G_2, G_3 \in N(H)$ where $|G_i| \geq 4$ for $i \in [3]$. For every edge e of H let e' and e'' be the two edges of H adjacent to e. Let F, F', F'' be the faces, distinct from H, containing respectively edges e, e', e''. Suppose $|F| < 4$. If $|F'| \leq 3$ and $|F''| \leq 3$ then move 1 charge from H to F, otherwise move $1/2$ charge from H to F.

Let us assign to every vertex v an initial charge equal to one. Throughout this section we will assume that $\mathcal{G}$ has a fixed planar embedding. Let $v \in V(\mathcal{G})$. Since $d(v) = 5$ by going around v in a clockwise, or counterclockwise fashion we will visit in sequence faces $F_1, \ldots, F_5$ where for all

$i \in [5]$, F_i, F_{i+1} $(k+1=1)$ are adjacent faces. We write $NF(v) = (F_1, \ldots, F_k)$. We also write $NF_i(v)$ (resp. $NF_i^+(v)$) for the set of faces F of $NF(v)$ where $|F|$ is equal to (resp. at least) i.

Discharging rules for vertices v: Suppose that $NF(v) = (F_1, F_2, F_3, F_4, F_5)$.

(V1) if $|NF_4^+(v)| = 4$ then move 1 from v to the face in $NF(v) - NF_4^+(F)$.

(V2) if $|NF_4^+(v)| = 3$ then move $1/2$ from v to each face in $NF(v) - NF_4^+(F)$.

(V3) if $NF_4^+(v) = \{F_1\}$ then move $1/2$ from v to F_2 and F_5.

(V4) if $NF_4^+(v) = \{F_1, F_2\}$ then move $1/2$ from v to F_3 and F_5.

(V5) if $NF_4^+(v) = \{F_1, F_3\}$ and $|F_4| = |F_5| = 2$ then move $1/2$ from v to F_4 and F_5.

(V6) if $NF_4^+(v) = \{F_1, F_3\}$, and $|F_4| = |F_5| = 3$ then move 1 from v to F_2.

(V7) if $NF_4^+(v) = \{F_1, F_3\}$, $|F_4| = 2$, and $|F_5| = 3$ then move $1/2$ to F_2 and F_4.

Remark 4.1. Observe that in every case at most one of the rules (F1),(F2) and (V1)-(V7) will apply.

For each face F (resp. vertex v) let α_F (resp. α_v) denote the charge after discharging.

Remark 4.2. Rules (F1),(F2) imply that if a face H satisfies $|H| \geq 4$ then $\alpha_H \geq 4$. Rules (V1)-(V7) imply that $\alpha_v \geq 0$ for all $v \in V(\mathcal{G})$.

Suppose that for every face F of $\mathcal{G}$, $\alpha_F \geq 4$, then since $\alpha_v \geq 0$ for all $v \in V(\mathcal{G})$,

$$2m + n = \sum_{\text{faces } F} |F| + n = \sum_{\text{faces } F} \alpha_F + \sum_{v \in V(\mathcal{G})} \alpha_v \geq 4f$$

which contradicts, Lemma 2.11. Hence Remark 4.2 implies that it will suffice to show lemmas 4.9 and 4.11.

Lemma 4.3. *Let e be an edge and let F_1, F_2 be the two faces containing e. If $|F_1| \leq 5$ or $|F_2| \leq 5$ then for every e-colouring $B_1, \ldots, B_5$, $e \in (B_1 \cap B_2 \cap B_3) - B_4 - B_5$.*

Proof. Let $B_1, \ldots, B_5$ be an e-colouring. Lemmas 2.4 implies that an odd number, at least three, of $B_1, \ldots, B_5$ use e. Suppose that $e \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5$. Lemma 2.8 implies that for

$i \in [5]$, B_i has a mate $\delta(U_i)$. By definition of mates, for all $i \in [5]$, $\delta(U_i) - \{e\} \subseteq B_i$. In particular, $|F_1|, |F_2| \geq 6$. □

Let H be a face of $\mathcal{G}$, we define $val(H) := |\{F \in N(H) : |F| \geq 4\}|$.

Lemma 4.4. *Suppose that $\mathcal{G}$ has faces F_1 and F_2 with a single common edge e. Let $B_1, \ldots, B_5$ be an e-colouring. For $j = 1, 2$ let e_j be some edge in $F_j - \{e\}$ and let H_j be the face using e_j which is distinct from F_i. Suppose that $e_1 \in B_4$ and $e_2 \in B_5$ and let $I \subseteq [5]$ be a set of indices. If for all $i \in I$, B_i has a mate $\delta(U_i)$ with no bad triangles which uses both e_1, e_2 then $val(H_1), val(H_2) \geq |I|$.*

Proof. Let $j \in [2]$ and let $i \in [I]$. Since $e, e_1, e_2 \in \delta(U_i)$ it follows from the definition of mates that $\delta(U_i) - \{e, e_1, e_2\} \subseteq B_i$. For all $i \in I$, let g_i^j be an edge in $H_j \cap \delta(U_i) \subseteq B_i$ and let G_i^j be the face distinct from H_j with g_i^j. There is an edge $\hat{g}_i^j \in G_i^j \cap \delta(U_i)$ distinct from g_i^j. Observe that $\hat{g}_i^j \neq e$ for otherwise $\delta(U_i) = \{e, e_i, g_i^j\} < 5 = k$, a contradiction. Because $\delta(U_i)$ have no bad triangles, $|G_i^j| \geq 4$ for all $i \in I$. Finally, Lemma 2.10 implies that $\{G_i^j : i \in I\}$ are all distinct. Thus $val(H_j) \geq |I|$. □

Lemma 4.5. *Suppose that $\mathcal{G}$ has faces F_1 and F_2 with a common edge e and $|F_1|, |F_2| \leq 3$. Let $B_1, \ldots, B_5$ be an e-colouring. We may assume (after possibly relabeling B_4, B_5) that $F_1 - \{e\}$ contains exactly one edge in B_4 and that $F_2 - \{e\}$ contains exactly one edge in B_5.*

Proof. Lemma 4.3 implies that $e \in (B_1 \cap B_2 \cap B_3) - B_4 - B_5$. Lemma 2.8 implies that for $i \in [5]$, B_i has a mate $\delta(U_i)$. Mates $\delta(U_4), \delta(U_5)$ imply (after possibly relabeling B_4, B_5) that $(F_1 - \{e\}) \cap B_4 \neq \emptyset, (F_2 - \{e\}) \cap B_5 \neq \emptyset$. Finally, $F_1 - \{e\}$ (resp. $F_2 - \{e\}$) contain at most one edge in B_4 (resp. B_5) because of Lemma 2.4(1). □

Lemma 4.6. *Suppose that $\mathcal{G}$ has adjacent faces F_1 and F_2 where $|F_1| = 2$ and $|F_2| \leq 3$. Then F_1 and F_2 have two adjacent faces H_1 and H_2 respectively, where $val(H_1), val(H_2) \geq 4$. In particular $|H_1|, |H_2| \geq 5$ and all faces adjacent to H_1 or H_2 receive one charge from these faces.*

Proof. Suppose that F_1 consists of edges e, e_1 and that F_2 consists of either edges e, e_2 or e, e_2, e_2'. Let $B_1, \ldots, B_5$ be an e-colouring. Because of Lemma 4.5 we can assume that $e_1 \in B_4$, $e_2 \in B_5$, and $e_2' \in B_s$ where $s \in [4]$. It follows that for all $i \in [5] - \{s\}$, $\delta(U_i)$ uses both e_1, e_2. For $j = 1, 2$ let H_j be the face containing e_j which is distinct from F_j. Lemma 4.6 implies that $val(H_1), val(H_2) \geq 4$. Hence, $|H_1|, |H_2| \geq 5$ and the result follows from rule (F1). □

Lemma 4.7. *Let $v \in V(\mathcal{G})$ and $\delta(v) = \{e_1, e_2, e_3, e_4, e_5\}$. Suppose that $\{e_1, e_2\}$ and $\{e_2, e_3\}$ are faces of $\mathcal{G}$. Then the ends of e_1, e_4, e_5 distinct from v are independent.*

Proof. Let v_1 (resp. v_4; v_5) be the end of e_1 (resp. e_4; e_5) distinct from v. Lemma 4.6 implies that $v_1 \neq v_4$ and $v_1 \neq v_5$. Let $B_1, \ldots, B_5$ be an e_2-colouring. Because of Lemma 4.5 we can assume that $e_1 \in B_4$ and $e_3 \in B_5$. Since $v \in T$ there is $i \in [5]$ such that $e_4, e_5 \in B_i$. Then Lemma 2.4(1) implies that $v_4 \neq v_5$. □

Lemma 4.8. *Let e_1, e_2 be edges such that $\{e_1, e_2\}$ is a face and let v, v' be the ends of e_1, e_2. Consider, $\delta(v) = \{e_1, e_2, e_3, e_4, e_5\}$ and $\delta(v') = \{e_5', e_4', e_3', e_2, e_1\}$ where the labeling is obtained by visiting edges in a clockwise fashion around v and v'. Let F_{34} (resp. F_{45}, F_{34}', F_{45}') be the faces containing edges e_3, e_4 (resp. e_4e_5; $e_3'e_4'$; $e_4'e_5'$). Let H_1 and H_2 be the faces with respectively edges e_5, e_1, e_5' and e_3, e_2, e_3'.*

(a) *if $|H_2| = 3$ then $|H_1| \geq 5$ and $val(H_1) \geq 4$.*
(b) *if $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$ then $val(H_1) + val(H_2) \geq 5$.*
(c) *if $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$ and; $|F_{34}'| \neq 3$ or $|F_{45}'| \neq 3$ then $val(H_1) + val(H_2) \geq 6$.*

Proof. Let $B_1, \ldots, B_5$ be the e_1-colouring given in Lemma 4.3. Lemma 2.8 implies that for $i \in [5]$, B_i has a mate $\delta(U_i)$ with no bad triangles. For all $i \in [5]$ and $j \in [2]$, let g_i^j be an edge in $H^j \cap \delta(U_i)$ and let G_i^j be the unique (Lemma 2.10) face distinct from H with g_i^j. There is an edge $\hat{g}_i^j \in G_i \cap \delta(U_i)$ distinct from g_i^j. Because of $\delta(U_4)$ and $\delta(U_5)$ we may assume that $e_2 \in B_4$.

Claim: We have $|G_5^1|, |G_5^2| \geq 4$. Moreover for all $i \in [4]$, there is $j \in [2]$ such that $|G_i^j| \geq 4$ and $g_i^j, \hat{g}_i^j \in B_i$ unless one of the following statements (1) or (2) holds, where

(1) $\delta(U_i) = \delta(v)$ and $|F_{34}| = |F_{45}| = 3$;

(2) $\delta(U_i) = \delta(v')$ and $|F'_{34}| = |F'_{45}| = 3$.

Proof of claim: Note that $\delta(U_5) - \{e_1, e_2\} \subseteq B_5$. In particular, $g_5^1, \hat{g}_5^1, g_5^2, \hat{g}_5^2 \in B_5$. Hence, since $\delta(U_5)$ has no bad triangles, $|G_5^1|, |G_5^2| \geq 4$. For all $i \in [4]$, $\delta(U_i) - \{e_1, e_2\}$ is included in B_i except for a unique edge, say $f_i \in B_5$. Suppose that $|G_i^1|, |G_i^2| \leq 3$ we will show (1) or (2) holds. Suppose that for some $j \in [2]$, $f_i \neq g_i^j, \hat{g}_i^j$. Then the fact that $\delta(U_i)$ has no bad triangles implies that $|G_i^j| \geq 4$, a contradiction. Thus $f_i \in \{g_i^1, \hat{g}_i^1\} \cap \{g_i^2, \hat{g}_i^2\}$. It follows that $\hat{g}_i^1 = \hat{g}_i^2$ and thus that $\delta(U_i) = \{e_1, e_2, g_i^1, \hat{g}_i^1, g_i^2\}$. It follows from Lemma 2.1 that $\delta(U_i) = \delta(v)$ or $\delta(U_i) = \delta(v')$. We may assume that the former case occurs. Then $G_i^1 = F_{45}$ and $G_i^2 = F_{34}$. In particular we must have $|F_{34}|, |F_{45}| \leq 3$. Lemma 4.7 implies that $|F_{34}| = 3$ or $|F_{45}| = 3$. Suppose for a contradiction that $|F_{45}| = 2$. Since $B_1, \ldots, B_5$ are T-joins and since $e_1 \in (B_1 \cap B_2 \cap B_3) - B_4 - B_5$ and $e_2 \in B_4$, we must have $e_3, e_5 \in B_i$ and $e_4 \in B_5$. Let e be the edge in $F_{34} - \{e_3, e_4\}$. Then $C := \{e_3, e_5, e\}$ is a bad triangle for $\delta(U_i)$, a contradiction. ◇

Consider first the case where $|H_2| = 3$, i.e. $H_2 = \{e_2, e_3, e'_3\}$. Then $\delta(U_5)$ implies that one of e_3, e'_3, say e_3, is in B_5. Lemma 2.4(1) implies that e'_3 is in some B_s where $s \in [4]$. Let $i \in [5] - \{s\}$. To prove (a) it suffices to show that $|G_i^1| \geq 4$. Because of the Claim we may assume $i \neq 5$. Since $e'_3 \in B_s$, $e_3 \in \delta(U_i)$. It follows that $\delta(v') \neq \delta(U_i)$. Suppose $\delta(U_i) = \delta(v)$, then $e_4, e_5 \in B_i$. But since F_{45} is not a bad triangle of $\delta(U_i)$, $|F_{45}| \neq 3$. Thus neither statement (1) or (2) holds. Since $e_3 \in B_5 \cap \delta(U_i)$, it follows from the Claim that $|G_i^1| \geq 4$.

Suppose now $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$. Then statement (1) is not satisfied for any $i \in [4]$. It follows from the Claim that there exists at least three indices $i \in [4]$ for which $|G_i^j| \geq 4$ for some $j \in [2]$. Since $|G_5^1|, |G_5^2| \geq 4$ it implies $val(H_1) + val(H_2) \geq 5$, proving (b). Finally, suppose $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$ and; $|F'_{34}| \neq 3$ or $|F'_{45}| \neq 3$. Then neither statement (1) nor (2) is not satisfied for any $i \in [4]$. It follows from the Claim that for all $i \in [4]$, $|G_i^j| \geq 4$ for some $j \in [2]$. Since $|G_5^1|, |G_5^2| \geq 4$ it implies $val(H_1) + val(H_2) \geq 6$, proving (c). □

Lemma 4.9. *If H is a face with $|H| = 2$ then $\alpha_H \geq 4$.*

Proof. Let e_1, e_2 be the edges of H and let v, v' be the two ends of e_1, e_2.

Claim 1: We can assume that there is no edge e_3 such that $\{e_2, e_3\}$ is a face of $\mathcal{G}$.

Proof of claim: Suppose that there is such an edge e_3. Let H_1 (resp. H_3) be the face containing e_1 (resp. e_3) which is distinct from $\{e_1, e_2\}$ (resp. $\{e_2, e_3\}$). Lemma 4.6 implies that $\{e_1, e_2\}$ receives one charge from H_1 and that $|H_1|, |H_3| \geq 4$. Rules (V2) and (V5) imply that $\{e_1, e_2\}$ receives $1/2$ from both v and v'. Hence, $\alpha_{\{e_1,e_2\}} \geq 4$. ◇

Assume that the faces $NF(v), NF(v')$ and the edges $\delta(v), \delta(v')$ are as described in Lemma 4.8. Claim 1 implies that if $|H_2| \leq 3$ then $|H_2| = 3$. Moreover, Lemma 4.8(a) implies that if $|H_2| = 3$ then $|H_1| \geq 5$.

Claim 2: $\{e_1, e_2\}$ receives at least $1/2$ charge from both v and v'.

Proof of claim: We may assume $|NF_4^+(v)| \leq 2$ for otherwise rule (V2) implies that face $\{e_1, e_2\}$ receives $1/2$ charge from v. Consider first the case where $|H_2| = 3$, and hence $|H_1| \geq 5$. If $|NF_4^+(v)| = 1$ then rule (V3) implies that $\{e_1, e_2\}$ receives $1/2$ charge from v. Thus $|NF_4^+(v)| = 2$ and either $|F_{45}| \geq 4$ or $|F_{34}| \geq 4$. Then $\{e_1, e_2\}$ receives $1/2$ charge from v, in the former case because or rule (V4) and in the latter case because of rule (V7).

The remaining case is where $|H_1|, |H_2| \geq 4$. Since $|NF_4^+(v)| \leq 2$, $|F_{34}|, |F_{45}| \leq 3$. Lemma 4.7 implies that either $|F_{34}| \neq 2$ or $|F_{45}| \neq 2$. Thus either $|F_{34}| = |F_{45}| = 3$, or up to symmetry, $|F_{34}| = 2$ and $|F_{45}| = 3$. Then $\{e_1, e_2\}$ receives $1/2$ charge from v, in the former case because or rule (V6) and in the latter case because of rule (V7). ◇

Suppose $|H_2| = 3$. Then Lemma 4.8(a) implies that $val(H_1) \geq 4$ and $|H_1| \geq 5$. It follows from rule (F1) that $\{e_2, e_3\}$ receives one charge from H_1. Claim 2 implies that $\{e_1, e_2\}$ receives $1/2$ charge from v and by symmetry $1/2$ charge from v'. Hence, $\alpha_{\{e_1,e_2\}} \geq 4$. Hence, we will assume that $|H_1|, |H_2| \geq 4$. Up to symmetry, it suffices to consider the following cases.

Case 1: $|F_{34}| = |F_{45}| = |F'_{34}| = |F'_{45}| = 3$.

Then rule (V6) implies that $\{e_1, e_2\}$ receives one charge from both v and v'. Hence, $\alpha_{\{e_1,e_2\}} \geq 4$.

Case 2: $|F'_{34}| = |F'_{45}| = 3$. Moreover, $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$.

It follows from Lemma 4.8(b) that $val(H_1) + val(H_2) \geq 5$. Thus $val(H_j) \geq 3$ for some $j \in [2]$. Because of F'_{34}, F'_{45} we have $|H_j| \geq 5$. It follows from rule (F2) that $\{e_1, e_2\}$ receives $1/2$ charge from H_j. Rule (V6) implies that $\{e_1, e_2\}$ receives one charge from v'. Finally, Claim 2 implies $\{e_1, e_2\}$ receives $1/2$ charge from v. Hence, $\alpha_{\{e_1,e_2\}} \geq 4$.

Case 3: $|F_{34}| \neq 3$ or $|F_{45}| \neq 3$. Moreover, $|F'_{34}| \neq 3$ or $|F'_{45}| \neq 3$.

Lemma 4.8(c) implies that $val(H_1) + val(H_2) \geq 6$. Suppose $val(H_j) \geq 4$ for some $j \in [2]$. Then rule (F1) implies that $\{e_1, e_2\}$ receives one charge from H_j. Finally, Claim 2 implies that $\{e_1, e_2\}$ receives at least $1/2$ charge from both v and v'. Hence, $\alpha_{\{e_1,e_2\}} \geq 4$. Thus we can assume that $val(H_1) = val(H_2) = 3$. Suppose $|F_{34}| \leq 3$ and $|F_{45}| \leq 3$. Then $|H_1| \geq 5$ and $|H_2| \geq 5$. Rules (F1) or (F2) imply that face $\{e_1, e_2\}$ receives at least $1/2$ from both H_1, H_2. Together with Claim 2 this implies that $\alpha_{\{e_1,e_2\}} \geq 4$. Hence, we may assume, $|F_{34}| \geq 4$ or $|F_{45}| \geq 4$. Similarly we have $|F'_{34}| \geq 4$ or $|F'_{45}| \geq 4$.

Suppose $|F_{34}| \geq 4$ and $|F_{45}| \geq 4$. Then rule (V1) implies that $\{e_1, e_2\}$ receives one charge from v. If $|F'_{34}| \geq 4$ and $|F'_{45}| \geq 4$ then rule (V1) implies that $\{e_1, e_2\}$ receives one charge from v. But then, $\alpha_{\{e_1,e_2\}} \geq 4$. Thus we may assume (after possibly relabeling), that $|F'_{34}| \leq 3$. It follows that $|H_2| \geq 5$ and that rule (F2) implies that $\{e_1, e_2\}$ receives $1/2$ charge from H_2. But then Claim 2 implies that $\{e_1, e_2\}$ receives $1/2$ charge from v' and $\alpha_{\{e_1,e_2\}} \geq 4$.

Thus we can assume that exactly one of F_{34}, F_{45} has size ≥ 4 and that exactly one of F'_{34}, F'_{45} has size ≥ 4. Consider first the case where $|F_{34}| \leq 3$ and $|F'_{34}| \leq 3$. Then (F2) implies that $\{e_1, e_2\}$ receives 1 charge from H_2 and Claim 2 implies that $\alpha_{\{e_1,e_2\}} \geq 4$. Consider now the case where $|F_{34}| \leq 3$ and $|F'_{45}| \leq 3$. It follows that $|H_1| \geq 5$ and $|H_2| \geq 5$. Thus rule (F2) implies that face $\{e_1, e_2\}$ receives at least $1/2$ from both H_1, H_2. Together with Claim 2 this implies that $\alpha_{\{e_1,e_2\}} \geq 4$. □

Let $B_1, \ldots, B_5$ be an e-colouring and let i_1, i_2, i_3, i_4, i_5 be distinct elements of $[5]$. The plane dual of $\mathcal{G}$ is denoted $\mathcal{G}^*$. Consider C a circuit of $\mathcal{G}^*$ included in $B_{i_1} \cup B_{i_2} \cup B_{i_3}$. Then C partitions the plane into two regions, say $\mathcal{R}, \mathcal{R}'$. Let E_{i_4} (resp. E_{i_5}) be the set of edges of B_{i_4} (resp. B_{i_5}) with both ends in $\mathcal{R}$. Define, $B'_{i_4} := B_{i_4} \triangle E_{i_4} \triangle E_{i_5}$ and $B'_{i_5} := B_{i_5} \triangle E_{i_4} \triangle E_{i_5}$. Since B_{i_4} and B_{i_5} are T-joins, $B_{i_4} \triangle B_{i_5}$ is an Eulerian subgraph which can be decomposed into a collection $\mathcal{F}$ of edge disjoint circuits. Consider any circuit $C \in \mathcal{F}$. Since $\delta(U) \subseteq B_{i_1} \cup B_{i_2} \cup B_{i_3}$, if $C \in \mathcal{F}$ has an edge in common with $\delta(U)$ then it must be e. Because $\mathcal{G}$ is planar it follows that C is either entirely inside $\mathcal{R}$ or entirely inside $\mathcal{R}'$. Let $C_1, \ldots, C_q$ be the circuits of $\mathcal{F}$ inside $\mathcal{R}$. Then $B'_{i_4} = B_{i_4} \triangle C_1 \triangle \ldots \triangle C_q$ and $B'_{i_5} = B_{i_5} \triangle C_1 \triangle \ldots \triangle C_q$. Hence, in particular B'_{i_4}, B'_{i_5} are T-joins and $B_1, B_2, B_3, B'_{i_4}, B'_{i_5}$ is an e-colouring. We will say that this new e-colouring is obtained by *swapping colours* B_{i_4}, B_{i_5} *inside region* $\mathcal{R}$. Let $\mathcal{R}$ be a region and suppose that a new e-colouring is obtained by sequentially swapping colours inside regions which are all contained in $\mathcal{R}$. Then we say that the new e-colouring is obtained by a *changes which are local to* $\mathcal{R}$.

Lemma 4.10. *Let F be a face with $|F| = 3$ then either $\alpha_F \geq 4$ or for all $H \in N(F)$, $|H| \geq 4$.*

Proof. If some face adjacent to F has only two edges then Lemma 4.6 implies that F receives one charge from another adjacent face. But then $\alpha_F \geq 4$ thus we may assume all faces adjacent to F have at least three edges. Let e, e_1, e_2 be the edges of F and suppose that there is a face F' with edges e, e'_1, e'_2. For $j = 1, 2$ let H_j (resp. H'_j) be the face distinct from F (resp. F') with e_j. We can assume that

$$(*) \qquad val(H_1), val(H_2) \leq 3.$$

For otherwise $val(H_j) \geq 4$ for some $j \in [2]$ and rule (F1) implies that F receives one charge from H_j, so in particular $\alpha_F \geq 4$. Let $\mathcal{G}^*$ denote the plane dual of $\mathcal{G}$. $\mathcal{G}^*$ contains the following edges,

$$e = FF' \quad e_1 = FH_1 \quad e_2 = FH_2 \quad e'_1 = F'H'_1 \quad e'_2 = F'H'_2.$$

For $j = 1, 2$ let $K_1^j, \ldots, K_{t_j}^j$ be the vertices of $\mathcal{G}^*$ adjacent to H_j, but distinct from F, which correspond to faces of $\mathcal{G}$ with at least four edges. Equation (*) states that $t_1, t_2 \leq 3$. Consider an e-colouring $B_1, \ldots, B_5$. Lemma 4.3 implies that $e \in (B_1 \cup B_2 \cup B_3) - B_4 - B_5$. We call an H_jH_j'-path P in $\mathcal{G}^*$ a *j-link* if $P \subseteq B_i$ for some $i \in [3]$ and $H_jK_l^j$ is an edge of P for some $l \in \{1, \ldots, t_j\}$.

Claim 1:

(a) Internal vertices of j-links, $j = 1, 2$, are distinct from F, F'.

(b) 1-links and 2-links are vertex disjoint.

Proof of claim: Suppose F is a vertex of a j-link P. Then P and e_j contain a circuit C of $\mathcal{G}^*$ which does not intersect all of $B_1, \ldots, B_5$ with the same parity. As C is a cut of $\mathcal{G}$ it contradicts Remark 2.9. Suppose a 1-links P_1 and a 2-link P_2 have an internal vertex in common. Then P_1, P_2, e_1, e_2 contain a circuit C. We have $P_1 \subseteq B_i, P_2 \subseteq B_{i'}$ for some $i, i' \in [3]$. Since C must intersect all of $B_1, \ldots, B_5$ with the same parity we must have $\{e_1, e_2\} \in B_i \cup B_{i'}$. But then neither B_4 nor B_5 has a mate, a contradiction. $\diamond$

Claim 1(a) implies that given an j-link P, (P, e_j, e, e_j') forms a circuit of $\mathcal{G}^*$ which partitions the plane into two regions. Denote by $\mathcal{R}(P)$ the region which does not contain H_2. Denote by $\mathcal{K}(P)$ the vertices among $\{K_1^j, \ldots, K_{t_j}^j\}$ which are in (but possibly on the boundary of) the region $\mathcal{R}(P)$. A j-link P is *extreme* if we cannot make changes local to $\mathcal{R}(P)$ to obtain a j-link P' where $\mathcal{K}(P') \supset \mathcal{K}(P)$.

Claim 2: Let $j \in [2]$. Suppose $e_j \in B_4$ and $e_j' \in B_5$. Let P be an extreme j-link, where $P \subseteq B_i$ for some $i \in [3]$. Consider any $i' \in [3] - \{i\}$. Then after changes which are local to $\mathcal{R}(P)$ there is no j-link included in $B_{i'}$.

Proof of claim: Let $K_1, \ldots, K_s$ be the vertices of $\mathcal{G}^*$ in $\mathcal{K}(P)$. Note that because P is extreme, when make changes local to $\mathcal{R}(P)$ we will not create a new j-link using an edge H_jK where K is a vertex not in $\mathcal{K}(P)$. We may assume that H_1K_1 is an edge of P. (*) implies $s \leq 3$. Let i'' be the element in $[3]-\{i,i'\}$. Suppose $s\le 2$. If $H_jK_2\notin B_{i'}$ we are done. Otherwise, swap colours $B_{i'},B_{i''}$ inside $\mathcal{R}(P)$ to reduce it to that case. Thus we can assume $s=3$. If $H_jK_2,H_jK_3\notin B_{i'}$ then we are done. Since we can swap colours $B_{i'},B_{i''}$ inside $\mathcal{R}(P)$ we can assume $H_jK_2\in B_{i'}$ and $H_jK_3\in B_{i''}$. Moreover, may assume there is a path $P_2\subseteq B_{i'}$ using H_jK_2 from H_j to some vertex of $\mathcal{G}^*$ (possibly H'_j) in P (otherwise we are done). Since we can swap colours $B_{i'},B_{i''}$ inside $\mathcal{R}(P)$ we can also assume there is a path $P_3\subseteq B_{i''}$ using H_jK_3 from H_j to some vertex on P. Let z be the first vertex common to P,P_3 starting from H_j which is distinct from H_j. Then the subpaths of P,P_3 between H_j and z define the boundary of a region $\mathcal{R}'$ which is contained in $\mathcal{R}(P)$. Since we can swap colours $B_{i'},B_{i''}$ inside $\mathcal{R}(P)$ we can assume that K_2 is inside $\mathcal{R}'$. Then swap colours $B_{i'}$ and B_4 inside $\mathcal{R}'$. ◇

Because of Lemma 4.5 we can assume that $e_1\in B_4$, $e'_1\in B_5$, $e_2\notin B_4$, and $e'_2\notin B_5$.

Claim 3: We can assume $e_2\in B_5$ and $e'_2\in B_4$.

Proof of claim: Consider first the case where $e_2\in B_i$ and $e'_2\in B_{i'}$ where $i,i'\in[3]$. Let $i''\in[3]-\{i,i'\}$ and let $L=\{i'',4,5\}$. Then for $l\in L$ mates of B_l, with no bad triangles, consist of edges e_1,e'_1,e and an H_1,H'_1-path $P_l\subseteq B_l$ containing an edge $H_1K^1_t$ for some $t\le t_1$. This implies that $t_1=3$ and vertices K^1_1,K^1_2,K^1_3 are in the paths $P_l, l\in L$. Let l_1,l_2,l_3 be distinct elements of L. Let z be the first vertex common to P_{l_1},P_{l_3} starting from H_1 which is distinct from H_1. Then the subpaths of P_{l_1},P_{l_3} between H_1 and z define the boundary of a region $\mathcal{R}'$ which does not contain F'. We may assume, (after possibly relabeling l_1,l_2,l_3) that the vertex in $\{K^1_1,K^1_2,K^1_3\}$ of P_{l_2} is in $\mathcal{R}'$. Then swap colours B_{l_2} and B_i inside that region. Claim 1(a) implies that e_2 and e'_2 remain in respectively, B_i and $B_{i'}$. Hence, for the resulting e-colouring, B_{l_2} has no mate with no bad triangles, a contradiction. Thus either, $e_2\in B_5$ or $e'_2\in B_4$. Assume the former case occurs as the latter can be dealt with similarly. We can assume that $e'_2\in B_i$ for some $i\in[3]$. Then

(†) For all $l\in[3]-\{i\}$, mates of B_l with no bad triangles
consist of edges e_1,e'_1,e and a 1-link included in B_l.

Hence, there is an extreme 1-link P included in $B_{i'}$ for some $i' \in [3]$. We claim that statement (†) remains true after changes local to $\mathcal{R}(P)$. If vertex H_2' of $\mathcal{G}^*$ is not in $\mathcal{R}(P)$ then (†) follows from the fact that e_2' remains in B_i. If vertex H_2' is inside $\mathcal{R}(P)$, then (†) follows from Claim 1(b). Let $i'' \in [3] - \{i, i'\}$. Claim 2 states that there are changes local to $\mathcal{R}(P)$ such that afterwards there does not exist a 1-link included in $B_{i''}$. But (†) implies that $B_{i''}$ does not have a mate with no bad triangles, a contradiction. ◇

Consider first the case where there is no j-link for some $j \in [2]$. Then for all $i \in [3]$ there must be a $(3-j)$-link included in B_i. Let P be an extreme $(3-j)$-link included in some $B_i, i \in [3]$. Following a similar argument as in the previous claim, it can be shown that there still won't be any j-link after changes local to $\mathcal{R}(P)$. Choose $i' \in [3] - \{i\}$. Claim 2 implies that after changes local to $\mathcal{R}(P)$ there does not exist a $(3-j)$-link included in $B_{i'}$. But then $B_{i'}$ has no mate with no bad triangle, a contradiction. Otherwise for $j = 1, 2$ let P_j be an extreme link and let $i_j \in [3]$ be such that $P_j \subseteq B_{i_j}$. Claim 1(b) implies that P_1 and P_2 do not intersect. It follows that for $j = 1, 2$, changes local to $\mathcal{R}(P_j)$ do not change the e-colouring inside $\mathcal{R}(P_{3-j})$ and will not change the fact that P_{3-j} is extreme (since otherwise the new extreme $(3-j)$-link would intersect P_j). Let i'' be the element in $[3] - \{i_1, i_2\}$. Claim 2 implies that after changes local to $\mathcal{R}(P_1)$ and $\mathcal{R}(P_2)$ there does not exist, for $j = 1, 2$, a j-link, included in $B_{i''}$. It follows that $B_{i''}$ does not have a mate with no bad triangles, a contradiction. □

Lemma 4.11. *Let H be a face with $|H| = 3$ then $\alpha_H \geq 4$.*

Proof. Let e_1, e_2, e_3 be the edges of H. For each $i \in [3]$ let F_i be the face containing e_i which is distinct from H. Lemma 4.10 implies that $|F_i| \geq 4$ for all $i \in [3]$. Let v_{12} (resp. v_{23}; v_{31}) be the common end of e_1, e_2 (resp. e_2, e_3; e_3, e_1). We will show that face H receives $1/2$ charge from v_{12} (and by symmetry from v_{13} and v_{23}). Since $d(v_{12}) = 5$ there are two faces F, F' so that $NF(v_{12}) = (F_1, H, F_2, F, F')$. We can assume that $|F| \neq 2$ or that $|F'| \neq 2$ for otherwise there exists three edges incident to v_{12} which are parallel and Lemma 4.6 implies that H receives one charge from F_1 and from F_2. We can assume that $|F|, |F'| \leq 3$, for otherwise rules (V1) or (V2), would imply that H receives at least $1/2$ charge from v_{12}. If $|F| = |F'| = 3$ then (V6) implies H receives one charge from v_{12}. If either $|F| = 2$ or $|F'| = 2$ then (V7) implies that H receives $1/2$ charge from v_{12}. □

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