PACKING ODD CIRCUIT COVERS: A CONJECTURE

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ABSTRACT. We conjecture that the length of the shortest odd circuit in a signed graph is equal to the maximum number of pairwise disjoint odd circuit covers, as long as some necessary parity condition is satisfied and as long as the signed graph does not contain $\widetilde{K}_5$ as a signed minor.

1. THE CONJECTURE

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq EG$. A subset of edges B is said to be *odd* (resp. even) if $|B \cap \Sigma|$ is odd (resp. even). Thus an edge e is *odd* if $e \in \Sigma$ and *even* otherwise. We think of the odd (resp. even) edges of (G, Σ) as having odd (resp. even) length. A set $\Gamma \subseteq EG$ is a *signature* of (G, Σ) if (G, Σ) and (G, Γ) have the same set of odd circuits. A signed graph is *bipartite* if it has no odd circuit. A set B of edges of (G, Σ) is an (odd circuit) *cover* if every odd circuit of (G, Σ) contains at least one edge of B.

Given a signed graph we define two parameters. Denote by $\tau(G, \Sigma)$ the odd girth of (G, Σ), i.e. the length of the shortest odd circuit. Denote by $\nu(G, \Sigma)$ the maximum number of pairwise disjoint covers of (G, Σ). Since every odd circuit intersects every cover we must have,

$$\tau(G, \Sigma) \geq \nu(G, \Sigma). \tag{*}$$

We will say that (G, Σ) *packs* if equality holds in (*). We are interested in this note in signed graphs which pack.

The *double triangle* is the signed graph obtained from a triangle by replacing every edge by two parallel edges where one is odd and one is even. The odd girth of the double triangle is two but it does not contains two disjoint covers. For if it had disjoint covers B_1 and B_2 then each of B_1 and B_2 would correspond to a triangle. However, as there are an odd number of odd edges for some $i \in [2]$ we would have B_i odd. But then B_{3-i} would not be a cover, a contradiction. Thus the double triangle does not pack.

We now introduce a parity condition which will excludes examples such as the double triangle. A signed graph (G, Σ) is *consistent* if the parity of the length of every odd cycle of G is the same. Observe that the double triangle is not consistent since its has odd circuits of length two and of length three. A characterization of consistent signed graphs is given in the next proposition:

Proposition 1.1. *The following statements are equivalent for a non-bipartite signed graph* (G, Σ).

(1) (G, Σ) *is consistent,*
(2) *either* G *is bipartite or* EG *is a signature of* (G, Σ).

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Proof. Suppose that (1) holds. Let C be an even cycle of (G, Σ). Since (G, Σ) is not bipartite, there exists an odd circuit C'. Then $C \triangle C'$ and C' are both odd cycles (where $C \triangle C'$ denote the symmetric difference of C and C'). It follows that the parity of $|C'|$ and $|C \triangle C'|$ is the same. Thus $|C' \triangle (C \triangle C')| = |C|$ is even. If every odd circuit of (G, Σ) has even length then G is bipartite. Thus every odd circuit of (G, Σ) has odd length and every even circuit of (G, Σ) has even length, i.e. EG is a signature. Suppose that (2) holds. If G is bipartite then all cycles have even length, in particular so do all the odd cycles. If EG is a signature then every odd cycle has odd length since $|C \cap EG| = |C|$. □

We denote by $\widetilde{G}$ the signed graph (G, EG). Pr. 1.1 implies that $\widetilde{K}_5$ is consistent. We claim that $\widetilde{K}_5$ does not pack. Clearly the odd-girth of $\widetilde{K}_5$ is three. However, since every cover of $\widetilde{K}_5$ contains at least 4 edges and since K_5 has 10 edges, there are no three disjoint set of covers. We conjecture that every consistent signed graph which does not pack contains $\widetilde{K}_5$ as an "obstruction". We need to introduce the notion of minors to clarify what is meant by "obstruction".

We say that the signed graph (G, Σ) contains the signed graph (H, Γ) as a *signed minor*, if we can obtain (H, Γ) from (G, Σ) by a sequence of the following operations: (i) delete an edge (and remove it from the signature if it is present), (ii) contract an edge which is not in the signature, (iii) replace the signature Γ by another signature. We say that (G, Σ) is (H, Γ)-free if it does not contain (H, Γ) as a signed minor. Here is the main conjecture,

Main Conjecture: Consistent signed graphs which are $\widetilde{K}_5$-free, pack.

In Section 2 we point out that the Main Conjecture is a special case of a conjecture on binary clutters. We restate the Main Conjecture in Sections 3. Section 4 present special cases of the Main conjecture.

2. Cycling clutters

A *clutter* $\mathcal{H}$ is a finite family of sets, over some finite ground set $E\mathcal{H}$, with the property that no set of $\mathcal{H}$ contains, or is equal to, another set of $\mathcal{H}$. The *blocker* $b(\mathcal{H})$ of $\mathcal{H}$ is the clutter defined as follows: $Eb(\mathcal{H}) := E\mathcal{H}$ and $b(\mathcal{H})$ is the set of inclusion-wise minimal members of $\{B : B \cap C \neq \emptyset, \forall C \in \mathcal{H}\}$. It is well known that for a clutter, $\mathcal{H}$, $b(b(\mathcal{H})) = \mathcal{H}$. A clutter is said to be *binary* if, for any $C_1, C_2, C_3 \in \mathcal{H}$, their symmetric difference $C_1 \triangle C_2 \triangle C_3$ contains, or is equal to, a set of $\mathcal{H}$. Given $w \in Z_+^{E\mathcal{H}}$ we define the following two parameters:

$$\tau(\mathcal{H}, w) = \min\{w^T x : x(S) \geq 1 \; \forall S \in \mathcal{H}, x \in \{0,1\}^{E\mathcal{H}}\},$$
$$\nu(\mathcal{H}, w) = \max\{\sum_{S \in \mathcal{H}} y_S : \sum_{S : e \in S \in \mathcal{H}} y_S \leq w_e \; \forall e \in E\mathcal{H}, y \in Z_+^{\mathcal{H}}\}.$$

It follows from linear programming duality that,

$$\tau(\mathcal{H}, w) \geq \nu(\mathcal{H}, w). \tag{*}$$

We say that $\mathcal{H}$ *packs* with weights w if equality holds in (*). We say that weights $w \in Z_+^{E\mathcal{H}}$ are *Eulerian* if for all pairs $T, T' \in b(\mathcal{H})$, $w(T \triangle T')$ is even. Clutter $\mathcal{H}$ is *cycling* if it packs for all Eulerian weights.

Let $\mathcal{P}_{10}$ be the clutter whose ground set correspond to the Petersen graph and where elements of $\mathcal{P}_{10}$ correspond to the postman sets of the Petersen graph (i.e. sets of edges which induce a graph whose odd degree vertices correspond to the odd degree vertices of the Petersen graph). Let $\mathcal{O}_{K_5}$ denote the clutter, where $E\mathcal{O}_{K_5}$ corresponds to the edges of the complete graph K_5 and the elements of $\mathcal{O}_{K_5}$ are each of the odd circuits of K_5 (the triangles or the circuits of length five). The ground set of the clutter $\mathcal{L}_{F_7}$ are the elements of the Fano matroid and the sets in $\mathcal{L}_{F_7}$ are the circuits of length three (the lines) of the Fano matroid. It can be readily checked that none of $\mathcal{L}_{F_7}, \mathcal{O}_{K_5}, b(\mathcal{O}_{K_5}), \mathcal{P}_{10}$ are cycling.

Let $\mathcal{H}$ be a clutter and $i \in E\mathcal{H}$. The *contraction* $\mathcal{H}/i$ and *deletion* $\mathcal{H} \setminus i$ are clutters with ground set $E\mathcal{H} - \{i\}$ where: $\mathcal{H}/i$ is the set of inclusion-wise minimal members of $\{S - \{i\} : S \in \mathcal{H}\}$ and; $\mathcal{H} \setminus i := \{S : i \notin S \in \mathcal{H}\}$. Contractions and deletions can be performed sequentially, and the result does not depend on the order. A clutter obtained from $\mathcal{H}$ by a sequence of deletions and a sequence of contractions is called a *minor* of $\mathcal{H}$. It can be readily checked that if a clutter is cycling then so are all its minors.

Cycling conjecture:
A binary clutter is cycling if and only if it has none of the following minors: $\mathcal{L}_{F_7}, \mathcal{O}_{K_5}, b(\mathcal{O}_{K_5}), \mathcal{P}_{10}$.

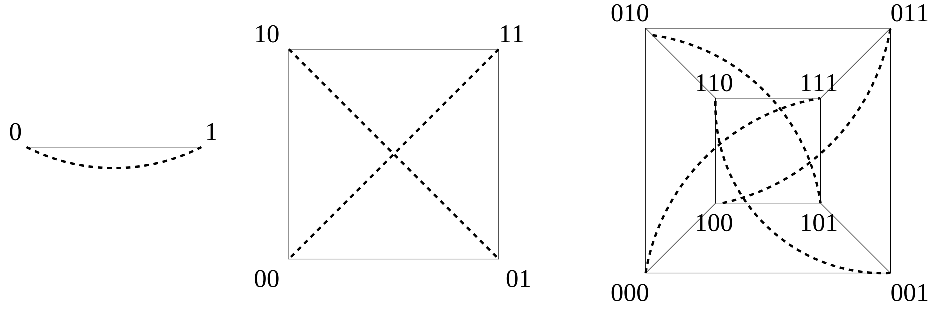
Cycling clutters where introduced by Seymour [9] who also proposed an excluded minor characterization for these clutters. However, his proposed characterization was not correct. The above conjecture can be found in [8].

Suppose the Cycling Conjecture holds. We will show that the Main Conjecture must hold as well. Consider a consistent signed graph (G, Σ) which is $\widetilde{K}_5$-free and let $\mathcal{H}$ be the *clutter of odd circuits* of (G, Σ), i.e. $E\mathcal{H} = EG$ and the elements of $\mathcal{H}$ are the odd circuits of (G, Σ). Then $b(\mathcal{H})$ is the *clutter of covers* of (G, Σ), i.e. the elements of $b(\mathcal{H})$ are the (inclusion-wise) minimal covers of (G, Σ). It can be readily checked that none of $\mathcal{L}_{F_7}, \mathcal{O}_{K_5}, \mathcal{P}_{10}$ are minors of clutters of covers. Moreover, since (G, Σ) is $\widetilde{K}_5$-free, $b(\mathcal{H})$ does not contain the minor $b(\mathcal{O}_{K_5})$. Consider any $S_1, S_2 \in \mathcal{H}$. Then S_1, S_2 are odd circuits of (G, Σ). Since (G, Σ) is consistent, the parities of $|S_1|$ and $|S_2|$ are the same. It follows that $|S_1 \triangle S_2| = |S_1| + |S_2| - 2|S_1 \cap S_2|$ is even. Hence, $\mathbf{1}$ (the vector of all ones) form Eulerian weights. It would follow from the Cycling Conjecture that $\tau(b(\mathcal{H}), \mathbf{1}) = \nu(b(\mathcal{H}), \mathbf{1})$. But $\tau(b(\mathcal{H}), \mathbf{1}) = \tau(G, \Sigma)$ and $\nu(b(\mathcal{H}), \mathbf{1}) = \nu(G, \Sigma)$, thus the Main Conjecture must hold.

3. Homeomorphism and vertex colouring

We will restate the Main Conjecture in terms of graph isomorphisms. Using this reformulation we will show in the subsequent section that the 4-colour theorem [1, 6] is a special case of the Main Conjecture.

Let (G, Σ) and (H, Γ) be signed graphs. We say that (G, Σ) is *homeomorphic* to (H, Γ) if for some suitable choice of signature of (G, Σ) there exists a mapping of the vertices of G to the vertices of H such that all odd edges of G get mapped to odd edges of H and all even edges of G get mapped to even edges of H. The *augmented hypercube of order* k is the signed graph with even edges corresponding to the hypercube of order k and odd edges connecting opposite points of the hypercube. In the next figure we draw the augmented hypercubes of dimensions one, two, and three (where solid edges are even and dashed edges are odd).



Proposition 3.1. *Let (G, Σ) be a signed graph and let $k \in Z_+$. Suppose k is even and G is bipartite or k is odd and EG is a signature. Then the following statements are equivalent,*

(1) *there exists k disjoint covers of (G, Σ),*
(2) *(G, Σ) is homeomorphic to the augmented hypercube of order $k - 1$.*

We will need the following easy observations,

Remark 3.2. The set $\Gamma \subseteq EG$ is a signature of (G, Σ) if and only if $\Gamma = \Sigma \triangle \delta(U)$ for some cut $\delta(U)$.

Remark 3.3. An (inclusion-wise) minimal cover is a signature.

Proof of Pr. 3.1. Suppose (1) holds. Re. 3.3 implies that there are k-disjoint signatures $B_1, \ldots, B_k$. We choose such signatures $B_1, \ldots, B_k$ such that $| \cup_{i=1}^k B_i|$ is maximized. Because the definition of homeomorphism allows us to choose the signature, we may assume that $B_1 = \Sigma$. Because of Re. 3.2 we can assume that for $i = 2, \ldots, k$, $B_i = \Sigma \triangle \delta(U_{i-1})$ for some cut $\delta(U_{i-1})$.

Claim. $\cup_{i=1}^k B_i = EG$.

Proof. Let $T = EG \triangle B_1 \triangle \ldots \triangle B_k$. We will first show T is a cut. Let $U = U_1 \triangle \ldots \triangle U_{k-1}$. Note that $\delta(U_1) \triangle \ldots \triangle \delta(U_{k-1}) = \delta(U)$. Suppose G is bipartite and k is even. Then $T = EG \triangle \delta(U)$. Since G is bipartite EG is a cut and so is T. Suppose EG is a signature and k is odd. Then $T = EG \triangle \Sigma \triangle \delta(U)$. Since Σ, EG are signatures $EG \triangle \Sigma$ is a cut (it intersects every odd and even circuits with even parity). Thus for both cases T is a cut. Define $B_1' := B_1 \triangle T = B_1 \cup T$. Re. 3.2 implies B_1' is a signature. Then $T = \emptyset$ for otherwise $B_1', B_2, \ldots, B_k$ contradicts the choice of $B_1, \ldots, B_k$. $\diamond$

The vertices of the augmented hypercube (H, Γ) of order $k-1$ are all the $0,1$-strings of length $k-1$. We map vertices of G to H as follows: if v is in U_i then digit i of v is 1 otherwise it is 0. Let $uv \in \Sigma$. Suppose for a contradiction, uv is not mapped to opposite points of (H, Γ). Then for some digit i, vertices u, v have the same value, i.e. either both $u, v \in U_i$ or both $u, v \in \bar{U}_i$, thus $uv \in \Sigma \triangle \delta(U_i)$. It follows that $uv \in \Sigma \cap (\Sigma \triangle \delta(U_i)) = B_1 \cap B_{i+1}$, a contradiction since B_1, B_{i+1} are disjoint. Let $uv \notin \Sigma$. Suppose for a contradiction both u, v are mapped to the same vertex of H. Then $uv \notin \delta(U_i)$ for any $i \in [k-1]$. It follows that $uv \notin B_1 \cup \ldots B_k$, contradicting the Claim. Suppose for a contradiction, uv is not mapped to adjacent point of (H, Γ). Then for a pair of digits i, j

vertices u and v have distinct values for both digits i and j. It follows that $uv \in \delta(U_i) \cap \delta(U_j)$. Since uv is even, $uv \in (\Sigma \triangle \delta(U_i)) \cap (\Sigma \triangle \delta(U_j)) = B_{i+1} \cap B_{j+1}$, a contradiction as B_{i+1}, B_{j+1} are disjoint.

Suppose (2) holds, i.e. (G, Σ) is homeomorphic to the augmented hypercube (H, Γ) of order $k-1$. Observe that all odd circuits of (G, Σ) are mapped into odd cycles of (H, Γ). Hence, it suffices to show that (H, Γ) has k disjoint covers. Let Σ be the set of edges connecting opposite points of H. For $i = 1, \ldots, k-1$, let U_i be the set of vertices of H where digit i is zero. Let $B_1 = \Sigma$ and $B_i = \Sigma \triangle \delta(U_{i-1})$ for $i = 2, \ldots, k$. Re. 3.2 implies that $B_1, \ldots, B_k$ are covers. Consider B_i for $i \in \{2, \ldots, k\}$. Let $uv \in \Sigma = B_1$. Since uv connects opposite points of H, $uv \in \delta(U_{i-1})$ and $uv \notin \Sigma \triangle \delta(U_{i-1}) = B_i$. Hence B_1, B_i disjoint. Consider B_i, B_j for $i, j \in \{2, \ldots, k\}$ and $i \neq j$. If $uv \in \Sigma$ then we showed already $uv \notin B_i, B_j$. Suppose $uv \notin \Sigma$ and $uv \in B_j$. Then $uv \in \delta(U_{j-1})$. Since uv connects adjacent points, $uv \notin \delta(U_{i-1})$. Then $uv \notin \Sigma \triangle \delta(U_{i-1}) = B_i$. Hence B_i, B_j are disjoint. □

We are ready to restate the Main Conjecture,

Main Conjecture (restated). *Consider a signed graph (G, Σ) where either $\Sigma = EG$ or G is bipartite. Suppose (G, Σ) is $\widetilde{K}_5$-free. Then (G, Σ) is homeomorphic to the augmented hypercube of order $k-1$ where k denotes the odd girth of (G, Σ).*

Pr. 1.1 implies that (G, Σ) is consistent. The Main Conjecture would imply that $k = \tau(G, \Sigma) = \nu(G, \Sigma)$. Hence, there exists k disjoint covers in (G, Σ) and the result follows from Pr. 3.1.

4. Special cases

We present special cases of the Main Conjecture in this section.

4.1. **Vertex colouring.** We say that G contains H as an *odd minor* if H can be obtained from G by first deleting edges and then contracting *every* edge on some cut. A graph G is *odd-K_5-free* if it does not contain K_5 as an odd minor. Clearly if a graph is K_5-free it is odd-K_5-free. However the converse is not true in general as the graph obtained from K_5 by replacing a single edge by two series edges illustrates. Bert Gerards [2] conjectured the following generalization of the Four-Colour Theorem which is now a theorem [4].

Theorem 4.1. *Odd-K_5-free graphs are 4-colourable.*

We claim that the Main Conjecture implies the previous theorem. Consider a simple graph G which is odd-K_5-free. It can be readily checked that the signed graph $\widetilde{G}$ is $\widetilde{K}_5$-free. Choose $k = 3$. Then the restated version of the Main Conjecture would imply that G is homeomorphic to K_4, hence that G is 4-colourable. Note that if G is bipartite and $\tau(G, \Sigma) = 2$ then the restated version of the Main Conjecture is trivial since being homeomorphic to the augmented hypercube of order 1 is the same as G being bipartite. Thus the first interesting case for the case where G is bipartite is $k = 4$.

4.2. **Edge colouring and planar graphs.** A graph G is a *k-graph*, if G is k-regular, and for every cut $\delta(U)$ such that $|U|$ is odd, $|\delta(U)| \geq k$. An *edge colouring* of a graph G is an assignment of colours to the edges of G. A colouring is *proper* if edges incident to the same vertex are assigned different colours. We say that G can be *k-edge-coloured* if there exists a proper edge-colouring of G with k colours. Seymour [5] proposed the following conjecture.

Conjecture 4.2. *Planar k-graphs are k-edge-colourable.*

Proposition 4.3. *The Main Conjecture implies Conjecture 4.2.*

Proof. Let G be a plane k-graph. If k is even, then let Σ be a perfect matching of G (such a matching must exist since G is a k-graph). If k is odd, then let $\Sigma = EG$. This implies that in both cases, for every cut $\delta(U)$ of G, $|\delta(U) \cap \Sigma|$ is odd if and only if $|U|$ is odd. Let G^* be the plane dual of G. The odd cycles of (G^*, Σ) correspond to cuts $\delta(U)$ of G where $|\delta(U)|$ is odd. Since G is a k graph, $|\delta(U)| \geq k$. Hence, the odd girth of (G^*, Σ) is k. If k is odd then $\Sigma = EG$ and if k is even then G^* is bipartite (as G is Eulerian). It follows from Pr. 1.1 that (G^*, Σ) is consistent. Since G^* is planar it has no K_5 minors. In particular (G^*, Σ) is $\widetilde{K}_5$-free. If the Main Conjecture holds, there exists disjoint covers $B_1, \ldots, B_k$ of (G^*, Σ). Since $B_1, \ldots, B_k$ are disjoint and since B_i intersect every cut $\delta(v)$ of G, it follows that each $B_1, \ldots, B_k$ are perfect matchings of G. Hence G is k-edge-colourable. □

4.3. **Quasi orders and bounds.** We say that a graph G is homeomorphic to H if there is a mapping from VG to VH such that edges of G are mapped to edges of H. Given graphs G, H we write $G \preceq H$ to indicate that G is homeomorphic to H. Then the relation $\preceq$ defines a quasi order (a reflective and transitive binary relation) on the set of all graphs. We say that a graph H is a *bound* for a class of graphs $\mathcal{G}$ it for every graph G in $\mathcal{G}$ we have $G \preceq H$. Naserasr [7] conjectured that,

Conjecture 4.4. *The class of planar graphs with girth $2k+1$ is bounded by the augmented hypercube of order $2k$.*

In [7] the statement is given in terms of a Cayley graph but it can be readily checked that this is the same graph as the augmented hypercube. Suppose the (restated) Main Conjecture holds for $\widetilde{G}$ where G has odd girth $2k+1$. Since G is planar, $\widetilde{G}$ is $\widetilde{K}_5$-free. Thus $\tilde{G}$, and hence G, is isomorphic to the augmented hypercube of order $2k$. Since this holds for every planar graph G with odd girth $2k+1$, Conjecture 4.4 must hold.

Naserasr [7] showed that Conjecture 4.4 for k is equivalent to Conjecture 4.2 for $r = 2k+1$. (This equivalence follows also from the proof of Pr. 4.3 and the equivalence between the two formulations of the Main Conjecture.) Guenin [3] proved Conjecture 4.2 for $r = 4, 5$, hence Conjecture 4.4 holds for $k = 2$ (the case $k = 1$ is the 4-colour theorem).

4.4. **The 2-commodity cut theorem.** Consider a graph G with pairs (s_1, t_1) and (s_2, t_2) of vertices. A *2-commodity-cut* is a set $B \subseteq EG$ such that $G \setminus B$ has no s_1t_1-path and no s_2t_2-paths. Seymour [10] proved the following result,

Theorem 4.5. *For a bipartite graph G, the length of the shortest path among all s_1t_1- and s_2t_2-paths is equal to the maximum number of pairwise disjoint 2-commodity cuts.*

Proposition 4.6. *The Main Conjecture implies Theorem 4.5*

Proof. For $i = 1, 2$ let l_i denote the length of the shortest s_it_i-path. We may assume $l_1 = l_2$, for if say $l_1 < l_2$ then add to the graph a path from a new vertex s_1' to s_1 of length $l_2 - l_1$ and prove the result for the pairs (s_1', t_1) and (s_2, t_2). Let G' be the graph obtained from G by adding edges s_1t_1 and s_2t_2. Let Γ be a signature of $(G', \{s_1t_1, s_2t_2\})$ which avoids both edges s_1t_1, s_2t_2. Let H be the graph obtained from G' by contracting edges s_1t_1, s_2t_2 and denote by r_1, r_2 the vertices of H corresponding to respectively s_1t_1, s_2t_2.

Claim 1. (H, Γ) is consistent.

Proof. Every odd circuit of (H, Γ) corresponds to an s_it_i-path of G. Since G is bipartite for $i \in [2]$ all s_it_i-paths of G have the same parity length. Since $l_1 = l_2$ the s_1t_1- and s_2t_2-paths of G have the same parity length. It follows that all odd circuits of (H, Γ) have the same parity length. $\diamond$

Claim 2. (H, Γ) is $\widetilde{K}_5$-free.

Proof. By construction vertices r_1, r_2 intersect all odd circuits. It follows that every minor of (H, Γ) will have a pair of vertices which intersect all odd circuits. However, $\widetilde{K}_5$ does not have such a pair. $\diamond$

If the Main Conjecture holds, then there exists $l_1 = l_2$ disjoint covers of (H, Γ). Since there is a one-to-one correspondence between odd circuits of (H, Γ) and s_it_i-paths of G, each cover is a 2-commodity-cut of G. $\square$

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