Stabilizer theorems for even cut matroids

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Abstract

Even cut matroids are binary lifts of cographic matroids, i.e. they are binary matroids whose matrix representation may be obtained by adding a row to the matrix representation of a cographic matroid. A more tangible way of representing an even cut matroid is via a grafts, that is, a graph together with a distinguished set of vertices.

In pathological cases, even cut matroids can have an arbitrary number of inequivalent graft representations (for a suitable definition of equivalence). However, we prove that if an even cut matroid is suitably connected and contains a minor with certain properties, then the number of its inequivalent representations is bounded by the number of inequivalent representations of the minor. In particular, we prove two stabilizer theorems, for two types of minors and different types of connectivity (namely 3-connected and connected matroids). For instance, we deduce that any connected even cut matroid which contains R_{10} as a minor has at most 10 inequivalent representations.

These results are used in a forthcoming paper to provide a polynomial time algorithm for the classes of even cycle and even cut matroids. We also believe that such results will be valuable tools for the determination of the excluded minors for the class of even cut matroids.

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1 Introduction

The non-standard terms used here are defined in the next section; at this point we aim to introduce the main results of the paper, leaving precise definitions for later. An *even cut* matroid is a binary lift of a cographic matroid, i.e. it is a binary matroid whose binary matrix representation may be obtained by adding a row to the binary matrix representation of a cographic matroid. To any such matrix representation one may associate a *graft*, i.e. a graph together with an even number of distinguished vertices (where the underlying graph is one representing the cographic matroid). Grafts provide a tangible way of dealing with even cut matroids. A given even cut matroid may be the lift of several different cographic matroids, each producing a different graft. Grafts arising from different graph representations of the same cographic matroid will be called *equivalent*, while grafts arising from different cographic matroids are inequivalent. In Section 2.3 we show that an even cut matroid may have an exponential number of inequivalent representations. However, if an even cut matroid M contains a minor N with certain properties, then the number of inequivalent representations of M is bounded in terms of the number of inequivalent representations of N (independently of how large M is). This is expressed in the two main results of the paper, where we employ the terms "non-degenerate" and "substantial" to encompass properties of the minor N which will be described precisely in the next section.

Theorem. *Let M be a 3-connected even cut matroid which contains as a minor a non-degenerate 3-connected matroid N. Then the number of equivalence classes of representations of M is at most twice the number of equivalence classes of representations of N.*

Theorem. *Let M be a connected even cut matroid which contains as a minor a connected matroid N that is substantial. Then the number of equivalence classes of representations of M is at most the number of equivalence classes of representations of N.*

We will see that matroid R_{10} (which figures prominently in Seymour's decomposition of regular matroids [11]) is substantial and has, up to equivalence, 10 graft representations. Thus, every connected even cut matroid containing R_{10} as a minor has, up to equivalence, at most 10 graft representations.

Even cut matroids are a natural class of matroids to study as they are the smallest minor closed class of binary matroids which contains all single element co-extensions of cographic matroids. Robertson and Seymour [9] proved that, for every infinite set of graphs, one of its members is isomorphic to a minor of another. Geelen, Gerards, and Whittle announced that an analogous result holds for matroids representable over a finite field. Hence, any minor closed class of binary matroids can be characterized by a finite set of excluded minors. In particular, this is the case for even cut matroids. Tutte [12] gave an explicit description of the excluded minors for the class of graphic matroids. By duality, this immediately provides an explicit description of the excluded minors for the class of cographic matroids.

No explicit description of the excluded minors is known for even cut matroids. The difficulty for this problem lies with the fact that we do not have a sufficient understanding of the graft representations of even cut matroids. Theorems 7 and 9 are a first step toward a better understanding of this problem. Eventually, we wish to extend the aforementioned theorems so as to have a compact description of the representations of arbitrary even cut matroids. We believe that there exists a constant k such that every even cut matroid with more than k inequivalent representations is constructed in a way analogous to that of the Shuffle described in Section 2.3.

Another typical problem for a minor-closed class of matroids is that of recognition. Tutte gave a polynomial time algorithm to check if a binary matroid (given by its binary matrix representation) is graphic [13], and by duality this leads to a recognition algorithm for cographic matroids. Unfortunately, not many efficient recognition algorithms are known for classes of matroids. The main results of this paper were recently used in [3] to provide a polynomial time recognition algorithm for even cycle and even cut matroids (even cycle matroids are binary lifts of graphic matroids).

2 Preliminaries

We assume that the reader is familiar with the basics of matroid theory. Unless otherwise stated, we follow the notation in Oxley [6]. We only consider binary matroids in this paper. Thus the reader should substitute the term "binary matroid" every time "matroid" appears in this text.

2.1 Graphs, grafts and even cut matroids

We will consider graphs with multiple edges and loops, but no isolated vertices. Let G be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident to an edge of X and $G[X]$ for the subgraph with vertex set $V_G(X)$ and edge set X. We denote by $V_{odd}(G)$ the set of vertices of G of odd degree. Given $X \subseteq E(G)$, we write $\mathcal{B}_G(X)$ for $V_G(X) \cap V_G(\bar{X})$, where $\bar{X} = E(G) - X$. (For any pair of sets A and B, $A - B$ denotes set difference, i.e. $A - B = \{a \in A : a \notin B\}$ and we write $A - b$ as shorthand for $A - \{b\}$). Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $\mathcal{B}(X)$ for $\mathcal{B}_G(X)$.

Next we define cycles and paths in a slightly non-standard way. A set of edges P of G is a *path* if either P is empty or $G[P]$ is connected and has two vertices of degree one and all other vertices of degree two. The two vertices of degree one are the *ends* of P. If P is a path and $a, b \in V(P)$, then $P(a, b)$ denotes the subpath of P between vertices a and b.

A subset C of edges of G is a *cycle* if $G[C]$ is a graph where every vertex has even degree (this is sometimes called an Eulerian subgraph). A *cycle* in a matroid is any symmetric difference of circuits; equivalently, for a binary matroid a cycle is a (possibly empty) disjoint union of circuits (see Theorem 9.1.2 in [6]). Thus the cycles of a graph G are the cycles of the graphic matroid of G.

Let G be a graph. Given a set of vertices U, we denote by $\delta_G(U)$ the *cut* induced by U, that is $\delta_G(U) := \{uv \in E(G) : u \in U, v \notin U\}$. A *bridge* is a single edge forming a cut. We denote by $\mathrm{cut}(G)$ the set of all cuts of G. Since the cuts of G correspond to the cycles of the *cographic matroid* of G, we identify $\mathrm{cut}(G)$ with that matroid and say that G is a *representation* of that matroid.

A *graft* is a pair (G,T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. The vertices in T are the *terminals* of the graft. A cut $\delta(U)$ is T*-even* (respectively T*-odd*) if $|T \cap U|$ is even (respectively odd). When there is no ambiguity we omit the prefix T when referring to T-even and T-odd cuts. We denote by $\text{ecut}(G,T)$ the set of all even cuts of (G,T). It can be verified that $\text{ecut}(G,T)$ is the set of cycles of a binary matroid, which is called the *even cut matroid* represented by (G,T). We identify $\text{ecut}(G,T)$ with that matroid and say that (G,T) is a *representation* of that matroid. When simply referring to a *representation* of an even cut matroid we will always mean a graft representation. Occasionally we will mention the *matrix representation* of an even cut matroid, meaning a binary matrix representing the matroid (as discussed later in this section). Observe that since $\text{cut}(G) = \text{ecut}(G,\emptyset)$, every cographic matroid is an even cut matroid. Even cut matroids are discussed with some detail in [8].

Given a graft (G,T) we say that $J \subseteq E(G)$ is a T*-join* of G if $T = V_{odd}(G[J])$. It is easy to see that if J is a T-join of G, then a cut C of G is T-even if and only if $|C \cap J|$ is even. If J_1 and J_2 are T-joins of G then the symmetric difference $J_1 \triangle J_2$ is a cycle of G_1. Thus every T-join of G may be generated, via symmetric difference, from a fixed T-join and a the cycles of G.

Given a graft (G,T), we say that an edge e of G is a *pin* if e is an odd bridge of G incident to a vertex of degree one, which we call the *head* of the pin. By definition the head of a pin is a terminal. We denote by $\text{pin}(G,T)$ the set of pins of (G,T). Note that every T-join of G contains every pin of (G,T).

Next we explain how to get a binary matrix representation of an even cut matroid. Let (G,T) be a graft and J a T-join of G. Let $A(G)$ be a binary matrix whose rows span the cycle space of G (i.e. $A(G)$ is a binary matrix representation of the cographic matroid of G). Let S be the transpose of the incidence vector of J; hence S is a row vector indexed by $E(G)$ and S_e is 1 if $e \in J$ and 0 otherwise. Construct a matrix A from $A(G)$ by adding row S. Let $M(A)$ be the binary matroid represented by A. and let C be a cycle of $M(A)$. Then C intersects every cycle of G and also set J with even parity. The sets that intersect every cycle of G with even parity are exactly the cuts of G. Moreover, a cut intersects J with even parity if and only if it is T-even. Thus $M(A) = \text{ecut}(G,T)$.

A matroid M is a *lift* of a matroid N if there exists a matroid M', with an element e such that $M = M' \backslash e$ and $N = M'/e$. Even cut matroids are lifts of cographic matroids: in the matrix construction above one may add a column e with a one in row S and zero everywhere else to get a matrix representation of matroid M' in the definition of lifts.

Consider a graft (G,T). What is a basis for $\text{ecut}(G,T)$? A set $F \subseteq E(G)$ is dependent if and only if it contains an even cut. Hence, if (G,T) does not contain any odd cut (i.e. T is empty) then $\text{ecut}(G,T) = \text{cut}(G)$ and a basis is just formed by the complement of a spanning tree. If (G,T) contains at least one odd cut, every basis for $\text{ecut}(G,T)$ is formed by the complement $\bar{B}$ of a spanning tree B together with an edge $f \in B$ forming an odd cut with edges in $\bar{B}$.

A cocycle of a matroid is The co-cycle space of $\text{ecut}(G,T)$ is the space spanned by the rows of A, where A is the binary matrix representation of $\text{ecut}(G,T)$. Note that the symmetric difference of a cycle and a T-join is a T-join. From the above construction we have the following (see also [4]).

Remark 1. *The cocycles of* $\text{ecut}(G,T)$ *are the cycles of* G *and the* T*-joins of* (G,T)*.*

We define minor operations on grafts as follows. Let (G,T) be a graft and let $e \in E(G)$. Then $(G,T) \setminus e$ is defined as $(G \setminus e, T')$, where $T' = \emptyset$ if e is an odd bridge of (G,T) and $T' = T$ otherwise. Note that the even cuts of $(G,T) \setminus e$ are either even cuts of (G,T) not using e or even cuts of (G,T) with the element e removed.

We define $(G,T)/e$ as $(G/e, T')$, where T' is defined as follows. Let u and v be the ends of e in G and let w be the vertex obtained by contracting e. If $x \neq w$, then $x \in T'$ if and only if $x \in T$; $w \in T'$ if and only if $|\{u,v\} \cap T| = 1$. With this definition we have that

Remark 2. $\text{ecut}(G,T)/C \setminus D = \text{ecut}\big((G,T)/D \setminus C\big)$.

In particular, this implies that being an even cut matroid is a minor closed property.

2.2 Representations of cographic matroids are nice

We will state a theorem by Whitney that shows, for a cographic matroid, how to construct the set of all its representations (as graphs) from a single representation. We require a number of definitions.

Suppose that $\mathcal{B}_G(X) = \{u_1, u_2\}$ for some $u_1, u_2 \in V(G)$ and $X \subset E(G)$. Let G' be the graph obtained by identifying vertices u_1 and u_2 of $G[X]$ with vertices u_2 and u_1 of $G[\bar{X}]$ respectively. Then G' is obtained from G by a *Whitney-flip* on X. We will also call Whitney-flip the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of G. We define two graphs to be *equivalent* if one can be obtained from the other by a sequence of Whitney-flips (it is easy to verify that this does indeed define an equivalence relation).

In a seminal paper [15], Whitney proved the following.

Theorem 3. *All representations of a cographic matroid are equivalent.*

It follows in particular that, if a cographic matroid is 3-connected, then it has a unique representation.

2.3 Representations of even cut matroids are naughty

The situation is considerably more complicated for even cut matroids than for cographic matroids, as we will illustrate in this section.

We say that two grafts (G_1, T_1) and (G_2, T_2) are *equivalent* if G_1 and G_2 are equivalent and a T_1-join of G_1 is a T_2-join of G_2. If G_1 and G_2 are equivalent, then $\text{cycle}(G_1) = \text{cycle}(G_2)$. Thus, if one T_1-join of G_1 is a T_2-join of G_2, then every T_1-join of G_1 is a T_2-join of G_2, so, by Remark 1 $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ whenever (G_1, T_1) and (G_2, T_2) are equivalent. Conversely, if G_1 and T_2 are equivalent and $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$, then every T_1-join of G_1 is a T_2-join of G_2, so (G_1, T_1) and (G_2, T_2) are equivalent. We summarise this in the following remark.

Remark 4. *Let G_1 and G_2 be equivalent graphs. Then for two sets of terminals T_1 for G_1 and T_2 for G_2 we have* $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ *if and only if* (G_1, T_1) *and* (G_2, T_2) *are equivalent.*

Equivalence of grafts does indeed define an equivalence relation. It follows that we can partition the representations of any even cut matroid N into equivalence classes $\mathcal{R}_1, \ldots, \mathcal{R}_k$. We will say that $\mathcal{R}_i$ $(i \in [k])$ [1] is an *equivalence class of representations of* N. There is no direct analogue to Whitney's theorem for even cut matroids, as the following result illustrates.

Remark 5. *For any integer k, there exists an even cut matroid M with $|E(M)| \leq 6k$ and 4^{k-1} equivalence classes of representations.*

We now describe a general operation to construct a matroid as in the previous result. Let (G, T) be a graft with $T = \{a, b, c, d\}$, for distinct vertices a, b, c, d. Let $X \subseteq E(G)$ with $\mathcal{B}_G(X) \subseteq T$. Construct a graph G' from $G[X]$ and $G[\bar{X}]$ by the following vertex identifications:

- identify vertex a of $G[X]$ with vertex b of $G[\bar{X}]$, producing vertex a';
- identify vertex b of $G[X]$ with vertex a of $G[\bar{X}]$, producing vertex b';
- identify vertex c of $G[X]$ with vertex d of $G[\bar{X}]$, producing vertex c';
- identify vertex d of $G[X]$ with vertex c of $G[\bar{X}]$, producing vertex d'.

Let $T' = \{a', b', c', d'\}$. We say that the grafts (G, T) and (G', T') are related by a *shuffle move* on X with *pairing* a, b. One may check that every cycle and every T-join of G corresponds to either a cycle or a T'-join of G' and vice versa. As cycles and T-joins of (G, T) form the co-cycles of $\text{ecut}(G, T)$ (by Remark 1) this implies that $\text{ecut}(G, T) = \text{ecut}(G', T')$. A shuffle move is illustrated in Figure 1: white vertices are terminals and dotted lines denote vertices that are identified.

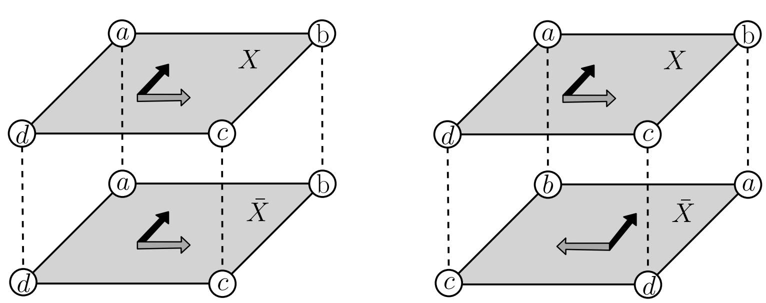

Figure 1: A shuffle move.

Using sequences of shuffle moves we can construct many inequivalent grafts representing the same even cut matroid. Suppose (G, T) is a graft with $|T| = 4$ and $E(G)$ partitions into

[1] $[k] = \{1, \ldots, k\}$

sets $X_1, \ldots, X_k$, with $\mathcal{B}_G(X_i) = T$ for every $i \in [k]$. Then, for every $i \neq 1$, we may apply a shuffle move on X_i or not. For a shuffle move we have three different choices of pairings. This leads to four choices for every $i \neq 1$, thus 4^{k-1} representations of M, which will be pairwise inequivalent as long as each X_i is sufficiently connected. For example, we may choose each X_i to be a copy of the graph K_4, producing a graph with $6k$ edges and 4^{k-1} equivalence classes of representations, as required.

The example we just constructed shows that if a graft (G, T) has only four terminals, then $\mathrm{ecut}(G, T)$ may have many inequivalent representations. One may wonder if having more than four terminals forces the representation to be unique, up to equivalence. Unfortunately, this is not the case, as stated in the following remark.

Remark 6. *For every integer k, there exists a graft (G, T) with the property that:*

(1) every graft equivalent to (G, T) has at least k terminals, and

(2) $\mathrm{ecut}(G, T)$ has at least two inequivalent representations.

An example of a construction for Remark 6 with $k = 12$ is given in Figure 2: (a) and (b) are

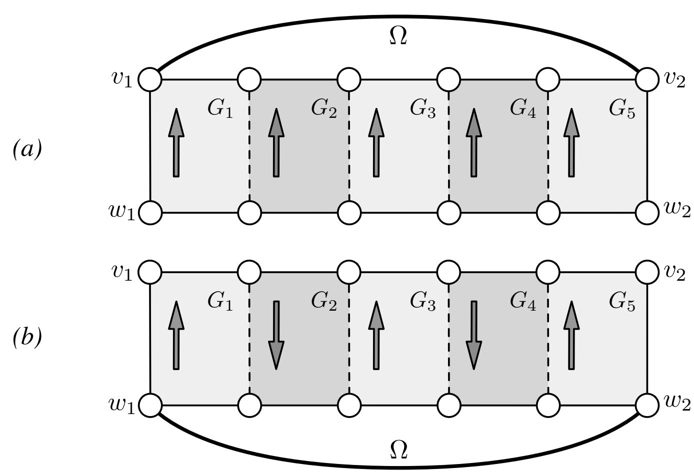

Figure 2: $|T| \geq 6$ and inequivalent representations

non-equivalent representations of the same even cut matroid. White vertices are terminals, dashed lines represent 2-separations. Each of the graphs $G_1, \ldots, G_5$ may be any graph. The arrows indicate how each piece is flipped. Note that every cycle and every T-join of (a) corresponds to either a cycle or a T-join of (b) and vice versa. As cycles and T-joins form the co-cycles of even cut matroids, this implies that (a) and (b) represent the same even cut matroid. This operation generalizes to any number of graphs $G_1, \ldots, G_r$ in which case we obtain $2r + 2$ terminals in both (a) and (b). This proves Remark 6. We will see that this construction is a special example of the clip operation defined in section 4.2. It is also possible to extend this construction to graphs that are 4-connected.

2.4 Main results

2.4.1 Non-degenerate minors

We say that a graft (G,T) is *degenerate* if there is a graft (G',T') equivalent to (G,T) with $|T'| \leq 4$. It follows from the definition that if a graft is degenerate then so are all grafts equivalent to it. We say that an equivalence class of representations of an even cut matroid is non-degenerate if all grafts in the class are non-degenerate.

An even cut matroid M is *degenerate* if some representation (G,T) of M is degenerate, it is non-degenerate otherwise. If a graft has less than six terminals, then so do all of its minors, thus being degenerate is a minor closed property. If a matroid is cographic, then it has a representation $(G,\emptyset)$ as an even cut matroid, hence it is degenerate.

An example of an even cut matroid which is non-degenerate is given by the matroid R_{10} (introduced in [11]). R_{10} has, up to equivalence, 10 representations as an even cut matroid, which are all isomorphic to the graft in Figure 3. (As always, terminals are represented by white vertices.) Hence R_{10} is a non-degenerate even cut matroid.

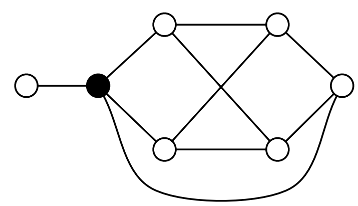

Figure 3: Graft representation of R_{10}.

We are now ready to restate the first main result of the paper.

Theorem 7. *Let M be a 3-connected even cut matroid which contains as a minor a non-degenerate 3-connected matroid N. Then the number of equivalence classes of M is at most twice the number of equivalence classes of N.*

This result implies, in particular, that every 3-connected even cut matroid containing R_{10} as a minor has, up to equivalence, at most 20 representations. We will strengthen this result in Section 2.4.2.

We will show that degenerate even cut matroids are "close" to being cographic matroids. We require a number of definitions to formalize this notion.

Let N and M be matroids where $E(N) = E(M)$. Then N is a *lift* of M if, for some matroid M' where $E(M') = E(M) \cup \{\Omega\}$, $M = M'/\Omega$ and $N = M' \setminus \Omega$. If N is a lift of M then M is a *projection* of N. Lifts and projections were introduced in [2]. As discussed in Section 2.1, every even cut matroid N is a lift of a cographic matroid; indeed, for any representation (G,T) of N we may construct a graft (G',T') by adding an odd bridge Ω. Then $N = \text{ecut}(G,T) = \text{ecut}((G',T')/\Omega) = \text{ecut}(G',T')\backslash\Omega$ and $\text{ecut}(G',T')/\Omega = \text{ecut}((G',T')\backslash\Omega) = \text{ecut}(G,\emptyset)$ is a cographic matroid. The following result shows that degenerate even cut matroids are also projections of cographic matroids.

Remark 8. *Let (H,S) be a graft.*

(1) If $|S| \le 2$*, then* $\mathrm{ecut}(H,S)$ *is a cographic matroid.*

(2) If $|S| = 4$*, then* $\mathrm{ecut}(H,S)$ *is a projection of a cographic matroid.*

Proof. **(1)** The result is obvious if $S = \emptyset$. Now suppose that $|S| = 2$. Let G be obtained from H by identifying the two vertices in S. Then $\mathrm{cut}(G) = \mathrm{ecut}(H,S)$. **(2)** Suppose that $|S| = 4$. Let G be obtained from H by adding an edge $\Omega = (s_1, s_2)$, for distinct $s_1, s_2 \in S$. Let $M' = \mathrm{ecut}(G,T)$. Then by construction $(G,T) \setminus \Omega = (H,S)$, hence $M'/\Omega = M$. By definition of the minor operations on grafts, $(G,T)/\Omega$ has two terminals. Therefore, by (1), $\mathrm{ecut}((G,T)/\Omega) = M' \setminus \Omega$ is a cographic matroid. □

2.4.2 Substantial minors

Consider a graft (G,T) and suppose that there exist graphs G_1 and G_2 equivalent to G and paths P_1 and P_2 in G_1 and G_2 respectively, such that $T = V_{odd}(G[P_1 \triangle P_2])$. We call the pair (G_1,P_1) and (G_2,P_2) a *reaching pair* for (G,T). If (G,T) is degenerate, then there exist (possibly empty) paths P_1 and P_2 in G such that $T = V_{odd}(G[P_1 \triangle P_2])$; hence (G,P_1) and (G,P_2) form a reaching pair for (G,T). It follows that having no reaching pair is a stronger property than being non-degenerate. An example of a reaching pair is given in Figure 4. Here $G_1 = G$, while G_2 is obtained from G by two Whitney flips, one on each 2-separation. Path P_1 is dashed, while path P_2 is dotted.

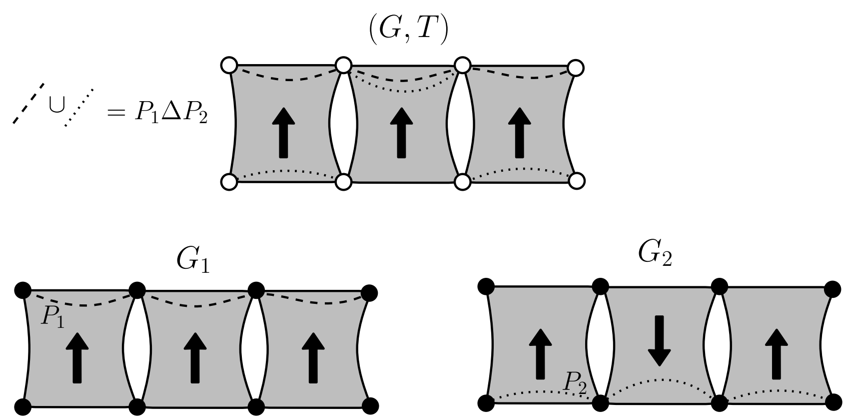

Figure 4: A graft with a reaching pair.

It follows from the definition that a reaching pair for a graft (G,T) is also a reaching pair for all grafts that are equivalent to (G,T). We say that an equivalence class of representations $\mathcal{R}$ has no reaching pair if none of the grafts in $\mathcal{R}$ have a reaching pair.

An even cut matroid is *substantial* if none of its representations has a reaching pair. Hence, if an even cut matroid is degenerate, it is not substantial. In particular, substantial matroids are not cographic. We will see (in Lemma 17) that not being substantial is also a minor closed property.

It follows immediately from the definition that an even cut matroid M is substantial if, for every representation (G,T) of M, the graph G is 3-connected and $|T| \ge 6$. Recall that every representation of R_{10} is isomorphic to the graft in Figure 3 and the representations of

R_{10} partition into 10 equivalence classes. The graft obtained by contracting the bridge in the graft in Figure 3 is 3-connected and has six terminals, hence it has no reaching pair; it follows that no representation of R_{10} has a reaching pair, hence R_{10} is a substantial even cut matroid.

We are now ready to present the second main result of the paper.

Theorem 9. *Let M be a connected even cut matroid which contains as a minor a connected matroid N that is substantial. Then the number of equivalence classes of M is at most the number of equivalence classes of N.*

This result implies, in particular, that every connected even cut matroid containing R_{10} as a minor has, up to equivalence, at most 10 representations.

2.5 Organization of the paper

Section 3 introduces generalizations of Theorems 7 and 9. Sections 3.3 and 3.4 contain the proof of these theorems, modulo the exclusion of three key lemmas (namely Lemmas 16, 18 and 19). Lemma 16 is proved in Section 3.2. Lemma 18 is proved in Section 4. In Section 5 we give a characterization of special pairs of representations of the same matroid. This characterization is used to prove Lemma 19, in Section 6.

3 The proofs (modulo the exclusion of two lemmas)

If N is a minor of a matroid M then M is a *major* of N. Consider an even cut matroid M with a representation (G, T). Let I and J be disjoint subsets of $E(M)$ and let $N := M \setminus I/J$. Then $(H, S) := (G, T)/I \setminus J$ is a representation of N. We say that (G, T) is an *extension* to M of the representation (H, S) of N, or alternatively that (H, S) *extends* to M.

The following result implies Theorem 7.

Theorem 10. *Let N be a 3-connected non-degenerate even cut matroid. Let M be a 3-connected major of N. For every equivalence class of representations $\mathcal{R}$ of N, the set of extensions of $\mathcal{R}$ to M is the union of at most two equivalence classes of representations.*

The"at most two" in the previous theorem is best possible. Consider for instance the example in Figure 2. Observe that the grafts obtained from (a) and (b) by deleting the edge Ω are equivalent. However, (a) and (b) are not equivalent.

The following result implies Theorem 9.

Theorem 11. *Let N be a connected substantial even cut matroid. Let M be a connected major of N. For every equivalence class of representations $\mathcal{R}$ of N, the set of extensions of $\mathcal{R}$ to M is contained in one equivalence class of representations.*

The proofs of Theorems 10 and 11 are constructive. Thus, given a description of the inequivalent representations of N, it is possible to construct the set of all inequivalent representations of M.

3.1 Definitions

First an easy observation.

Remark 12. *If G_1 and G_2 are equivalent graphs and I and J are disjoint subsets of $E(G)$, then $G_1 \setminus I/J$ and $G_2 \setminus I/J$ are equivalent.*

Proof. Since G_1 and G_2 are equivalent, $\text{cut}(G_1) = \text{cut}(G_2)$. Hence, $\text{cut}(G_1)/I\backslash J = \text{cut}(G_2)/I\backslash J$. As the minor operations on graphs and matroids commute, we have, $\text{cut}(G_1 \setminus I/J) = \text{cut}(G_2 \setminus I/J)$. The result now follows from Theorem 3. □

Consider a matroid M and let N be a minor of M. If $N = M\backslash e$ for some element e, then M is a *column major* of N. If $N = M/e$ for some element e, then M is a *row major* of N. A set $\mathcal{R}$ of representations of an even cut matroid is *closed under equivalence* if, for every $(H, S) \in \mathcal{R}$ and (H', S') equivalent to (H, S), we have that $(H', S') \in \mathcal{R}$.

Remark 13. *Let $\mathcal{R}$ be a set of representations of an even cut matroid N and let M be a major of N. If $\mathcal{R}$ is closed under equivalence, then so is the set $\mathcal{R}'$ of extensions of $\mathcal{R}$ to M.*

Proof. Let $(G, T) \in \mathcal{R}'$ and let (G', T') be equivalent to (G, T). We have $N = M \setminus D/C$ for some $D, C \subseteq E(M)$. Moreover, $(H, S) := (G, T)/D \setminus C$ and $(H', S') := (G', T')/D \setminus C$ are equivalent (see Remark 12). Since $(G, T) \in \mathcal{R}'$, we have $(H, S) \in \mathcal{R}$. As $\mathcal{R}$ is closed under equivalence, $(H', S') \in \mathcal{R}$. Hence, by definition, $(G', T') \in \mathcal{R}'$. □

Let $\mathcal{R}$ be an equivalence class of representations of an even cut matroid N. We say that $\mathcal{R}$ is *row stable* (resp. *column stable*) if for all row (resp. column) majors M of N, where M has no loop and no co-loop and M is not cographic, the set of extensions of $\mathcal{R}$ to M is an equivalence class.

We say that two grafts (G_1, T_1) and (G_2, T_2) are *siblings* if $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ and the graphs G_1 and G_2 are not equivalent.

3.2 Column stable equivalence classes

In this section we prove that every equivalence class of representations of an even cut matroid is column stable. The next result, proved in [4], is an easy consequence of Theorem 3.

Remark 14. *Suppose that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. If any odd cut of (G_1, T_1) is a cut of G_2, then G_1 and G_2 are equivalent.*

The following result is proved in [5]. We report the proof here for completeness.

Lemma 15. *Let (G_1, T_1) and (G_2, T_2) be graft siblings and let $\Omega \in E(G_1)$. For $i = 1, 2$, let $(H_i, S_i) := (G_i, T_i)/\Omega$. Suppose that (H_1, S_1) and (H_2, S_2) are equivalent. Then, for $i = 1, 2$, either Ω is a loop of G_i or $T_i = \{v_i, w_i\}$ and v_i and w_i are the ends of Ω in G_i. In particular, Ω is a co-loop of $\text{ecut}(G_1, T_1)$.*

Proof. For $i = 1, 2$, denote by v_i and w_i the endpoints of edge Ω in G_i. We prove the statement for $i = 1$. Since (G_1, T_1) and (G_2, T_2) are siblings, they are not equivalent, so Remark 14 implies that no odd cut of (G_1, T_1) is a cut of G_2. Since H_1 and H_2 are equivalent, $\text{cut}(H_1) = \text{cut}(H_2)$. It follows that all odd cuts of (G_1, T_1) use Ω. Hence, $T_1 \subseteq \{v_1, w_1\}$. Similarly, we have that $T_2 \subseteq \{v_2, w_2\}$. If Ω is a loop of G_1, we are done. Suppose otherwise. If $T_1 = \emptyset$, then there exists an even cut of (G_1, T_1) using Ω; hence Ω is not a loop of G_2 and $T_2 \neq \{v_2, w_2\}$. But then $T_1 = T_2 = \emptyset$ and $\text{cut}(G_1) = \text{cut}(G_2)$ and it follows by Theorem 3 that G_1 and G_2 are equivalent, a contradiction. We conclude that $T_1 = \{v_1, w_1\}$, completing the proof. □

Lemma 16. *Every equivalence class of representations of an even cut matroid is column stable.*

Proof. Let $\mathcal{R}$ be an equivalence class of representations of an even cut matroid N. Let M be a column major of N, i.e. for some $\Omega \in E(M)$, $N = M \setminus \Omega$. Suppose moreover that M has no loops or co-loops. Let $\mathcal{R}'$ be the set of all extensions of $\mathcal{R}$ to M. We need to show that $\mathcal{R}'$ is an equivalence class. For otherwise there exist siblings $(G_1, T_1), (G_2, T_2) \in \mathcal{R}'$. For $i = 1, 2$, let $(H_i, S_i) := (G_i, T_i)/\Omega$. Then $(H_1, S_1), (H_2, S_2) \in \mathcal{R}$. In particular, (H_1, S_1) and (H_2, S_2) are equivalent. Hence, by Lemma 15, Ω is a loop or co-loop of $\text{ecut}(G_1, T_1)$, a contradiction. □

3.3 A sketch of the proof of Theorem 11

The following implies that if a matroid is not substantial then neither are any of its minors.

Lemma 17. *If (G, T) has a reaching pair, so does every minor (H, S) of (G, T).*

Proof. Since (G, T) has a reaching pair, there exist, for $i = 1, 2$, a graph G_i equivalent to G and a path P_i in G_i such that $T = V_{odd}(G[P_1 \triangle P_2])$. By induction, it suffices to prove the statement for the cases $(H, S) = (G, T) \setminus e$ and $(H, S) = (G, T)/e$, for some $e \in E(G)$.

First, suppose that $(H, S) = (G, T)/e$. For $i = 1, 2$, define $H_i := G_i/e$ and let Q_i be the (possibly empty) path in H_i obtained by removing all the cycles from $H_i[P_i - e]$. As H and H_i are equivalent, every cycle of H_i is a cycle of H, hence $V_{odd}(H[P_i - e]) = V_{odd}(H[Q_i])$. As $P_1 \triangle P_2$ is a T-join of G, $(P_1 \triangle P_2) - e$ is an S-join of H. Hence $S = V_{odd}(H[Q_1 \triangle Q_2])$, and the statement follows.

Now suppose that $(H, S) = (G, T) \setminus e$. If e is an odd bridge of G, then S is empty and the statement is trivially true (taking as reaching pair $(H, \emptyset), (H, \emptyset)$). If e is not an odd bridge of (G, T), then $S = T$. If e is an even bridge of (G, T), then $(G, T)/e$ is equivalent to $(G, T) \setminus e$ (joining the two components of $G \setminus e$ on the endpoints of e is a Whitney-flip). It follows (by the first part of the proof) that (G, T) has a reaching pair. Thus we may assume that e is not a bridge of G.

For $i = 1, 2$, let v_i and w_i be the ends of P_i in G_i and $H_i := G_i \setminus e$. Let Q_i be a (v_i, w_i)-path in H_i (Q_i exists, as e is not a bridge of G, hence e is not a bridge of G_i). Then $P_i \triangle Q_i$ is a cycle of G_i, hence a cycle of G. It follows that $V_{odd}(G[P_i]) = V_{odd}(G[Q_i])$, for $i = 1, 2$. Therefore $T = V_{odd}(H[Q_1 \triangle Q_2])$ and (H_1, Q_1), (H_2, Q_2) is a reaching pair for (H, T).

□

We postpone the proof of the following result until Section 4.

Lemma 18. *Equivalence classes of representations without reaching pairs are row stable.*

Proof of Theorem 11. Let N be a connected non-degenerate even cut matroid. Let M be a connected major of N. Then (see [1, 10]) there is a sequence of connected matroids $N_1, \dots, N_k$, where $N \cong N_1$, $M = N_k$ and, for all $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loops or co-loops, for any $i \in [k]$. Since N_1 is substantial, it is not cographic, hence neither are $N_2, \dots, N_k$. Let $\mathcal{R}$ be an equivalence class of N which extends to M and, for every $j \in [k]$, define $\mathcal{R}_j$ to be the set of extensions of $\mathcal{R}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathcal{R}_j$ is an equivalence class. Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. By Lemma 17, $\mathcal{R}_j$ does not have a reaching pair. Therefore, by Lemma 16 and Lemma 18, $\mathcal{R}_j$ is row and column stable. It follows that $\mathcal{R}_{j+1}$ is an equivalence class. □

3.4 A sketch of the proof of Theorem 10

We postpone the proof of the following result until Section 6.

Lemma 19. *Let N be an even cut matroid and let $\mathcal{R}$ be an equivalence class of representations of N that is non-degenerate. Let M be a row major of N. Suppose that N and M are 3-connected and suppose that the set $\mathcal{R}'$ of extensions of $\mathcal{R}$ to M is non-empty. Then $\mathcal{R}'$ is either an equivalence class of representations or the union of two equivalence classes of representations $\mathcal{R}_1$ and $\mathcal{R}_2$, each without reaching pairs.*

Proof of Theorem 10. Let N be a 3-connected non-degenerate even cut matroid. Let M be a 3-connected major of N. It follows (by [11]) that there is a sequence of 3-connected matroids $N_1, \dots, N_k$, where $N \cong N_1$, $M = N_k$ and, for every $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loops or co-loops for any $i \in [k]$. Since N_1 is non-degenerate, it is not cographic, hence neither are $N_2, \dots, N_k$. Let $\mathcal{R}$ be an equivalence class of N that extends to M. For every $j \in [k]$, define $\mathcal{R}_j$ to be the set of extensions of $\mathcal{R}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathcal{R}_j$ is either

(a) an equivalence class, or

(b) the union of two equivalence classes without reaching pairs.

Let us proceed by induction. As $N_1 \cong N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. Consider the case where N_{j+1} is a column major of N_j. If (a) holds for $\mathcal{R}_j$, then Lemma 16 implies that (a) holds for $\mathcal{R}_{j+1}$. If (b) holds for $\mathcal{R}_j$, then Lemma 16 and Lemma 17 imply that either (a) or (b) holds for $\mathcal{R}_{j+1}$. Consider the case where N_{j+1} is a row major of N_j. If (a) holds for $\mathcal{R}_j$, then Lemma 19 implies that either (a) or (b) holds for $\mathcal{R}_{j+1}$. If (b) holds for $\mathcal{R}_j$, then Lemma 18 implies that either of (a) or (b) holds for $\mathcal{R}_{j+1}$. □

The rest of the paper is devoted to prove Lemmas 18 and 19. Lemma 18 is proved in Section 4, while Lemma 19 is proved in Section 6.

4 Row extensions and reaching pairs

Before we proceed with the proof of Lemma 18 we establish some preliminaries in Sections 4.1 and 4.2.

4.1 Even cycle matroids

Given a graph G, we denote by $\mathrm{cycle}(G)$ the set of all cycles of G. Since the cycles of G correspond to the cycles of the *graphic matroid* of G, we identify $\mathrm{cycle}(G)$ with that matroid and say that G is a *representation* of that matroid. A *signed graph* is a pair (G, Σ) where $\Sigma \subseteq E(G)$. We call Σ a *signature* of G. A subset $B \subseteq E(G)$ is Σ-*even* (respectively Σ-*odd*) if $|B \cap \Sigma|$ is even (respectively odd). When there is no ambiguity we omit the prefix Σ when referring to Σ-even and Σ-odd sets. Given a signed graph (G, Σ), we denote by $\mathrm{ecycle}(G, \Sigma)$ the set of all even cycles of (G, Σ). It can be verified that $\mathrm{ecycle}(G, \Sigma)$ is the set of cycles of the *even cycle matroid*. We identify $\mathrm{ecycle}(G, \Sigma)$ with that matroid and say that (G, Σ) is a *representation* of that matroid.

Given a signed graph (G, Σ), we say that Σ' is obtained from Σ by a *signature exchange* if $\Sigma \triangle \Sigma'$ is a cut of G (where $\triangle$ denotes symmetric difference). Every set Σ' which may be obtained from Σ by a signature exchange is a *signature* of (G, Σ).

We will make repeated use of the following result (which is proved in [4]).

Theorem 20. *Let G_1 and G_2 be inequivalent graphs on the same edge set.*

(1) Suppose that there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$*. For $i = 1, 2$, if (G_i, Σ_i) has no Σ_i-odd cycle, define $C_i := \emptyset$; otherwise let C_i be an odd cycle of (G_i, Σ_i). Let $T_i := V_{odd}(G_i[C_{3-i}])$. Then* $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$*.*

(2) Suppose that there exists a pair $T_1 \subseteq V(G_1), T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$*. For $i = 1, 2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$; otherwise let $t_i \in T_i$ and $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then* $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$*.*

Moreover, if it exists, the pair Σ_1, Σ_2 is unique (up to signature exchange) and, if it exists, the pair T_1, T_2 is unique.

4.2 Clip siblings

We now introduce an operation on grafts which preserves even cuts. Consider a pair of equivalent graphs H_1 and H_2. Suppose that $P_i \subset E(H_i)$ is a path in H_i, for $i = 1, 2$. For $i = 1, 2$, let G_i be the graph obtained from H_i by adding an edge Ω with endpoints the ends of P_i. Since H_1 and H_2 are equivalent, Theorem 3 implies that $\mathrm{cut}(H_1) = \mathrm{cut}(H_2)$ and hence that $\mathrm{cycle}(H_1) = \mathrm{cycle}(H_2)$. Thus,

$$\mathrm{ecycle}(G_1, \{\Omega\}) = \mathrm{cycle}(H_1) = \mathrm{cycle}(H_2) = \mathrm{ecycle}(G_2, \{\Omega\}).$$

Theorem 20 implies that there exist $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$ such that $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$. If G_1 and G_2 are inequivalent, such pair is unique (again by Theorem 20); in this case we say that the tuple $\mathbb{T} = (H_1, P_1, H_2, P_2)$ is a *clip-template* and that (G_1, T_1) and (G_2, T_2) are *clip siblings* which *arise* from $\mathbb{T}$. An explicit characterization of clip siblings is given in Section 5. Such characterization is needed to prove Lemma 19.

Remark 21. *Let $\mathbb{T} = (H_1, P_1, H_2, P_2)$ be a clip-template and let (G_1, T_1) and (G_2, T_2) be clip siblings that arise from $\mathbb{T}$. Then, for $i = 1, 2$, we have $T_i = V_{odd}(G_i[P_1 \Delta P_2])$.*

Proof. As $P_i \cup \Omega$ is an odd cycle of $(G_i, \{\Omega\})$ for $i = 1, 2$, by Theorem 20 we have $T_i = V_{odd}(G_i[P_{3-i} \cup \Omega]) = V_{odd}(G_i[P_{3-i}]) \Delta V_{odd}(G_i[\Omega])$. As Ω and P_i have the same ends in G_i, we have $V_{odd}(G_i[\Omega]) = V_{odd}(G_i[P_i])$. It follows that $T_i = V_{odd}(G_i[P_{3-i}]) \Delta V_{odd}(G_i[P_i]) = V_{odd}(G_i[P_1 \Delta P_2])$. □

We illustrate this construction in Figure 5. The dashed lines indicate 2-separations. H_2 is obtained from H_1 by doing a Whitney flips on each of the 2-separations. White vertices represent terminal vertices T_1 in G_1 and T_2 in G_2.

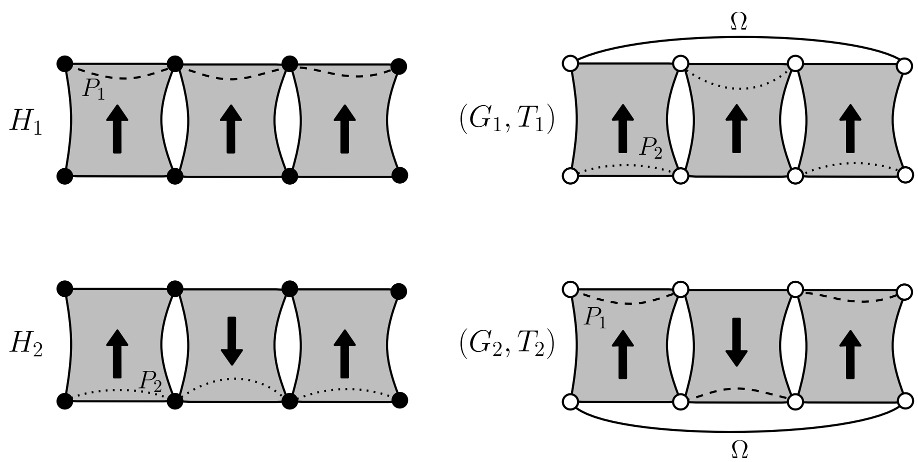

Figure 5: Clip siblings.

4.3 Proof Lemma 18

The following easy observation is the analogue to Remark 14 for the case of even cycle matroids (see [4] for a proof).

Remark 22. *Suppose that* $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. *If any odd cycle of (G_1, Σ_1) is a cycle of G_2, then G_1 and G_2 are equivalent.*

We define minor operations on signed graphs as follows. Let (G, Σ) be a signed graph and let $e \in E(G)$. Then $(G, \Sigma) \setminus e$ is defined as $(G \setminus e, \Sigma - \{e\})$. We define $(G, \Sigma)/e$ as $(G \setminus e, \emptyset)$ if e is an odd loop of (G, Σ) and as $(G \setminus e, \Sigma)$ if e is an even loop of (G, Σ); otherwise $(G, \Sigma)/e$ is equal to $(G/e, \Sigma')$, where Σ' is any signature of (G, Σ) which does not contain e. Observe that (see [8] for instance),

Remark 23. $\text{ecycle}(G, \Sigma) \setminus I/J = \text{ecycle}\big((G, \Sigma) \setminus I/J\big)$.

In particular, this implies that being an even cycle matroid is a minor closed property.

The following result is the analogue to Lemma 15 for even cycle matroids. This result is proved, for example, in [5]; we report the proof here for completeness.

Lemma 24. *Consider signed graphs (G_1, Σ_1) and (G_2, Σ_2) such that* $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$ *and G_1 and G_2 are inequivalent. Let $\Omega \in E(G_1)$. For $i = 1, 2$, let $(H_i, \Gamma_i) := (G_i, \Sigma_i) \setminus \Omega$. Suppose that H_1 and H_2 are equivalent. Then, for $i = 1, 2$, Ω is either a bridge of G_i or a signature of (G_i, Σ_i). In particular, Ω is a co-loop of* $\mathrm{ecycle}(G_1, \Sigma_1)$.

Proof. We prove the statement for $i = 1$. Remark 22 implies that no odd cycle of (G_1, Σ_1) is a cycle of G_2. Since H_1 and H_2 are equivalent, $\mathrm{cycle}(H_1) = \mathrm{cycle}(H_2)$. It follows that all odd cycles of (G_1, Σ_1) use Ω. Hence, after possibly a signature exchange, $\Sigma_1 \subseteq \{\Omega\}$. Similarly, we may assume that $\Sigma_2 \subseteq \{\Omega\}$. If Ω is a bridge of G_1, we are done. Suppose otherwise. If $\Sigma_1 = \emptyset$, then there exists an even cycle C of (G_1, Σ_1) using Ω; hence Ω is not a bridge of G_2 and $\Sigma_2 \neq \{\Omega\}$. But then $\Sigma_1 = \Sigma_2 = \emptyset$ and $\mathrm{cycle}(G_1) = \mathrm{cycle}(G_2)$, so $\mathrm{cut}(G_1) = \mathrm{cut}(G_2)$. It follows by Theorem 3 that G_1 and G_2 are equivalent, a contradiction. □

Lemma 25. *Let N be an even cut matroid that is not cographic and let $\mathcal{R}$ be an equivalence class of representations of N. Let M be a row major of N with no loops or co-loops. Suppose that the set $\mathcal{R}'$ of extensions of $\mathcal{R}$ to M is non-empty. Then $\mathcal{R}'$ is either an equivalence class or the union of two equivalence classes $\mathcal{R}_1$ and $\mathcal{R}_2$ and any $(G_1, T_1) \in \mathcal{R}_1$ and $(G_2, T_2) \in \mathcal{R}_2$ are clip siblings.*

Proof. We may assume that $\mathcal{R}'$ is not an equivalence class. Hence, there exist siblings (G_1, T_1) and (G_2, T_2) in $\mathcal{R}'$. Let Ω denote the unique element in $E(M) - E(N)$. Then $(G_1, T_1) \setminus \Omega$ and $(G_2, T_2) \setminus \Omega$ are in $\mathcal{R}$. By Theorem 20, there is a unique (up to resigning) pair of signatures Σ_1 and Σ_2 for G_1 and G_2 such that $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_2, \Sigma_2)$. For $i = 1, 2$, let $(H_i, \Gamma_i) = (G_i, \Sigma_i) \setminus \Omega$. As H_1 and H_2 are equivalent, Lemma 24 implies that, for $i = 1, 2$, either Ω is a bridge of G_i or a signature of (G_i, Σ_i). If the latter case occurs for both $i = 1$ and $i = 2$, then (G_1, T_1) and (G_2, T_2) are clip siblings and we are done. Now suppose that Ω is a bridge of G_i, for $i = 1$ or $i = 2$. Then every cycle of G_i is a cycle of H_i, hence a cycle of H_{3-i} (as H_1 and H_2 are equivalent). It follows that every cycle of G_i is a cycle of G_{3-i}. By Remark 22, every cycle of (G_i, Σ_i) is even. Therefore $\Sigma_i' = \emptyset$ is a signature of (G_i, Σ_i). By Theorem 20, T_{3-i} is empty and M is cographic, a contradiction.

It remains to show that $\mathcal{R}'$ can be partitioned into at most two equivalence classes. Suppose, for a contradiction, that this is not the case. Then there exist, for $i = 1, 2, 3$, $(G_i, T_i) \in \mathcal{R}'$, where G_1, G_2 and G_3 are pairwise inequivalent. For $i = 1, 2, 3$, let $\Sigma_i := \Omega$. It follows from the argument in the previous paragraph that $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_3, \Sigma_3)$. Similarly, we have $\mathrm{ecycle}(G_2, \Sigma_2) = \mathrm{ecycle}(G_3, \Sigma_3)$. For $i = 1, 2$, let C_i be an odd cycle of (G_i, Σ_i); note that, by the definition of Σ_i, we have $\Omega \in C_i$ for $i = 1, 2$. Theorem 20 applied to the pair G_1, G_3 implies that $T_3 = V_{odd}(C_1)$. Similarly, Theorem 20 applied to the pair G_2, G_3 implies that $T_3 = V_{odd}(C_2)$. Therefore $V_{odd}(G_3[C_1 \Delta C_2]) = T_3 \Delta T_3 = \emptyset$, i.e. $C_1 \Delta C_2$ is a cycle of G_3. Moreover $\Omega \notin C_1 \Delta C_2$, as $\Omega \in C_1 \cap C_2$. By definition $\Sigma_3 = \{\Omega\}$, so $C_1 \Delta C_2$ is an even cycle of (G_3, Σ_3). As $\mathrm{ecycle}(G_1, \Sigma_1) = \mathrm{ecycle}(G_3, \Sigma_3)$, the set $C_1 \Delta C_2$ is a cycle of G_1. Hence $C_2 = (C_1 \Delta C_2) \Delta C_1$ is a cycle of G_1. Remark 22 implies that G_1 and G_2 are equivalent, a contradiction. □

We are now ready for the main result of this section,

Proof of Lemma 18. Let $\mathcal{R}$ be an equivalence class of representations of an even cut matroid N without reaching pairs. Let M be a row major of N which is not cographic and has no loops or co-loops. Let Ω be the unique element in $E(M) - E(N)$. Suppose by contradiction that the set of extensions of $\mathcal{R}$ to M is non-empty and is not an equivalence class. By Lemma 25 the set of extensions of $\mathcal{R}$ to M is the union of two equivalence classes $\mathcal{R}_1$ and $\mathcal{R}_2$ and any $(G_1, T_1) \in \mathcal{R}_1$ and $(G_2, T_2) \in \mathcal{R}_2$ are clip siblings that arise from some template $\mathbb{T} = (H_1, P_1, H_2, P_2)$ where, for $i = 1, 2$, $H_i = G_i \setminus \Omega$. Moreover, $(H_i, T_i) \in \mathcal{R}_i$, for $i = 1, 2$. Remark 21 states that $T_i = V_{odd}(G_i[P_1 \Delta P_2])$, for $i = 1, 2$. Hence $(H_i, T_i) \in \mathcal{R}$, for $i = 1, 2$. It follows that (H_1, P_1) and (H_2, P_2) form a reaching pair of (H_1, T_1), a contradiction. $\square$

5 A characterization of clip siblings

We only need to prove Lemma 19 to complete the paper. One ingredient will be Lemma 25. The other ingredient is a theorem (namely Theorem 35) that gives a structural characterization of clip siblings.

5.1 Connectivity

Let M be a matroid with rank function r. Given $X \subseteq E(M)$ we define $\lambda_M(X)$, the *connectivity function* of M, to be equal to $r(X) + r(\bar{X}) - r\big(E(M)\big) + 1$. The set X is a *k-separation* of M if $\min\{|X|, |\bar{X}|\} \geq k$ and $\lambda_M(X) = k$. M is k-connected if it has no r-separations for any $r < k$. Let G be a graph and let $X \subseteq E(G)$. The set X is a *k-separation* of G if $\min\{|X|, |\bar{X}|\} \geq k$, $|\mathcal{B}_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. A graph G is k-connected if it has no r-separations for any $r < k$.

Given a separation X of G, we define the *interior* of X in G to be $\mathcal{I}_G(X) = V_G(X) - \mathcal{B}_G(X)$.We say that X is a *k-(i,j)-separation* of a graft (G, T), where $i, j \in \{0, 1\}$, if the following hold:

- X is a k-separation of G;
- $i = 0$ when $T \cap \mathcal{I}_G(X)$ is empty and $i = 1$ otherwise;
- $j = 0$ when $T \cap \mathcal{I}_G(\bar{X})$ is empty and $j = 1$ otherwise.

Lemma 26. *Let (G, T) be a graft, where T is non-empty and G is connected. Let $M :=$ $\mathrm{ecut}(G, T)$. For every k-(i,j)-separation X of (G, T), we have $\lambda_M(X) = k + i + j - 1$.*

Proof. Let $N = \mathrm{cut}(G)$ with rank function r. As the dual of N is cycle(G), we have:

$$\lambda_N(X) = r(X) + r(\bar{X}) - r\big(E(N)\big) + 1 = k. \qquad (\star)$$

As T is non-empty and G is connected, a basis for M consists of the complement of the edge set of a spanning tree plus an edge forming an odd cut with this set of edges. Hence, the rank of M is one larger than the rank of N. To compute the rank of X for N we delete the elements $\bar{X}$ of N (i.e. we contract the edges $\bar{X}$ of G) and consider the size of the complement of a spanning forest in this new graph. To compute the rank of X for M we contract the set of edges in $\bar{X}$ (in G) and consider the size of the complement of a spanning forest plus possibly an edge forming an odd cut with this set of edges. Hence, if $T \cap \mathcal{I}_G(X)$ is non-empty, then the rank of X in M is one more that in N, otherwise the rank of X is the same in both matroids. A similar argument holds for $\bar{X}$. Thus the rank of X in M is $r(X) + i$ and the rank of $\bar{X}$ in M is $r(\bar{X}) + j$. The result follows from ($\star$). □

Recall that a pin of a graft (G, T) is an odd bridge of G incident to a vertex of degree one and $\text{pin}(G, T)$ denotes the set of all pins of (G, T).

Lemma 27. *Suppose that* $\text{ecut}(G, T)$ *is* 3*-connected. Then:*

(1) $|\text{pin}(G, T)| \leq 1$*;*

(2) $G/\text{pin}(G, T)$ *is* 2*-connected;*

(3) if G *has a* 2*-separation* X *then* $T \cap \mathcal{I}_G(X)$ *and* $T \cap \mathcal{I}_G(\bar{X})$ *are both non-empty.*

Proof. Let $M := \text{ecut}(G, T)$. We may assume that G is connected as we can identify vertices in distinct blocks of G without changing the even cut matroid. (Moreover, as we will prove that $G/\text{pin}(G, T)$ is 2-connected, this implies that originally, $G/\text{pin}(G, T)$ was connected.) As M is 3-connected, it has no loops, no co-loops and no parallel elements. We may assume that T is non-empty, for otherwise $M = \text{cut}(G)$ and G is 3-connected. **(1)** There do not exist distinct pins e, f of G, for otherwise $\{e, f\}$ would be an even cut of G and e, f would be in parallel in M. **(2)** Suppose that X is a 1-(i, j)-separation of (G, T). By Lemma 26, $\lambda_M(X) = 1+i+j-1 \leq 2$. As M is 3-connected, X is not a 2-separation; hence either $|X| = 1$ or $|\bar{X}| = 1$. The single element in X (or $\bar{X}$) is not a loop of G, for otherwise it is a co-loop of M. Hence X or $\bar{X}$ is a pin of G. **(3)** Suppose that X is a 2-(i, j)-separation of (G, T). As M is 3-connected, $\lambda_M(X) \geq 3$. By Lemma 26, $2 + i + j - 1 \geq 3$, hence $i = j = 1$. □

5.2 Types of clip siblings

In this section we define two types of clip siblings. This will require several preliminary definitions. By a *sequence* $(X_1, \ldots, X_k)$ we mean a family of sets $\{X_1, \ldots, X_k\}$ where X_i precedes X_j when $i < j$. We say that $\mathbb{S} = (X_1, \ldots, X_k)$ is a *w-sequence* of G if, for all $i \in [k]$, X_i is a 2-separation of the graph obtained from G by performing Whitney-flips on $X_1, \ldots, X_{i-1}$ (in this order). We denote by $\text{W}_{\text{flip}}[G, \mathbb{S}]$ the graph obtained from G by performing Whitney-flips on $X_1, \ldots, X_k$ (in this order). If $\mathbb{S}$ consists of a single set X, then we write $\text{W}_{\text{flip}}[G, X]$ in lieu of $\text{W}_{\text{flip}}[G, \mathbb{S}]$. If G and G' are equivalent graphs that are 2-connected, then $G' = \text{W}_{\text{flip}}[G, \mathbb{S}]$ for some w-sequence $\mathbb{S}$ of G. Consider a clip-template (H_1, P_1, H_2, P_2) where $H_2 = \text{W}_{\text{flip}}[H_1, \mathbb{S}]$ for some w-sequence $\mathbb{S}$ of H_1. We slightly abuse terminology and call the tuple $(H_1, P_1, H_2, P_2, \mathbb{S})$ a clip-template.

We say that two sets X and Y are *crossing* if the sets $X - Y$, $Y - X$, $X \cap Y, \bar{X} \cap \bar{Y}$ are non-empty. We say that a w-sequence is non-crossing if no two sets in the w-sequence are crossing. A sequence $(X_1, \ldots, X_k)$ is *nested* if $X_i \subset X_{i+1}$ for all $i \in [k-1]$. When considering a nested w-sequence $(X_1, \ldots, X_k)$, we will always assume that X_i and X_{i+1} have distinct boundaries, for all $i \in [k-1]$. If this is not the case, we could just remove the sets X_i and X_{i+1} from the sequence.

If a w-sequence is nested then it is non-crossing. If a w-sequence is non-crossing the graph obtained by performing the Whitney-flips on this sequence does not depend on the order in which the Whitney-flips are performed.

Remark 28. *Suppose that $H_2 = \mathrm{W}_{\mathit{flip}}[H_1, \mathbb{S}]$ for some non-crossing w-sequence $\mathbb{S}$ of H_1. Then for any sequence $\mathbb{S}'$ obtained by reordering $\mathbb{S}$ we have that, for $i = 1, 2$, $\mathbb{S}'$ is a w-sequence of H_i and $H_{3-i} = \mathrm{W}_{\mathit{flip}}[H_i, \mathbb{S}']$.*

We leave the proof of this last result as an exercise.

5.2.1 Basic twins

Consider a clip-template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$. If $\mathbb{S} = \emptyset$ (that is $H_1 = H_2$) then $\mathbb{T}$ is a *basic-template* and (G_1, T_1) and (G_2, T_2) arising from $\mathbb{T}$ are *basic twins*. By Remark 21, $T_i = V_{odd}(H_i[P_1 \triangle P_2])$, for $i = 1, 2$. As P_1 and P_2 are both paths in H_1 and in H_2, this implies that $|T_1|, |T_2| \leq 4$. Therefore:

Remark 29. *Basic twins are degenerate.*

We say that (G_1, T_1) and (G_2, T_2) are *basic siblings* if, for $i = 1, 2$, there exists (G_i', T_i') equivalent to (G_i, T_i) such that (G_1', T_1') and (G_2', T_2') are basic twins.

5.2.2 Nested twins

We say that a clip-template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$ is a *nested-template* if the following hold:

(A1) $\mathbb{S} = (X_1, \ldots, X_k)$ is a nested w-sequence for H_1 (where $k \geq 1$);

(A2) for $i = 1, 2$, P_i has one endpoint in $\mathcal{I}_{H_i}(X_1)$ and one endpoint in $\mathcal{I}_{H_i}(\bar{X}_k)$;

(A3) P_1 and P_2 have no common endpoint in both H_1 and H_2.

Moreover, if $e \in \mathrm{pin}(H_i, T_i)$ for $i = 1, 2$ and p_i denotes the endpoint of e that is not the head of e in H_i, then

(A4) $e \in P_1 \cup P_2$, $p_i \notin T_i$ and either $p_i \in \mathcal{I}_{H_i}(X_1)$ or $p_i \in \mathcal{I}_{H_i}(\bar{X}_k)$.

In this case we say that the grafts (G_1, T_1) and (G_2, T_2) arising from $\mathbb{T}$ are *nested twins*. Note that Remark 28 implies that (A1) is equivalent to the statement that $\mathbb{S}$ is a nested w-sequence for H_2. An example of nested twins is shown in Figure 5.

We say that (G_1, T_1) and (G_2, T_2) are *nested siblings* if, for $i = 1, 2$, there exists (G_i', T_i') equivalent to (G_i, T_i) such that (G_1', T_1') and (G_2', T_2') are nested twins.

In the case of nested twins we can give an explicit characterization of the set of terminals. This is stated in Lemma 32, at the end of the section. First we need some tools to deal with sequences of Whitney-flips.

A *caterpillar* is a tree obtained by taking a path and adding edges which have exactly one endpoint in common with the path. Let G be a graph and let $\mathbb{S} = (X_1, \ldots, X_k)$ be a nested w-sequence for G. We denote by $\mathrm{Cat}(G, \mathbb{S})$ the graph defined on the vertex set $\cup_{i=1}^k \mathcal{B}_G(X_i)$ with edge set $\{e_1, \ldots, e_k\}$, where the endpoints of e_i are the vertices in $\mathcal{B}_G(X_i)$. Note that $\mathrm{Cat}(G, \mathbb{S})$ is a vertex-disjoint union of caterpillars. An example of nested twins is given in Figure 6: grafts (a) and (b) are nested twins arising from some clip-template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$. White vertices are terminals, dashed lines represent the nested 2-

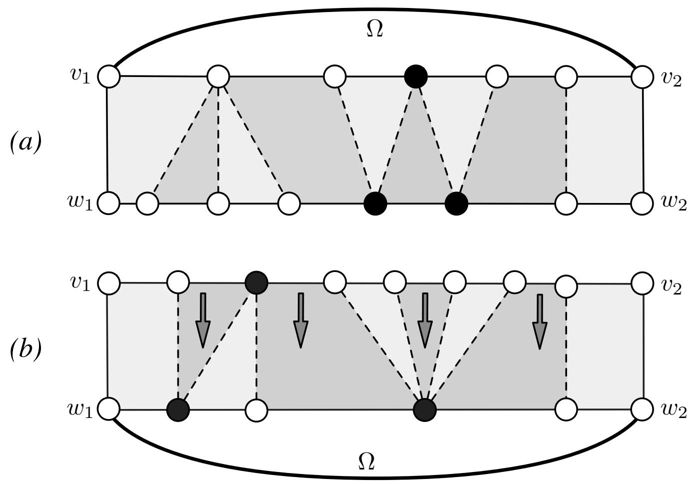

Figure 6: Nested twins.

separations $X \in \mathbb{S}$. Then $\mathrm{Cat}(H_1, \mathbb{S})$ is the graph in (a) where edges correspond to dashed lines. We will see in Lemma 32 that the odd degree vertices in that graph together with the vertices v_1, v_2, w_1, w_2 form the the set of terminals in graft (a). Similarly for (b) the odd degree vertices of $\mathrm{Cat}(H_2, \mathbb{S})$ together with v_1, v_2, w_1, w_2 form the set of terminals.

Next we need to specify the way we relabel vertices in graphs when doing Whitney-flips. Let G be a graph and let X be a 2-separation with $\mathcal{B}_G(X) = \{u_1, u_2\}$, let G' be obtained from G by doing a Whitney-flip on X, i.e G' is obtained by (i) identifying vertex u_1 of $G[X]$ with vertex u_2 of $G[\bar{X}]$; and (ii) identifying vertex u_2 of $G[X]$ with vertex u_1 of $G[\bar{X}]$. Throughout the remainder of the paper we will use the convention that the resulting vertex in (i) is labeled u_1 and that the resulting vertex in (ii) is labeled u_2 (i.e. vertices of G' in $\mathcal{B}_{G'}(X)$ are labeled according to $G[X]$). Given a graft (G, T) and a w-sequence $\mathbb{S}$ for G, we denote by $\mathrm{W}_{\text{flip}}[(G, T), \mathbb{S}]$ the graft (G', T'), where $G' = \mathrm{W}_{\text{flip}}[G, \mathbb{S}]$ and (G, T) and (G', T') are equivalent.

Remark 30. *Let (H, T) be a graft and let X be a 2-separation of H with $\mathcal{B}_H(X) = \{u_1, u_2\}$. If $|\mathcal{I}_H(\bar{X}) \cap T|$ is odd, then for $(H', T') = \mathrm{W}_{\text{flip}}[(H, T), X]$ we have $T' = T \triangle \{u_1, u_2\}$.*

Proof. Let J be a T-join of H. Since $H[J \cap \bar{X}]$ has an even number of vertices of odd degree, and since by hypothesis there are an odd number of such vertices in $\mathcal{I}_H(\bar{X})$, exactly one of the following sets has odd cardinality: $\delta_H(u_1) \cap J \cap \bar{X}$ or $\delta_H(u_2) \cap J \cap \bar{X}$. Thus J is a $T \triangle \{u_1, u_2\}$-join of H'. $\square$

Lemma 31. *Let H be a graph and let $\mathbb{S} = (X_1, \ldots, X_k)$ be a nested w-sequence for H. Let $v \in \mathcal{I}_H(X_1)$, let $w \in \mathcal{I}_H(\bar{X}_k)$ and let $(H', T) = \mathrm{W}_{\mathit{flip}}[(H, \{v, w\}), \mathbb{S}]$. Then $T = \{v, w\} \cup V_{odd}\big(\mathrm{Cat}(H', \mathbb{S})\big)$.*

Proof. For all $r \in [k]$, let $\mathbb{S}_r = (X_1, \ldots, X_r)$ and let $(H_r, T_r) = \mathrm{W}_{\mathrm{flip}}[(H, \{v, w\}), \mathbb{S}_r]$. By induction we will show that for all such r we have $T_r = \{v, w\} \cup V_{odd}\big(\mathrm{Cat}(H_r, \mathbb{S}_r)\big)$. The result holds for $r = 1$ by Remark 30. Suppose now that the result holds for $r < k$. Let u_1, u_2 denote the vertices in $\mathcal{B}_{H_r}(X_r)$. Then

$$\begin{aligned} T_{r+1} &= T_r \triangle \{u_1, u_2\} \\ &= \{v, w\} \cup V_{odd}\big(\mathrm{Cat}(H_r, \mathbb{S}_r)\big) \triangle \{u_1, u_2\} \\ &= \{v, w\} \cup V_{odd}\big(\mathrm{Cat}(H_{r+1}, \mathbb{S}_{r+1})\big), \end{aligned}$$

where the first equality follows from Remark 30 (as $\{w\} = \mathcal{I}_{H_r}(\bar{X}_r) \cap T$), the second equality follows by induction, and the third equality follows from the fact that $\mathrm{Cat}(H_{r+1}, \mathbb{S}_{r+1})$ is obtained from $\mathrm{Cat}(H_r, \mathbb{S}_r)$ by adding the edge $u_1 u_2$. □

Lemma 32. *Let $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$ be a nested-template where $\mathbb{S} = (X_1, \ldots, X_k)$ and, for $i = 1, 2$, let v_i denote the endpoint of P_i in $\mathcal{I}_{H_1}(X_1)$ and w_i denote the endpoint of P_i in $\mathcal{I}_{H_1}(\bar{X}_k)$. Let (G_1, T_1) and (G_2, T_2) be the nested twins arising from $\mathbb{T}$. Then, for $i = 1, 2$,*

$$T_i = \{v_1, v_2, w_1, w_2\} \cup V_{odd}\big(\mathrm{Cat}(H_i, \mathbb{S})\big).$$

Proof. Remark 21 implies that $T_i = V_{odd}(G_i[P_1 \triangle P_2]) = V_{odd}(G_i[P_1]) \triangle V_{odd}(G_i[P_2])$. Since P_i is a path of G_i with endpoints v_i, w_i, $V_{odd}(G_i[P_i]) = \{v_i, w_i\}$. Finally, Lemma 31 implies that $V_{odd}(G_i[P_{3-i}]) = \{v_{3-i}, w_{3-i}\} \cup V_{odd}\big(\mathrm{Cat}(H_i, \mathbb{S})\big)$ and the result follows by (A3) in the definition of nested templates. □

5.3 Characterizing clip siblings

Let G be a graph and P a path in G. We say that a Whitney-flip on a 2-separation X *preserves* P if P is a path of $\mathrm{W}_{\mathrm{flip}}[G, X]$. Note that this occurs if and only if the endpoints of P are both in $V_G(X)$ or both in $V_G(\bar{X})$. We say that a w-sequence $\mathbb{S}$ of G *preserves* P if P is a path in $\mathrm{W}_{\mathrm{flip}}[G, \mathbb{S}]$. The next result is proved in [8, Proposition 5.4].

Proposition 33. *Let H_1 and H_2 be equivalent 2-connected graphs and let P be a path in H_1. Then there exists a graph H such that:*

(1) $H = \mathrm{W}_{\mathit{flip}}[H_1, \mathbb{S}_1]$, for some w-sequence $\mathbb{S}_1$ of H_1 which preserves P, and

(2) $H_2 = \mathrm{W}_{\mathit{flip}}[H, \mathbb{S}_2]$, for some nested w-sequence $\mathbb{S}_2$ of H, where no $X \in \mathbb{S}_2$ preserves P in H.

Lemma 34. *Let H_1 and H_2 be equivalent 2-connected graphs with paths P_1 and P_2 respectively. Then there exist graphs H_1' and H_2' and a nested w-sequence $\mathbb{S}$ of both H_1' and H_2' such that, for $i = 1, 2$:*

(1) H_i' *is equivalent to* H_i *and* P_i *is a path of* H_i'*, and*

(2) $H_2' = \mathrm{W}_{\mathit{flip}}[H_1', \mathbb{S}]$ *and no* $X \in \mathbb{S}$ *preserves path* P_i *of* H_1'*.*

Proof. Proposition 33 implies that there exist H_1' equivalent to H_1 where P_1 is a path of H_1' and a nested w-sequence $\mathbb{S}_2$ of H_1' such that $H_2 = \mathrm{W}_{\text{flip}}[H_1', \mathbb{S}_2]$ where no $X \in \mathbb{S}_2$ preserves path P_1 of H_1'. Because of Remark 28, we can partition $\mathbb{S}_2$ into $\mathbb{S}$ and $\mathbb{S}'$ such that $H_2' = \mathrm{W}_{\text{flip}}[H_2, \mathbb{S}']$ where P_2 is a path of H_2' and $H_1' = \mathrm{W}_{\text{flip}}[H_2', \mathbb{S}]$ where no $X \in \mathbb{S}$ preserves path P_2 of H_2'. Because of Remark 28 we may assume that $\mathbb{S}$ is nested, and it is a w-sequence of both H_1' and H_2'. □

Theorem 35. *Let* M *be a* 3*-connected even cut matroid with representations* (G_i, T_i) *for* $i = 1, 2$*. Suppose that* (G_1, T_1) *and* (G_2, T_2) *are clip siblings arising from a clip-template* $\mathbb{T} = (H_1, P_1, H_2, P_2)$*, where* $\mathrm{ecut}(H_1, T_1)$ *is* 3*-connected and is not cographic. Then* (G_1, T_1) *and* (G_2, T_2) *are either basic siblings or nested siblings.*

Proof. First we consider the case where (H_1, T_1) and (H_2, T_2) have no pins. It follows from Lemma 27 that H_1 and H_2 are 2-connected. Hence, H_1, H_2, P_1, P_2 satisfy the hypothesis of Lemma 34. Let H_1', H_2' and $\mathbb{S}$ be obtained as in that lemma. Then properties (1) and (2) of the lemma imply that $\mathbb{T}' = (H_1', P_1, H_2', P_2, \mathbb{S})$ is a clip-template. Let $(G_1', T_1'), (G_2', T_2')$ be the clip siblings arising from $\mathbb{T}'$. Claim 1 and Claim 2 will imply that (G_1, T_1) and (G_2, T_2) are basic siblings or nested siblings.

Claim 1. *For* $i = 1, 2$*,* (G_i, T_i) *and* (G_i', T_i') *are equivalent.*

Proof. Let $i \in [2]$. Let Ω denote the edge in $E(G_i) - E(H_i)$. By construction,

$$\mathrm{cycle}(G_i) = \mathrm{span}\Big(\mathrm{cycle}(H_i) \cup \{P_i \cup \{\Omega\}\}\Big)$$
$$\mathrm{cycle}(G_i') = \mathrm{span}\Big(\mathrm{cycle}(H_i') \cup \{P_i \cup \{\Omega\}\}\Big).$$

By Theorem 3, $\mathrm{cut}(H_i) = \mathrm{cut}(H_i')$ or equivalently, $\mathrm{cycle}(H_i) = \mathrm{cycle}(H_i')$. It follows that $\mathrm{cycle}(G_i) = \mathrm{cycle}(G_i')$, or equivalently, $\mathrm{cut}(G_i) = \mathrm{cut}(G_i')$. Hence, by Theorem 3, G_i and G_i' are equivalent. Since G_i and G_i' are equivalent, for some set of terminals R_i, the grafts (G_i, R_i) and (G_i', T_i') are equivalent. Since (G_1', T_1') and (G_2', T_2') are clip siblings, $\mathrm{ecut}(G_1', T_1') = \mathrm{ecut}(G_2', T_2')$. Hence, $\mathrm{ecut}(G_1, R_1) = \mathrm{ecut}(G_2, R_2)$. Since (G_1, T_1) and (G_2, T_2) are clip siblings, $\mathrm{ecut}(G_1, T_1) = \mathrm{ecut}(G_2, T_2)$. It follows from Theorem 20 that $T_i = R_i$ for $i = 1, 2$. ◊

Claim 2. $\mathbb{T}' = (H_1', P_1, H_2', P_2, \mathbb{S})$ *is either a basic-template or a nested-template.*

Proof. If $\mathbb{S} = \emptyset$ then by definition $\mathbb{T}'$ is a basic-template. Thus we may assume that $\mathbb{S} = (X_1, \ldots, X_k)$ for some $k \geq 1$. Property (2) of Lemma 34 implies that, for $i = 1, 2$, P_i has one endpoint in $\mathcal{I}_{H_i}(X_1)$ and the other in $\mathcal{I}_{H_i}(\bar{X}_k)$. Thus properties (A1) and (A2) of nested-templates are satisfied. Suppose for a contradiction that (A3) does not hold, i.e. P_1 and P_2 have a common endpoint in H_i for some $i \in [2]$. Up to relabeling we may assume that P_1 and P_2 have the same endpoint $v \in \mathcal{I}_{H_1}(X_1)$. By Remark 21, $T_1' = V_{odd}(G_1'[P_1 \triangle P_2])$, hence

$v \notin T_1'$. It follows from Lemma 27 that X_1 is a 2-separation of $\text{ecut}(H_1', T_1')$. By Claim 1, $\text{ecut}(H_1', T_1') = \text{ecut}(H_1, T_1)$, a contradiction as this matroid is 3-connected. As by hypothesis (H_i, T_i) has no pins for $i = 1, 2$, property (A4) is vacuously true. $\Diamond$

It remains to consider the case where (H_1, T_1) and thus (H_2, T_2) has a pin e.

Claim 3. *We may assume $e \in P_1 - P_2$ and $P_1 \neq \{e\}$.*

Proof. Let h denote the head of the pin e in H_1. Suppose for a contradiction that $e \in P_1 \cap P_2$ or that $e \notin P_1 \cup P_2$. By Remark 21, $T_1 = V_{odd}(G_1[P_1 \triangle P_2])$. Thus $h \notin T_1$, a contradiction as e is a pin of (H_1, T_1). Thus we may assume that $e \in P_1 - P_2$. Suppose for a contradiction that $P_1 = \{e\}$. Remark 21 implies that $T_2 = V_{odd}(G_2[P_1 \triangle P_2])$. Hence, the only terminals in (H_2, T_2) are the endpoints of path P_2 and the endpoints of $P_1 = \{e\}$. It follows that the graft obtained from (H_2, T_2) by moving the pin e to an endpoint of P_2 has exactly two terminals. Hence, by Remark 8, $\text{ecut}(H_2, T_2)$ is cographic, contradicting the hypothesis. $\Diamond$

For $i = 1, 2$ let $(\hat{H}_i, \hat{T}_i) = (H_i, T_i)/e$ and let $\hat{P}_1 = P_1 - \{e\}$. Claim 3 implies that $\hat{\mathbb{T}} = (\hat{H}_1, \hat{P}_1, \hat{H}_2, P_2)$ is a clip-template. Proposition 33 implies that $\hat{H}_1$ and $\hat{H}_2$ are 2-connected. Thus we can now apply the previous argument to $\hat{\mathbb{T}}$. At the end we uncontract the pin e. It suffices to observe that as property (A2) holds before uncontracting e, property (A4) will hold after uncontracting e. $\square$

6 Row extensions and non degenerate matroids

The goal of this section is to prove Lemma 19.

6.1 The proof (modulo the exclusion of one lemma)

A graft (G, T) is *nice* if there exist an edge Ω that is not a bridge of G and a nested w-sequence $\mathbb{S} = (X_1, \ldots, X_k)$ of $H = G \setminus \Omega$ such that the following hold:

(B1) there exist $v_1, v_2 \in \mathcal{I}_H(X_1) \cap T$ and $w_1, w_2 \in \mathcal{I}_H(\bar{X}_k) \cap T$;

(B2) $T = \{v_1, v_2, w_1, w_2\} \cup T_c$ where $T_c \subseteq \bigcup_{i=1}^k \mathcal{B}_H(X_i)$ and v_1, v_2, w_1, w_2 are all distinct;

(B3) Ω has endpoints v_1, w_1;

(B4) $(H, T) := (G, T) \setminus \Omega$ is non-degenerate;

(B5) H is 2-connected.

Lemma 36. *Suppose that (G_1, T_1) and (G_2, T_2) are siblings arising from a nested-template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$, where (H_1, T_1) is non-degenerate and* $\text{ecut}(H_1, T_1)$ *is* 3*-connected and not cographic. Then, for $i = 1, 2$, (G_i, T_i) contains as a minor a nice graft.*

Proof. We may assume $i = 1$. Let Ω denote the element in $E(G_1) - E(H_1)$. $\mathbb{T}$ satisfies properties (A1)-(A4) of nested templates (see Section 5.2.2). (A1) states that $\mathbb{S} = (X_1, \ldots, X_k)$ is a nested w-sequence for H_1 ($k \geq 1$). (A2) implies that for $i = 1, 2$, path P_i has an endpoint $v_i \in \mathcal{I}_{H_1}(X_1)$ and an endpoint $w_i \in \mathcal{I}_{H_1}(\bar{X}_k)$. By (A3), $v_1 \neq v_2$ and $w_1 \neq w_2$. Thus v_1, v_2, w_1, w_2 are all distinct. Lemma 32 implies that $T_1 = \{v_1, v_2, w_1, w_2\} \cup V_{odd}\big(\text{Cat}(H_1, \mathbb{S})\big)$. Hence, (B1) and (B2) hold for (H_1, T_1). Moreover, since the endpoints of P_1 are v_1, w_1, (B3) holds as well. By hypothesis (H_1, T_1) is non-degenerate, i.e. (B4) holds for (H_1, T_1) as well.

Lemma 27 implies that H_1 is 2-connected except for a possible pin e of (H_1, T_1). If there is no pin e, then H_1 is 2-connected and (H_1, T_1) satisfies (B1)-(B5), i.e. is the required nice graft. Thus we may assume that there exists a pin e. Clearly, H_1/e is 2-connected. Hence, to show that $(H_1, T_1)/e$ is a nice graft, it suffices to verify that (B1)-(B4) hold for $(H_1, T_1)/e$. (B1)-(B3) follow from the fact that (H_1, T_1) satisfy (B1)-(B3) and the fact that $\mathbb{T}$ satisfies (A4). Finally, it can be readily checked that contracting a pin in a graft preserves non-degeneracy, i.e. $(H_1, T_1)/e$ satisfies (B4). □

Let (G, T) be a graft and let X be a 2-separation of G where $\mathcal{B}_G(X) = \{u_1, u_2\}$. We say that X is *simple* if $\mathcal{I}_G(X) = \{v\}$, $v \in T$ and either $X = \{u_1v, u_2v\}$ or $X = \{u_1v, u_2v, u_1u_2\}$. A graft (G, T) is *nearly* 3*-connected* if G is 2-connected and for every 2-separation at least one of X or $\bar{X}$ is simple.

Remark 37. *In a nearly* 3*-connected non-degenerate graft* (G, T) *no pair of* 2*-separations* X_1 *and* X_2 *cross.*

Proof. Suppose for a contradiction that X_1 and X_2 cross. Then there exist series edges e, f, g with $X_1 = \{e, f\}$ and $X_2 = \{f, g\}$. Since (G, T) is nearly 3-connected $E(G) - (X_1 \cup X_2)$ is simple. It follows that (G, T) is degenerate, a contradiction. □

Lemma 38. *Nearly* 3*-connected grafts with a reaching pair are degenerate.*

Proof. Suppose that a nearly 3-connected graft (G, T) has a reaching pair (G_1, P_1) and (G_2, P_2). By Remark 37 we may assume that any two 2-separations of G are non-crossing. In particular, simple separations of G are simple separations of G_1 and G_2 (and vice versa) and G_1 and G_2 are nearly 3-connected. Hence, G_1 and G_2 are 2-connected, and $G_2 = \mathrm{W}_{\text{flip}}[G_1, \mathbb{S}]$ for some w-sequence of G_1. Let $X \in \mathbb{S}$; we may assume (by possibly swapping X with its complement) that X is simple. Denote by u the vertex in $\mathcal{I}_G(X) = \mathcal{I}_{G_1}(X) = \mathcal{I}_{G_2}(X)$. By Remark 28 we may assume that X is the first element in the sequence $\mathbb{S}$. Thus we may assume that X does not preserve P_1 for otherwise we may replace G_1 with $\mathrm{W}_{\text{flip}}[G_1, X]$. This implies that u is the endpoint of P_1 in G_1, hence in G. Similarly, applying the argument for G_2, we deduce that u is the endpoint of P_2 in G. Since $T = V_{odd}(G[P_1 \triangle P_2])$, $u \notin T$, a contradiction as X is simple. Hence $\mathbb{S} = \emptyset$, P_1 and P_2 are paths of G, and $|T| \leq 4$. □

We will postpone the proof of the following key lemma until the next section.

Lemma 39. *Every nice graft has a minor that is a nearly* 3*-connected nice graft.*

Proof of Lemma 19. Let N, M, $\mathcal{R}$ and $\mathcal{R}'$ be as in the statement of the lemma. We may assume that $\mathcal{R}'$ is not an equivalence class. By Lemma 25, $\mathcal{R}'$ is the union of two equivalence classes $\mathcal{R}_1$ and $\mathcal{R}_2$ and any $(G_1, T_1) \in \mathcal{R}_1$ and $(G_2, T_2) \in \mathcal{R}_2$ are clip siblings. By Theorem 35, (G_1, T_1) and (G_2, T_2) are either basic or nested siblings, however the former is not possible because basic siblings are degenerate (Remark 29). Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence classes, we can choose $(G_1, T_1) \in \mathcal{R}_1$ and $(G_2, T_2) \in \mathcal{R}_2$ so that they are nested twins arising from a template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$.

Let $i \in [2]$. We need to show that (G_i, T_i) has no reaching pair. By Lemma 36, (G_i, T_i) has a minor $(\hat{G}, \hat{T})$ that is nice. By Lemma 39, $(\hat{G}, \hat{T})$ has a minor $(\hat{G}', \hat{T}')$ that is nice and nearly 3-connected. In particular, since it is nice, it is non-degenerate. It follows by Lemma 38 that $(\hat{G}', \hat{T}')$ has no reaching pair. Since $(\hat{G}', \hat{T}')$ is a minor of (G_i, T_i), Lemma 17 implies that (G_i, T_i) has no reaching pair. □

6.2 A few observations about 2-separations

For a graph H, we say that a sequence $\mathbb{F} = (B_1, \dots, B_t)$ with $t \geq 2$, where $B_1, \dots, B_t$ is a partition of $E(H)$, is a *flower* if there exist distinct $u_1, \dots, u_t \in V(H)$ such that,

- $H[B_i]$ is connected, for every $i \in [t]$, and
- $\mathcal{B}_H(B_i) = \{u_i, u_{i+1}\}$, for every $i \in [t]$ (where $t + 1$ denotes 1).

For $i \in [t]$, B_i is a *petal* with *attachments* u_i and u_{i+1}. We say that the flower is *maximal* if for no petal B, the graph $H[B]$ has a cut-vertex separating its attachments. For all $i \in [t]$, petals B_i and B_{i+1} are *consecutive*. Maximal flowers correspond to generalized circuits as introduced by Tutte in [14]. The term flower was introduced to describe crossing 3-separations in matroids (see [7]).

Lemma 40. *Let (H, T) be a graft and $\mathbb{F} = (B_1, \dots, B_t)$ be a flower of H with attachments U. Suppose that $T = T_a \cup T_b$ where $T_a \subseteq U$, $T_b \cap U = \emptyset$, $|T_b| \leq 4$, and no two vertices of T_b are contained in the same petal of $\mathbb{F}$. Then (H, T) is degenerate.*

Proof. $|T| = 2k$ for some integer k and we may choose a T-join $J = P_1 \triangle \dots \triangle P_k$ where $P_1, \dots, P_k$ are pairwise vertex-disjoint paths of H. Let $\mathbb{B} = \{B \in \mathbb{F} : B \cap P_i \neq \emptyset, \text{ for some } i \in [k]\}$. Let H' be obtained from H by rearranging the petals of $\mathbb{F}$ so that the petals in $\mathbb{B}$ are consecutive in H'. After possible Whitney flips on some of the petals in H' we may obtain a graph H'' where J is the union of at most two paths. Let $T'' := V_{odd}(H''[J])$. Then $|T''| \leq 4$ and (H'', T'') is equivalent to (H, T). □

We leave the following observation as an exercise,

Remark 41. *Let H be a 2-connected graph and let X and Y be crossing 2-separations of H. Then there exists a partition Z_1, Z_2, Z_3, Z_4 of the edges of G such that $X = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and either,*

(1) (Z_1, Z_2, Z_3, Z_4) is a flower of H, or

(2) $\mathcal{B}_G(Z_i) = \mathcal{B}_G(X) = \mathcal{B}_G(Y)$ *for all* $i \in [4]$.

Let (G,T) be a graft where G is 2-connected. Let X be a 2-separation of G and denote by u_1, u_2 the vertices in $\mathcal{B}_G(X)$. We say that X is a *Type I separation* if $\mathcal{I}_G(X) \cap T = \emptyset$, i.e. if X is a 2-$(0,i)$ separation for some $i \in \{0,1\}$. Suppose now that there exists a unique vertex v in $\mathcal{I}_G(X) \cap T$. We say that X is a *Type II separation* if v is a cut-vertex of $G[X]$ separating u_1 and u_2. We say that X is a *Type III separation* if v is not such a cut-vertex, i.e. if there exists a (u_1,u_2)-path of $G[X]$ avoiding v.

Consider a graft (G,T) with a 2-separation X with $\mathcal{B}_G(X) = \{u_1,u_2\}$. Suppose that X is a Type I separation. Let (H,T) be obtained from (G,T) by replacing X by an edge u_1u_2. We say that (H,T) is obtained from (G,T) by a *Type I simplification*. Suppose that X is a Type 2 separation where v is the vertex in $\mathcal{I}_G(X) \cap T$. Let (H,T) be obtained from (G,T) by replacing X by edges u_1v and vu_2. We say that (H,T) is obtained from (G,T) by a *Type II simplification*. Suppose that X is a Type 3 separation where v is the vertex in $\mathcal{I}_G(X) \cap T$. Let (H,T) be obtained from (G,T) by replacing X by edges u_1v, vu_2 and u_1u_2. We say that (H,T) is obtained from (G,T) by a *Type III simplification*. In all three cases we say that (H,T) is obtained by simplifying separation X of (G,T).

Lemma 42. *Let* (G,T) *be a graft where* G *is* 2*-connected, and let* (H,T) *be obtained from* (G,T) *by simplifying a separation. Then*

(1) (H,T) *is a minor of* (G,T) *and*

(2) if (G,T) *is non-degenerate then* (H,T) *is non-degenerate.*

Proof. **(1)** Let X be a 2-separation of G with $\mathcal{B}_G(X) = \{u_1,u_2\}$. Suppose that X is a Type I separation. Since G is 2-connected, there exists a (u_1,u_2)-path P in $G[X]$. Then we obtain a Type I simplification by contracting all but one edge of P. Suppose that X is a Type II separation. Since G is 2-connected, there exists a (u_1,u_2)-path P in $G[X]$ using vertex v. Then we obtain a Type II simplification by contracting all edges of P that are not adjacent to v. Suppose that X is a Type III separation. Since G is 2-connected, there exists a (u_1,u_2) path P in $G[X]$ using vertex v. Since v is not a cut-vertex of $G[X]$ separating u_1 and u_2, there exists a (u_1,u_2)-path Q in $G[X]$ avoiding v. For $i = 1,2$ let z_i be the last vertex of Q in the subpath $P(u_i,v)$ starting from u_i. Then we obtain a Type III simplification by deleting all edges of X outside $P \cup Q(z_1,z_2)$ and by contracting all edges in $P(u_1,z_1) \cup P(u_2,z_2)$, all edges in $P(z_1,z_2)$ not incident to v and all but one edge of $Q(z_1,z_2)$. **(2)** follows from the fact that (H,T) and (G,T) have the same set of terminals and every 2-separation of H corresponds to a 2-separation of G. Hence, if there exists a sequence of Whitney-flips that transforms the graft (H,T) into a graft with at most four terminals, then the corresponding Whitney-flips on (G,T) would transform (G,T) into a graft with at most four terminals. □

6.3 The proof of Lemma 39

We are now ready to prove the last lemma of the paper.

Proof of Lemma 39. Let (G,T) be a nice graft and let $\Omega, H = G \setminus \Omega$, $\mathbb{S} = (X_1,\dots,X_k)$, v_1, v_2, w_1, w_2, and T_c be as in the definition of nice grafts (see section 6.1). By choosing a minor minimal example, we may assume that no minor of (G,T) is equivalent to a nice graft. Throughout this proof we will use the fact that (by (B5)) H is 2-connected and, since Ω is not a bridge, G is also 2-connected.

We need to show that (G,T) is nearly 3-connected. Suppose for a contradiction this is not the case i.e. there exists a 2-separation Y of (G,T) with $\Omega \notin Y$ where neither Y, nor $\bar{Y}$ is simple. We will show that for some (G',T') equivalent to (G,T) and for some $Y' \subseteq Y$ that is a non-simple 2-separation of G', the following properties hold:

(P1) Y' is a non-simple separation of (G',T') of Type I, II, or III; and

(P2) the graft obtained from (G',T') by simplifying the separation Y' is equivalent to a nice graft.

Then (P1) and (P2) will contradict the fact that no minor of (G,T) is equivalent to a nice graft.

Note that we may swap the role of X_1 and $\bar{X}_k$, as we may replace $\mathbb{S}$ by $(\bar{X}_k, \bar{X}_{k-1}\dots,\bar{X}_1)$. Throughout the proof we will make repeated use of this symmetry.

Claim 1. *We may assume that* $\mathcal{I}_G(Y) \cap T \neq \emptyset$.

Proof. For otherwise Y is a Type I separation and let (G',T) be obtained from (G,T) by simplifying Y. Lemma 42 implies that (G',T) is a minor of (G,T) and that (G',T) is non-degenerate. It is easy to verify that properties (B1)-(B5) are preserved for (G',T'). Hence, (P1) and (P2) hold as required. ◊

By hypothesis, $\Omega = v_1w_1 \notin Y$. By (B1), $v_1 \in \mathcal{I}_H(X_1)$, and $w_1 \in \mathcal{I}_H(\bar{X}_k)$. Thus, $X_1 - Y \neq \emptyset$ and $\bar{X}_k - Y \neq \emptyset$. In particular one of the following three cases must hold.

<u>**Case 1:**</u> $Y \subseteq X_1$ or $Y \subseteq \bar{X}_k$.

Since we can interchange the role of X_1 and $\bar{X}_k$, it suffices to consider the case where $Y \subseteq X_1$. Then $v_1 \notin \mathcal{I}_H(Y)$ as the edge Ω is incident to v_1. By Claim 1, $T \cap \mathcal{I}_H(Y)$ is non-empty. (B1) and (B2) imply that $T \cap \mathcal{I}(X_1) = \{v_1, v_2\}$. It follows that $T \cap \mathcal{I}_H(Y) = \{v_2\}$. If v_2 is a cut-vertex of $G[Y]$ separating $\mathcal{B}_G(Y)$ then Y is a separation of Type II, otherwise Y is a separation of Type III. Hence, we proved (P1). Simplify Y by a simple separation Y', and let $X_1' = (X_1 - Y) \cup Y'$. Using Lemma 42 it is easy to verify that properties (B1)-(B5) are preserved, using X_1' instead of X_1. Thus (P2) holds as required.

<u>**Case 2:**</u> $Y \subseteq X_k - X_1$.

By Claim 1, there exists $x \in \mathcal{I}_H(Y) \cap T$. By (B2), $x \in \mathcal{B}_H(X_p)$ for some $p \in [k]$. Note that $p \neq 1, k$, as $x \in \mathcal{I}_H(Y)$ and $Y \subseteq X_k - X_1$. As $x \in \mathcal{I}_H(Y)$, the sets $X_p \cap Y$ and $Y - X_p$ are non-empty. Moreover, $X_1 \subset X_p$ and $Y \cap X_1$ is empty, so $X_p - Y$ is also non-empty. Finally $\bar{X}_k$ is contained in both $\bar{X}_p$ and in $\bar{Y}$, hence $\bar{X}_p \cap \bar{Y}$ is non-empty. It follows that Y and X_p cross. Therefore there exists a partition Z_1, Z_2, Z_3, Z_4 of $E(H)$ such that $X_p = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and by Remark 41 one of the following occurs:

(α) $\mathcal{B}_H(Z_i) = \mathcal{B}_H(X_p) = \mathcal{B}_H(Y)$, for every $i \in [4]$, or

(β) (Z_1, Z_2, Z_3, Z_4) is a flower of H.

However, (α) does not occur, as $x \in \mathcal{B}_H(X_p) - \mathcal{B}_H(Y)$; therefore ($\beta$) occurs. Since $Y = Z_2 \cup Z_3$, Z_2 and Z_3 are consecutive petals with common attachment x. In particular, we have shown that every vertex in $\mathcal{I}_H(Y) \cap T$ is a cut vertex of $H[Y]$ separating the vertices in $\mathcal{B}_H(Y)$. Applying this last result to each vertex in $\mathcal{I}_H(Y) \cap T$ it follows that Y can be partitioned into sets $B_1, \ldots, B_\ell$, for some $\ell \geq 2$, such that $(Z_1, B_1, \ldots, B_\ell, Z_4)$ form a flower. Moreover, the attachments of consecutive petals of $B_1, \ldots, B_\ell$ are in T and, for all $i \in [\ell]$, $\mathcal{I}_H(B_i) \cap T = \emptyset$. It follows by Claim 1 (applied to each B_i) that, for all $i \in [\ell]$, B_i consists of a single edge, say e_i, and that $e_1, \ldots, e_\ell$ are series edges. Since Y is not simple, $\ell \geq 3$. Let (G', T') be obtained from (G, T) by a Whitney flip on $\{e_1, e_2\}$. In (G', T') (see Remark 30) e_2 and e_3 are series edges and the vertex common to these edges in not in T'. Hence, $Y' = \{e_2, e_3\}$ is a non-simple separation of Type I of (G', T') and (P1) is satisfied. Proceeding as in Claim 1 we prove that (P2) holds as well.

Case 3: Y crosses X_1 or Y crosses $\bar{X}_k$.

Since we can interchange the role of X_1 and $\bar{X}_k$, it suffices to consider the case where Y crosses X_1. Therefore there exists a partition Z_1, Z_2, Z_3, Z_4 of $E(H)$ such that $X_1 = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and, by Remark 41, one of the following occurs:

(α) $\mathcal{B}_H(Z_i) = \mathcal{B}_H(X_1)$, for every $i \in [4]$, or

(β) (Z_1, Z_2, Z_3, Z_4) is a flower of H.

As $X_1 = Z_1 \cup Z_2$ and $\Omega = v_1 w_1 \notin Y$, we have $v_1 \in V_H(Z_1)$ and $w_1 \in V_H(Z_4)$.

Claim 2. *Case (α) does not occur.*

Proof. Suppose for a contradiction that case (α) occurs. We first claim that $\mathcal{I}_H(Z_3) \cap T \neq \emptyset$. Suppose that this is not the case; then, because of Claim 1, Z_3 consists of a single edge. As we obtained a contradiction in Case 1 for non-simple 2-separations, Z_2 is a simple separation or consists of a single edge. By Claim 1, H does not have parallel edges. Thus $Y = Z_2 \cup Z_3$ is simple, a contradiction.

Suppose that $k \geq 2$, i.e. there is a 2-separation X_2 in $\mathbb{S}$ with $X_1 \subset X_2$. We may assume that X_1 and X_2 have distinct boundaries. Let $\{a_1, a_2\}$ denote the vertices in $\mathcal{B}_H(X_2)$. As $X_1 \subset X_2$, $a_1, a_2 \in V_H(Z_3 \cup Z_4)$. If $a_1, a_2 \in V_H(Z_4)$, then $Z_3 \subset X_2 - X_1$ and $T \cap \mathcal{I}_H(Z_3)$ is empty, a contradiction. Hence, we may assume $a_1 \in \mathcal{I}_H(Z_3)$; then (as we are in case (α) and since H is 2-connected) there exists a (v_1, w_1)-path in $H \setminus \{a_1, a_2\}$, a contradiction. Thus $k = 1$, i.e. $\mathbb{S} = (X_1)$; hence, by (B2), $T \subseteq \{v_1, v_2, w_1, w_2\} \cup \mathcal{B}_H(X_1)$. As $|T| \geq 6$ (by (B4)), we have $T = \{v_1, v_2, w_1, w_2\} \cup \mathcal{B}_H(X_1)$. By Remark 30, it follows that $\mathrm{W}_{\text{flip}}[(H, T), X_1]$ has four terminals, contradicting (B4). $\diamondsuit$

Thus we may assume case (β) occurs.

Claim 3. *Y does not cross $\bar{X}_k$.*

Proof. Suppose for a contradiction it does. Then, by a similar argument to the one above (applied to $\bar{X}_k$), there exists a flower (W_1, W_2, W_3, W_4) of H with $\bar{X}_k = W_1 \cup W_2$ and $Y = W_2 \cup W_3$. Let $\mathbb{F}$ be the maximal flower that is a obtained from the flower (Z_1, Z_2, Z_3, Z_4) by partitioning the petals Z_i into as many petals as possible and let $\mathbb{F}'$ be the maximal flower that is a obtained from the flower (W_1, W_2, W_3, W_4) by partitioning the petals W_i into as many petals as possible. As Y crosses both X_1 and $\bar{X}_k$, we have $\mathbb{F} = \mathbb{F}'$. Hence $\mathbb{F}$ is a flower of H and X_1 and $\bar{X}_k$ are each the union of at least two petals of $\mathbb{F}$. We may assume that $H[X_1]$ does not partition into sets U_1, U_2, where $v_1, v_2 \in \mathcal{I}_H(U_1)$ and $(U_1, U_2, \bar{X}_1)$ is a flower of H, as otherwise we may redefine X_1 to be U_1 (by adding U_1 to the sequence $\mathbb{S}$). The analogue statement holds for $\bar{X}_k$. It follows that v_1 and v_2 are not in the interior of the same petal of $\mathbb{F}$. Similarly, w_1 and w_2 are not in the interior of the same petal of $\mathbb{F}$. Moreover, as we obtained $\mathbb{F}$ from (Z_1, Z_2, Z_3, Z_4), v_1 and w_1 are in distinct petals of $\mathbb{F}$. For every $X \in \mathbb{S}$, the vertices in $\mathcal{B}_H(X)$ are attachments of $\mathbb{F}$, as there is no (v_1, w_1)-path in $H - \mathcal{B}_H(X)$. Hence T_c is contained in the set of attachments of $\mathbb{F}$. By Lemma 40, (H, T) is degenerate, a contradiction to (B4). $\Diamond$

Thus Y does not cross $\bar{X}_k$. Hence, $Y \cap \bar{X}_k = \emptyset$ and thus $Z_3 \subseteq X_k - X_1$. As we obtained a contradiction in Case 2 for non-simple 2-separations, Z_3 either consists of a single edge e, or Z_3 is simple. If Z_3 is simple, the vertex in $T \cap \mathcal{I}_H(Z_3)$ is a cut-vertex of $H[Z_3]$, because of condition (B2). Therefore either Z_3 consists of a single edge e or it consists of two series edges, say e, f incident to a vertex $x \in T$. If Z_2 consists of a single edge, say g, then, as Y is non-simple, $Z_3 = \{e, f\}$. It follows that e, f, g are three series edges and we obtain a contradiction as in the proof of Case 2.

Thus we may assume that $|Z_2| > 1$. Because of Claim 1, $\mathcal{I}_H(Z_2) \neq \emptyset$. As we obtained a contradiction in Case 1 for non-simple 2-separations, Z_2 is simple, hence it is a Type II or Type III separation. We may assume that e has an endpoint y in $\mathcal{B}_G(Z_2)$.

If $y \notin T$, then, by Lemma 42, properties (P1) and (P2) hold as required. Hence we may assume that $y \in T$. Let $(G', T') = \mathrm{W}_{\text{flip}}[(G, T), Z_2]$. Denote by y' the endpoint of e that is in $\mathcal{B}_{G'}(Z_2)$ in G'. Since Z_2 is a Type II or Type III separation, $|\mathcal{I}_G(Z_2) \cap T| = 1$. Thus, by Remark 30, $y' \notin T'$. It follows that $Z_2 \cup \{e\}$ is a Type II or Type III separation of (G', T'). Thus property (P1) holds. A Type II or Type III simplification will replace (G', T') by $(H, S) = (G', T')/e$. Observe, that $\mathrm{W}_{\text{flip}}[(H, S), Z_2] = (G, T)/e$. Using Lemma 42 it follows readily that $(G, T)/e$ is nice, hence property (P2) holds as required. $\square$

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