Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

The Set packing problem

A max-min relation

Poset $(V, \leq)$, i.e $\forall a, b, c \in V$,

- $a \leq a$
- $a \leq b$ & $b \leq a \implies a = b$
- $a \leq b$ & $b \leq c \implies a \leq a$.

def: a, b comparable if $a \leq b$ or $b \leq a$

a, b incomparable otherwise

def: chain = set of comparable elements

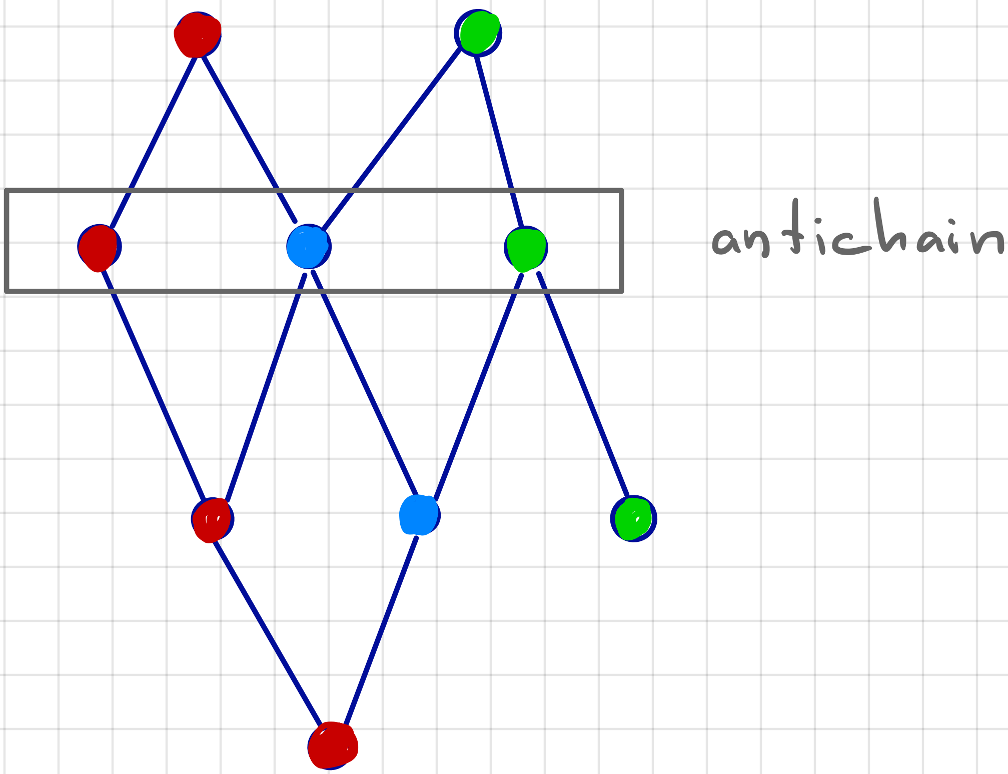
antichain = set of incomparable elements

Qu: maximum size of antichain ?

Th: [Dilworth]

max size of antichain =

min nb of chains needed to cover V.



An Integer Programming framework

Let M be $m \times n$ 0,1 matrix, $w \in \mathbb{R}^n_+$

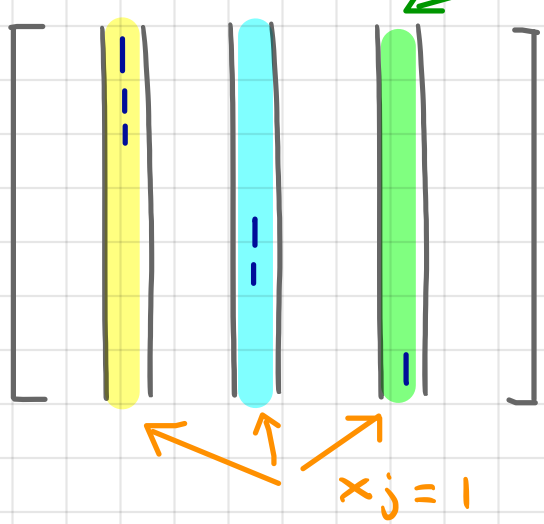
def: Set Packing IP

$$\max \{ w^T x : Mx \leq 1, \boxed{x \geq 0,\ x \text{ integer}} \} \qquad \text{(IP)}$$

$\longleftrightarrow x \in \{0,1\}^n$

Why the name?

incidence vector of set

$M =$ 

Finds max weight family of disjoint sets.

$$\max\{w^T x : Mx \le \mathbb{1}, x \ge 0, x \text{ integer}\} \qquad \text{(IP)}$$

$$\max\{w^T x : Mx \le \mathbb{1}, x \ge 0\} \qquad \text{(P)}$$

$$\min\{\mathbb{1}^T y : M^T y \ge w, y \ge 0\} \qquad \text{(D)}$$

$$\min\{\mathbb{1}^T y : M^T y \ge w, y \ge 0, y \text{ integer}\} \qquad \text{(ID)}$$

Let z_{IP}, z_P, z_D, z_{ID} be optimal values for (IP), (P), (D), (ID) respectively, then

$$\boxed{z_{IP} \le z_P = z_D \le z_{ID}}$$

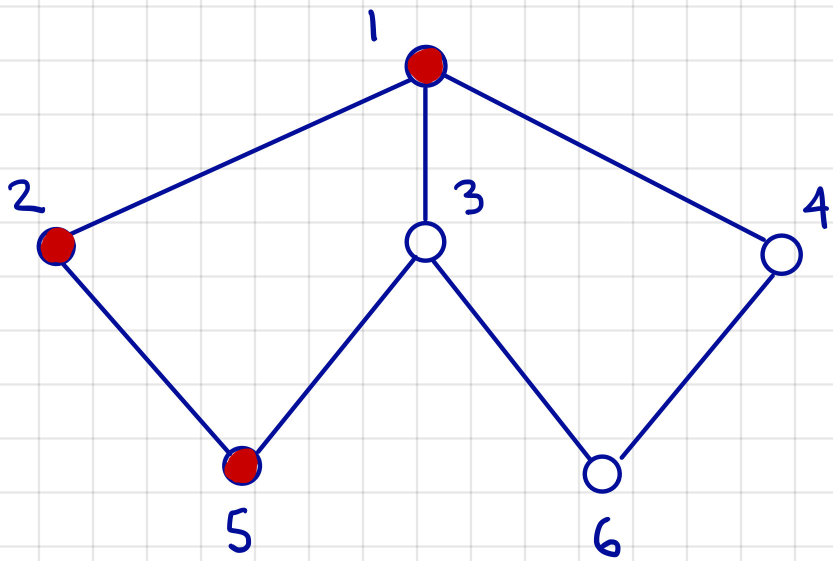
strong duality

Restating Dilworth's theorem

$(V, \leq)$ poset

M matrix where

- columns indexed by V
- rows = char. vectors of maximal chains.



$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & & & 1 & \\ 1 & & 1 & & 1 & \\ 1 & & 1 & & & 1 \\ 1 & & & 1 & & 1 \end{array}\right] \end{array}$$

M matrix where

- columns indexed by V
- rows = char. vectors of maximal chains.

Question: What does (IP) finds for $w = \mathbf{1}$?

$$\max \ \mathbf{1}^T x$$

st

char. vector of chain

$$\begin{bmatrix} & \\ 1\ 1\ 1\ 1 & \\ & \end{bmatrix} x \leq 1, \quad x \geq 0,\ x \text{ integer}$$

$\Longleftrightarrow x \in \{0,1\}^V$ ✓

$\Longrightarrow$ finds maximum size antichain

M matrix where

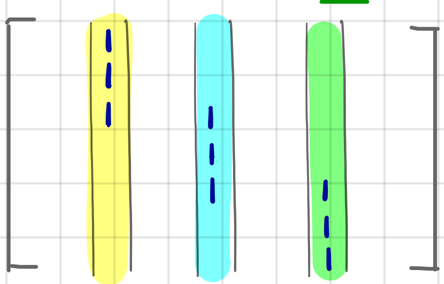
- columns indexed by V
- rows = char. vectors of maximal chains.

Question: What does (ID) finds when $w = 1$?

char. vector of chain

$$\min \ \mathbb{1}^T y$$

st



$$y \geq \mathbb{1}, \quad y \geq 0, \ y \text{ integer}$$

wma $\quad y \in \{0,1\}^e$

$\Longrightarrow$ finds minimum set of chains covering V

M matrix where

- columns indexed by V
- rows = char. vectors of maximal chains.

Then Dilworth's theorem

$\Longleftrightarrow$ $\boxed{z_{IP} = z_{ID} \quad \text{for} \quad w = 1}$

Exercise:

Show, $z_{IP} = z_{ID}$ $\forall\, w \in \mathbb{Z}_+^V$.

What does this say in terms of posets?

Three questions

$$\max\{w^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$
$$\max\{w^T x : Mx \leq 1, x \geq 0\} \qquad (P)$$
$$\min\{1^T y : M^T y \geq w, y \geq 0\} \qquad (D)$$
$$\min\{1^T y : M^T y \geq w, y \geq 0, y \text{ integer}\} \qquad (ID)$$

For all $w \in \mathbb{Z}_+^n$: $z_{IP} \leq z_P = z_D \leq z_{ID}$

Questions: For what class of matrices does

① $z_{IP} = z_{ID} \quad \forall w \in \mathbb{Z}_+^n$

② $z_{IP} = z_P \quad \forall w \in \mathbb{Z}_+^n$

③ $z_{ID} = z_D \quad \forall w \in \mathbb{Z}_+^n$

To address these questions we need to review some polyhedral theory

Elements of polyhedral theory

Pr: Let $P = \{x \geq 0 : Ax \leq b\} \subseteq \mathbb{R}^n$. TFAE

① $P = \text{conv.hull}(P \cap \mathbb{Z}^n)$

② $\bar{x}$ extreme point of $P \Rightarrow \bar{x} \in \mathbb{Z}^n$

③ $\bar{x} \in P$ & rank tight constraints $= n \Rightarrow \bar{x} \in \mathbb{Z}^n$

④ $\boxed{\forall w \in \mathbb{Z}^n, \ \max\{w^T x \mid x \in P\} \in \mathbb{Z} \text{ when max exists.}}$

def: P is integral when ① – ④ hold.

Consider primal/dual pair

$$\max\{w^T x : Ax \leq b,\ x \geq 0\} \qquad (P)$$

$$\min\{b^T y : A^T y \geq c,\ y \geq 0\} \qquad (D)$$

def: $Ax \leq b,\ x \geq 0$ Totally Dual Integral (TDI) if $\forall w \in \mathbb{Z}^n$ where (D) has optimal sol. it has an optimal <u>integer</u> sol.

property of systems not of polyhedra

Pr:

reverse not true

ⓐ $Ax \le b,\ x \ge 0$ TDI & b integer $\Longrightarrow$

ⓑ $\{x \ge 0 : Ax \le b\}$ integral

pf:

Pick $w \in \mathbb{Z}^n$ for which

$$z^* = \max\{w^T x : Ax \le b, x \ge 0\}$$
$$= \min\{b^T y : A^T y \ge c,\ y \ge 0\} \text{ exists.}$$

$Ax \le b,\ x \ge 0$ TDI $\Longrightarrow$

$\exists$ integer opt. sol. $\bar{y}$ $\Longrightarrow$ $z^* = b^T \bar{y} \in \mathbb{Z}$

By char. of integral polyhedra ✓ ▨

Three questions – revisited

$$\max\{w^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$
$$\max\{w^T x : Mx \leq 1, x \geq 0\} \qquad (P)$$
$$\min\{1^T y : M^T y \geq w, y \geq 0\} \qquad (D)$$
$$\min\{1^T y : M^T y \geq w, y \geq 0, y \text{ integer}\} \qquad (ID)$$

$$z_{IP} \leq z_P = z_D \leq z_{ID}$$

Questions: For what class of matrices

1. $z_{IP} = z_{ID} \quad \forall w \in \mathbb{Z}^n_+$ — matrices satisfying 2 & 3
2. $z_{IP} = z_P \quad \forall w \in \mathbb{Z}^n_+$ — $\{x \geq 0 : Mx \leq 1\}$ integral
3. $z_{ID} = z_D \quad \forall w \in \mathbb{Z}^n_+$ — $Mx \leq 1, x \geq 0$ TDI

Questions: For what class of matrices

① $z_{IP} = z_{ID} \quad \forall w \in \mathbb{Z}_+^n$ matrices satisfying ② & ③

② $z_{IP} = z_P \quad \forall w \in \mathbb{Z}_+^n$ $\{x \geq 0 : Mx \leq \mathbb{1}\}$ integral

③ $z_{ID} = z_D \quad \forall w \in \mathbb{Z}_+^n$ $Mx \leq \mathbb{1}, x \geq 0$ TDI

We will see:

$$\boxed{\begin{array}{l}\{x \geq 0 : Mx \leq \mathbb{1}\} \text{ integral} \implies \\ Mx \leq \mathbb{1}, x \geq 0 \quad \text{TDI}\end{array}}$$

Thus ①, ②, ③ equivalent.

Perfection

def: A 0,1 matrix M is perfect if

$$\{x \geq 0 : Mx \leq 1\} \text{ is integral.}$$

We will show, unique combinatorial object associated with perfect matrices,

$\boxed{\text{0,1 matrix is perfect if its maximal rows are the char. vectors of stable sets of a perfect graph}}$

$\Longrightarrow$ study of perfect matrices = study of perfect graphs.

The set covering problem

A min-max relation

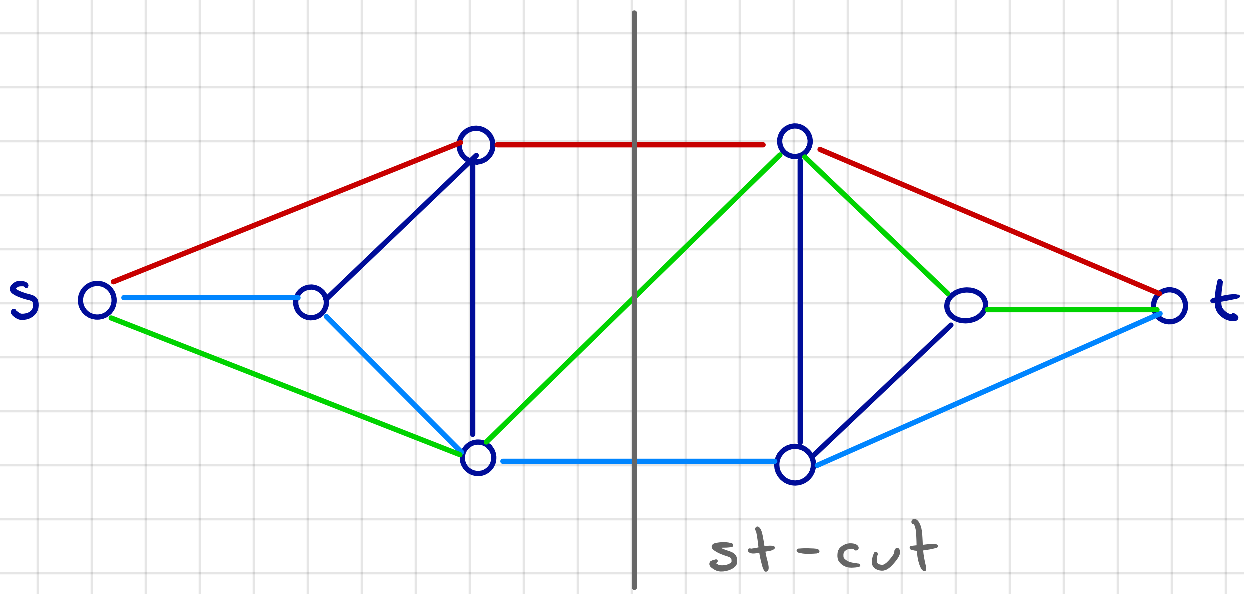
$G=(V,E)$ graph, $s,t \in V$ where $s \neq t$.

Th: [Menger]

Size of minimum st-cut =

edge

max nb of pairwise disjoint st-paths



An Integer Programming framework

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def: Set Covering IP

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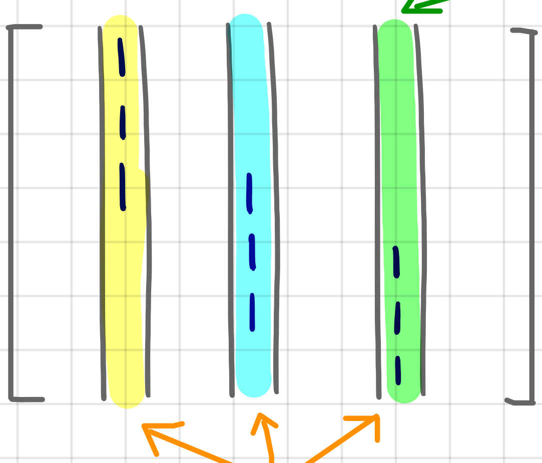
$\longleftrightarrow \ x \in \{0,1\}^n$

Why the name?

incidence vector of set

$M =$ 

$x_j = 1$

Finds min weight family of sets covering groundset

$$\min \{ w^T x : Mx \geq \mathbb{1}, x \geq 0, x \text{ integer} \} \qquad \text{(IP)}$$

$$\min \{ w^T x : Mx \geq \mathbb{1}, x \geq 0 \} \qquad \text{(P)}$$

$$\max \{ \mathbb{1}^T y : M^T y \leq w, y \geq 0 \} \qquad \text{(D)}$$

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Let z_{IP}, z_P, z_D, z_{ID} be optimal values for (IP), (P), (D), (ID) respectively, then

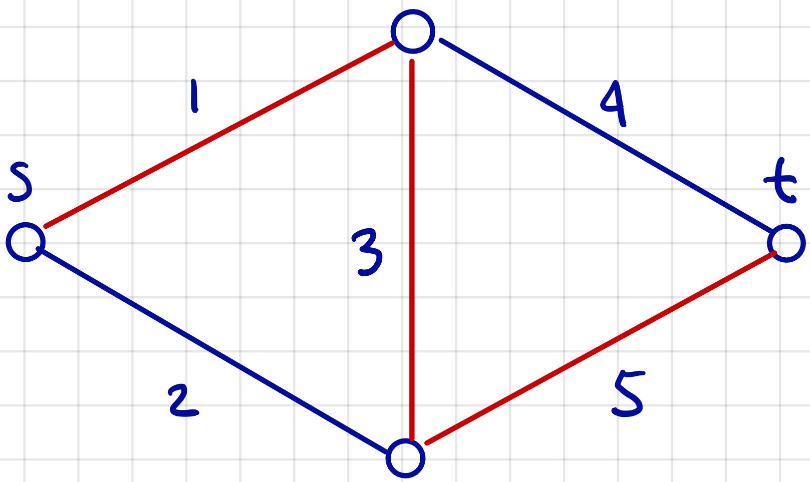
$$\boxed{z_{IP} \geq z_P = z_D \geq z_{ID}}$$

Restating Menger's th

$G = (V, E)$, $s, t \in V$

M matrix where

- columns indexed by E
- rows = char. vectors of st-paths



$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & & 1 & & 1 \\ 1 & & & 1 & \\ & 1 & 1 & 1 & \\ & 1 & & & 1 \end{bmatrix} \end{array}$$

M matrix where

- columns indexed by E
- rows = char. vectors of st-paths

Question: What does (IP) finds for $w = \mathbb{1}$?

$$\min \ \mathbb{1}^T x$$

st

char. vector of st-path

$$\begin{bmatrix} & \vdots & \\ 1 & 1 & 1 & 1 \\ & \vdots & \end{bmatrix} x \geq \mathbb{1}, \quad x \geq 0, \ x \text{ integer}$$

$$\Longleftrightarrow x \in \{0,1\} \checkmark$$

$\Longrightarrow$ finds min size st-cut

M matrix where

- columns indexed by E
- rows = char. vectors of st-paths

Question: What does (ID) finds when $w = \mathbf{1}$?

$$\max \ \mathbf{1}^T y$$

st

char. vector of st-path

$$\begin{bmatrix} 1 & & \\ 1 & & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} y \le \mathbf{1}, \quad \underline{y \ge 0,\ y \text{ integer}}$$

wma $y \in \{0,1\}^E$

$\Longrightarrow$ finds maximum nb of disjoint st-paths

M matrix where

- columns indexed by E
- rows = char. vectors of st-paths

Then Menger's theorem

$$\Longleftrightarrow \quad \boxed{z_{IP} = z_{ID} \quad \text{for} \quad w = 1}$$

Exercise:

Show, $z_{IP} = z_{ID} \quad \forall\, w \in \mathbb{Z}_+^n$

What does this say in terms of graphs?

undirected flows

Three questions

$$\min \{w^T x : Mx \geq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$
$$\min \{w^T x : Mx \geq 1, x \geq 0\} \qquad (P)$$
$$\max \{1^T y : M^T y \leq w, y \geq 0\} \qquad (D)$$
$$\max \{1^T y : M^T y \leq w, y \geq 0, y \text{ integer}\} \qquad (ID)$$

For all $w \in \mathbb{Z}_+^n$: $z_{IP} \geq z_P = z_D \geq z_{ID}$

Questions: For what class of matrices does

① $z_{IP} = z_{ID}$ $\forall w \in \mathbb{Z}_+^n$ matrices satisfying ② & ③

② $z_{IP} = z_P$ $\forall w \in \mathbb{Z}_+^n$ $\{x \geq 0 : Mx \geq 1\}$ integral

③ $z_{ID} = z_D$ $\forall w \in \mathbb{Z}_+^n$ $Mx \geq 1, x \geq 0$ TDI

Questions: For what class of matrices

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③ $z_{ID} = z_D$ $\forall w \in \mathbb{Z}^n_+$ $Mx \leq 1, x \geq 0$ TDI

We saw ③ implies ②

For $i = 1, 2, 3$ let $\mathcal{M}_i$ class of matrices satisfying ⓘ

$$\Longrightarrow \quad \mathcal{M}_3 \subseteq \mathcal{M}_2 \;\&\; \mathcal{M}_1 = \mathcal{M}_2 \cap \mathcal{M}_3 = \mathcal{M}_3$$

def: M ideal if $\{x \geq 1 : Mx \geq 1\}$ integral, i.e $M \in \mathcal{M}_2$

def: M Mengerian if $Mx \geq 1, x \geq 1$ TDI, i.e $M \in \mathcal{M}_1$.

Ideal versus Mengerian

We saw: Mengerian $\Longrightarrow$ Ideal.

Question: does converse hold? No

$$Q_6 = \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{bmatrix}$$

- ideal, why?
- not Mengerian: for $w = 1$ we have

$$2 = z_{IP} > z_{ID} = 1$$

No single combinatorial object associated to ideal or Mengerian matrices.

contrast with perfection

Remainder of lectures

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

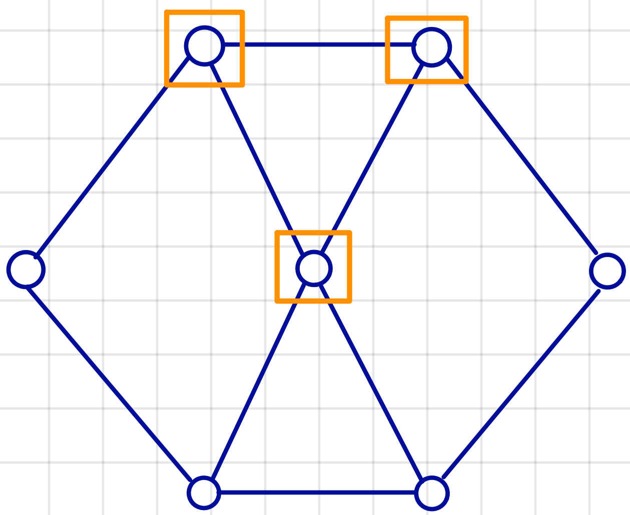
Part IV : The Mengerian property

What are perfect graphs?

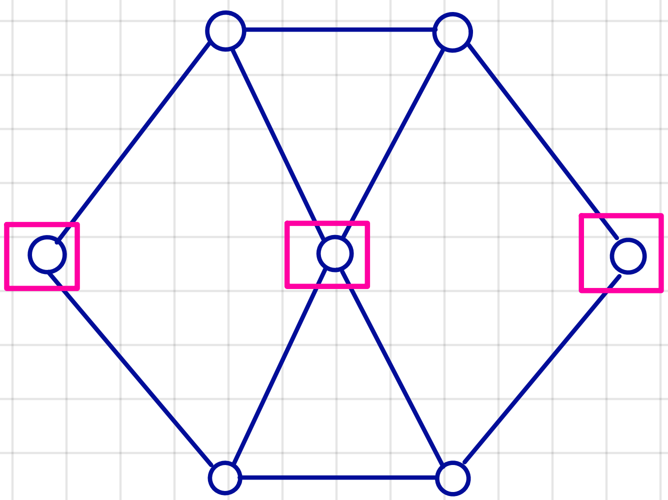
Let $G = (V, E)$ be a graph.

def: a complete subgraph is a clique

a stable set is a clique in complement



clique



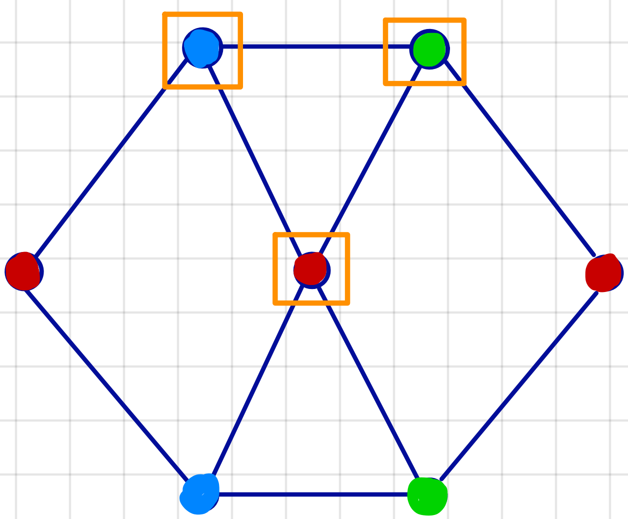
stable set

def: $\omega(G)$ = size of maximum clique of G

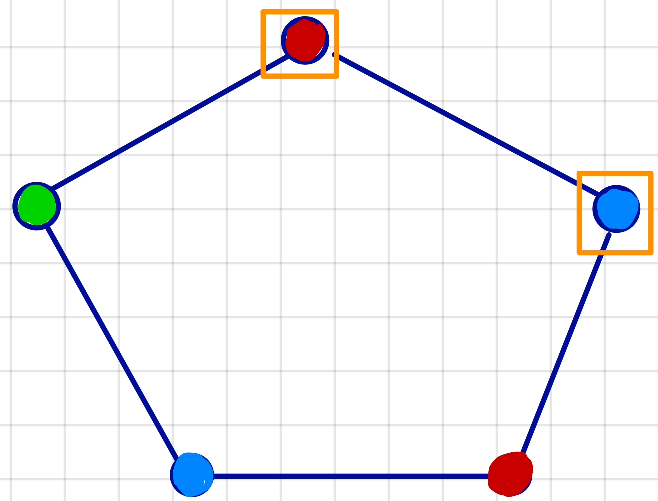
$\chi(G)$ = chromatic number of G

minimum nb of colours in proper colouring

Rem: $\omega(G) \leq \chi(G)$



$\omega(G) = \chi(G) = 3$

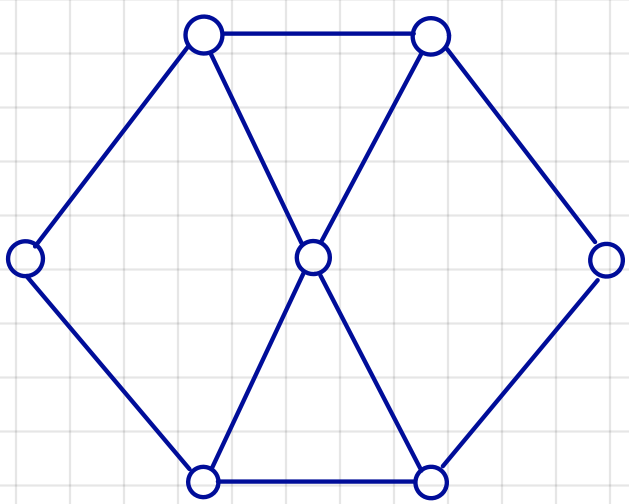


$2 = \omega(G) < \chi(G) = 3$

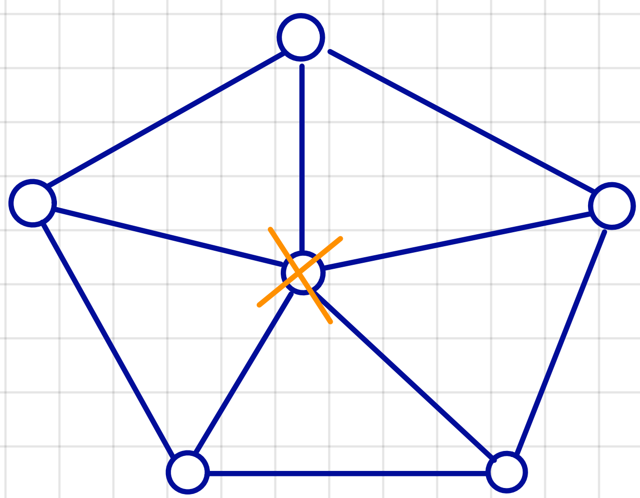
def: H is an induced subgraph of G if H is obtained from G by deleting a subset of vertices

possibly empty

def: a graph is perfect if for all induced subgraphs H: $\omega(H) = \chi(H)$.



perfect



NOT perfect

From perfect matrices to perfect graphs

Pr [Fulkerson]

M perfect $\Longrightarrow$ $Mx \leq 1,\ x \geq 0$ TDI

pf:

Pick $c \in \mathbb{Z}_+^n$

$$z(c) := \max\{c^T x : Mx \leq 1,\ x \geq 0\} \qquad (P_c)$$
$$= \min\{1^T y : M^T y \geq c,\ y \geq 0\} \qquad (D_c)$$

To show: $\boxed{(D_c) \text{ has optimal integer sol.}}$

By induction on $1^T c$.

If $1^T c = 0$ pick $\bar{y} = 0$. ✓

$$z(c) := \max\{c^T x : Mx \leq 1, x \geq 0\} \qquad (P_c)$$
$$= \min\{1^T y : M^T y \geq c, y \geq 0\} \qquad (D_c)$$

To show: (D_c) has optimal integer sol.

Let $\bar{y}$ be optimal sol. to (D_c)

Wma $\bar{y}_1 > 0$.

Define
$$y'_1 = \lceil \bar{y}_1 \rceil - 1$$
$$y'_j = \bar{y}_j \quad \forall j \neq 1$$

Let $a = \text{row}_1(M)$

$$\begin{array}{ll} \min & 1^T y \\ \text{st} & \begin{bmatrix} a & | & \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \end{bmatrix} \geq c \\ & y \geq 0 \end{array}$$

① y' feasible for (D_{c-a})

② $1^T y' = \underbrace{\lceil \bar{y}_1 \rceil - \bar{y}_1 - 1}_{<0} + \overbrace{\boxed{\sum_{j=1}^{n} \bar{y}_j}}^{z(w)} < z(c)$

$$z(c) = \min\{1^T y : M^T y \geq c, y \geq 0\} \quad (D_c)$$

$$z(c-a) = \min\{1^T y : M^T y \geq c-a, y \geq 0\} \quad (D_{c-a})$$

① y' feasible for (D_{c-a})

② $1^T y' < z(c)$

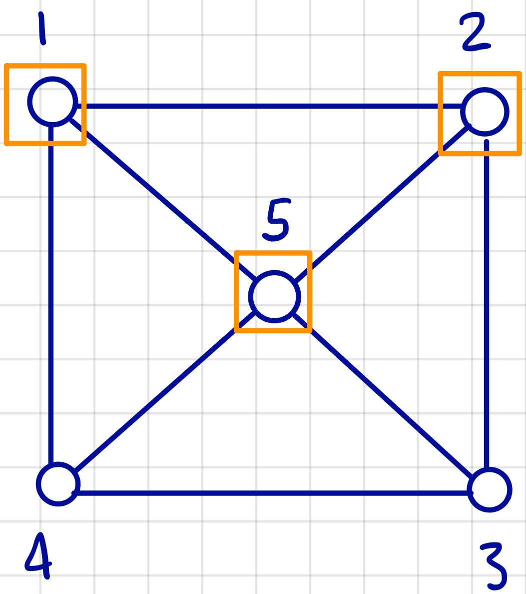
Weak duality $\Longrightarrow$ $z(c-a) < z(c)$

M perfect $\Longrightarrow$ $z(c-a) \leq z(c) - 1$

By induction, $\exists\, \hat{y} \in \mathbb{Z}^n$ optimal sol. to (D_{c-a})

$\Longrightarrow$ $\hat{y} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is optimal sol of (D_c) $\square$

def: M is a clique matrix if its maximal rows are the set of all maximal cliques of some graph G.



$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{c|c|c|c|c} 1 & 1 & & & 1 \\ \hline & 1 & 1 & & 1 \\ \hline & & 1 & 1 & 1 \\ \hline 1 & & & 1 & 1 \end{array}\right] \end{array}$$

def: M is a stable set matrix if its maximal rows are the set of all maximal stable sets of some graph G.

Pr [Padberg]

M perfect $\Longrightarrow$ M clique matrix of graph

$\Longleftrightarrow$ M stable set " " "

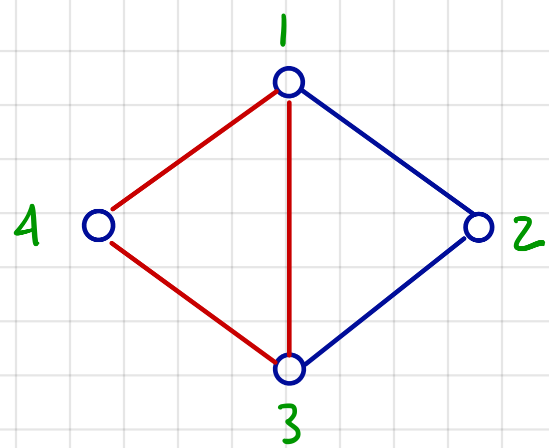
pf: Define $G=(V,E)$ where

$V=\{1,\dots,n\}$ &

$ij \in E$ if $\exists$ row a of M with $a_i = a_j = 1$.

$\Longrightarrow$ rows of M are cliques of G.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$
$$\left[\begin{array}{c|c|c|c} 1 & & 1 & 1 \\ \hline & 1 & 1 & \\ \hline 1 & 1 & & \end{array}\right]$$



Pr [Padberg]

M perfect $\Longrightarrow$ M clique matrix of graph

$\Longleftrightarrow$ M stable set " " "

pf: Define $G=(V,E)$ where

$V=\{1,\dots,n\}$ &

$ij \in E$ if $\exists$ row a of M with $a_i = a_j = 1$.

$\Longrightarrow$ rows of M are cliques of G.

To show:

$$\boxed{\begin{array}{l}\text{For every clique } S: \\ \text{char. vector of } S \leq \text{some row of } M\end{array}}$$

By induction on $|S|$. Wma $|S| \geq 3$.

Let c be char. vector of S.

$$z_P := \max\{c^T x : Mx \leq 1, x \geq 0\} \qquad (P)$$
$$z_{IP} := \max\{c^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$

S clique $\Longrightarrow$ $z_{IP} \leq 1$

Define $\bar{x}$ where

$$\bar{x}_j = \begin{cases} \frac{1}{|S|-1} & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases}$$

$\bar{x}$ feasible for (P) $\Longrightarrow$ $z_P \geq \frac{|S|}{|S|-1} > 1$

But $z_P = z_{IP}$ as M perfect ↯ $\blacksquare$

Pr [Chvátal]

M perfect $\Longrightarrow$

M stable set matrix of perfect graph.

pf: We proved: M stable set matrix of G.

To show: G perfect

① column submatrices of M perfect ✓

② column submatrices of M $\Longleftrightarrow$ stable set matrices of induced subgraphs of G.

By ① + ②, suffices to show:

$$\boxed{\omega(G) = \chi(G)}$$

Let

$$\max\{1^T x : Mx \leq 1,\ x \geq 0,\ x \text{ integer}\} \qquad \text{(IP)}$$

$$\min\{1^T y : M^T y \geq 1,\ y \geq 0,\ y \text{ integer}\} \qquad \text{(ID)}$$

What does (IP) finds ?

$\max\ 1^T x$

st

char. vector of stable set

$$\begin{bmatrix} & & & \\ 1 & 1 & 1 & 1 \\ & & & \end{bmatrix} x \leq 1, \quad x \geq 0,\ x \text{ integer}$$

$\Longleftrightarrow x \in \{0,1\}^V$

$\Longrightarrow$ finds maximum size clique, i.e $\omega(G)$

Let

$$\max\{\mathbb{1}^T x : Mx \le \mathbb{1},\ x \ge 0,\ x \text{ integer}\} \qquad (IP)$$

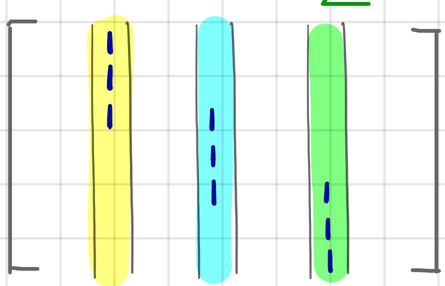
$$\min\{\mathbb{1}^T y : M^T y \ge \mathbb{1},\ y \ge 0,\ y \text{ integer}\} \qquad (ID)$$

What does (ID) finds ?

$\min\ \mathbb{1}^T y$

st

char. vector of stable set



$y \ge \mathbb{1}$, $\quad y \ge 0$, y integer

wma $y_S \in \{0,1\}$

$\Longrightarrow$ finds minimum set of stable sets covering V, i.e $\chi(G)$.

Thus

$$\omega(G) = \max\{1^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \quad (IP)$$
$$\chi(G) = \min\{1^T y : M^T y \geq 1, y \geq 0, x \text{ integer}\} \quad (ID)$$

M perfect & thus $Mx \leq 1, x \geq 0$ TDI

$\Longrightarrow \omega(G) = \chi(G)$

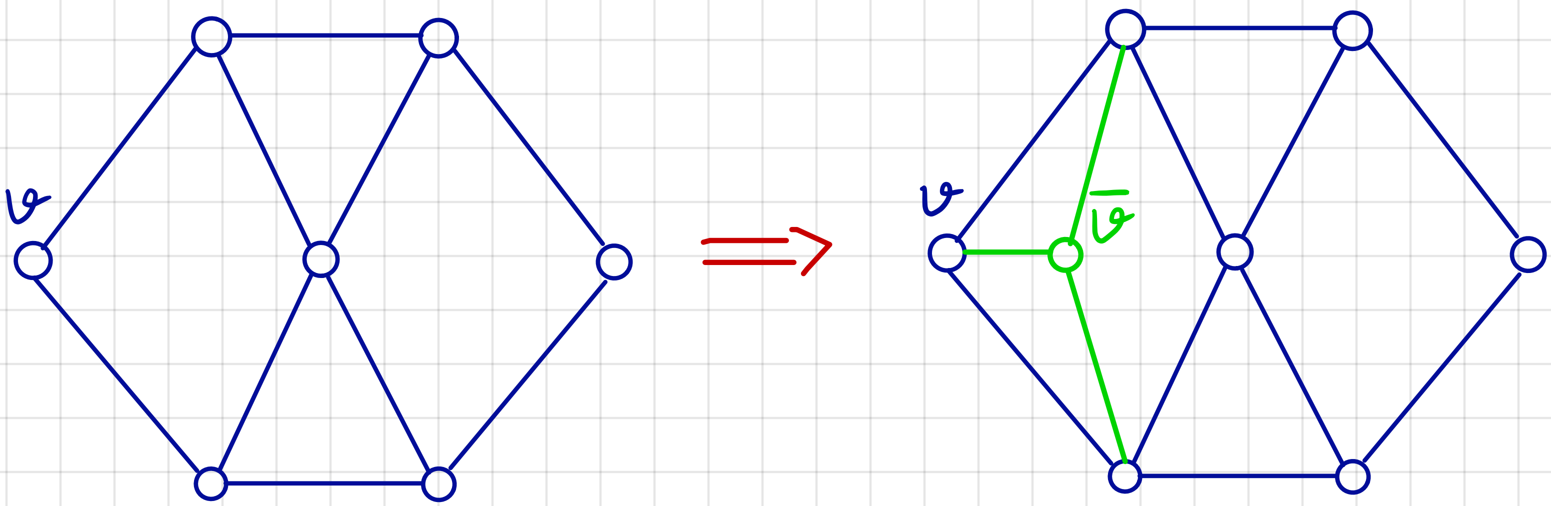
We showed:

To understand perfect matrices it suffices to understand perfect graphs!

Is the converse true?

From perfect graphs to perfect matrices

def: duplicating a vertex v is to add a new vertex $\bar{v}$, join it to all neighbors of v and to v.



Duplication preserve perfection !

Pr [Lovász]

G perfect, H obtained from G by duplicating vertex v. Then H is perfect.

pf: We show: $\omega(H) = \chi(H)$

Induced subgraphs? exercise

$\exists$ colour classes $C_1, \dots, C_{\omega(G)}$ of G.

Wma $v \in C_1$

Case 1: $v \in S$: maximum clique of G.

$S \cup \{\bar{v}\}$ clique of H $\Longrightarrow$ $\omega(H) = \omega(G) + 1$

As $\{\bar{v}\}, C_1, \dots, C_{\omega(G)}$ is proper colouring of H

$\omega(H) = \chi(H)$

Case 2: v is in no maximum clique of G.

Let $G' = G \setminus (C_1 - v)$

Consider maximum clique S of G

Since $|S \cap C_i| \leq 1$, $\forall i = 1, .., \omega(G)$

$\Longrightarrow$ $|S \cap C_i| = 1$,

$\Longrightarrow$ $|S \cap (C_1 - v)| = 1$

$\Longrightarrow$ $\omega(G) - 1 = \omega(G') = \chi(G')$.

Let $D_1, \dots, D_{\omega(G)-1}$ be colouring of G'

Then $D_1, \dots, D_{\omega(G)-1}, C_1 - v \cup \bar{v}$ is proper colouring of H and $\omega(H) = \chi(H)$. $\blacksquare$

Pr[Chvátal]

G perfect graph, M stable matrix of G

$\Longrightarrow$ M perfect

pf:

$$z_{IP} := \max\{c^Tx : Mx \leq 1, x \geq 0, x \text{ integer}\} \quad (IP)$$

$$\rightarrow z_P := \max\{c^Tx : Mx \leq 1, x \geq 0\} \quad (P)$$

$$z_D := \min\{1^Ty : M^Ty \geq c, y \geq 0\} \quad (D)$$

$$z_{ID} := \min\{1^Ty : M^Ty \geq c, y \geq 0, y \text{ integer}\} \quad (ID)$$

$$z_{IP} \leq z_P = z_D \leq z_{ID}$$

To show: $\forall c \in \mathbb{Z}_+^n$: z_P integer.

Let G_c be obtained from G by $\forall v \in V$:

- if $c_v = 0$: delete v
- if $c_v \geq 2$: duplicate v, $c_v - 1$ times

① $z_{IP} = \omega(G_c)$

pf:

$$\max \; c^T x$$

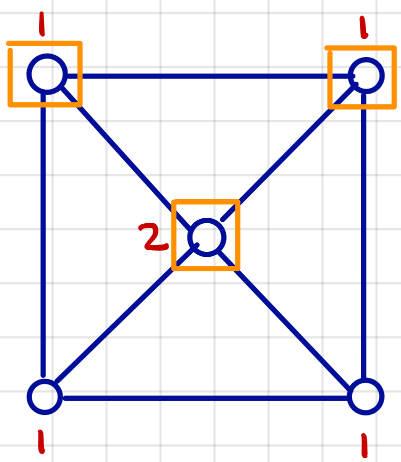
st

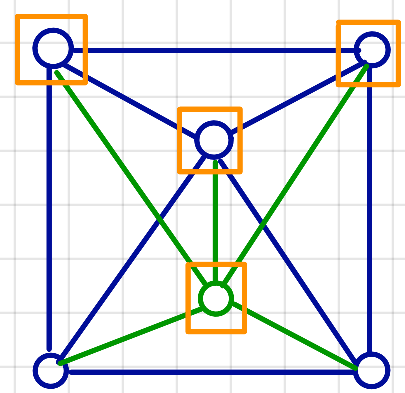
stable set

$$\begin{bmatrix} \\ 1\;1\;1\;1 \\ \\ \end{bmatrix} x \geq 1, \quad x \geq 0, \; x \text{ integer} \qquad (IP)$$

$\Longrightarrow$ find max weight clique of $G = \omega(G_c)$

G





G_c

② $z_{ID} = \chi(G_c)$

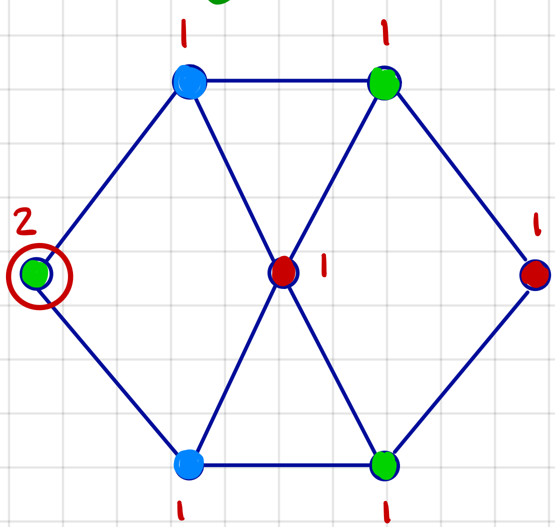
pf:

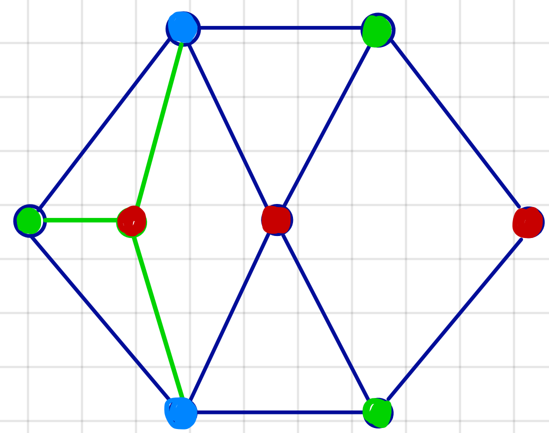
$$\min \; \mathbb{1}^T y$$

st $\begin{bmatrix} \vdots & \vdots & \vdots \end{bmatrix} y \geq c, \quad y \geq 0, \quad y \text{ integer} \qquad (ID)$

stable set

$\Longrightarrow$ finds minimum size family of stable sets covering each $v \in V$, $\geq c_v$ times $= \chi(G_c)$

G 

 G_c

Then $\forall c \in \mathbb{Z}_+^n$

$$\omega(G_c) \overset{①}{=} z_{IP} \le z_P = z_D \le z_{ID} \overset{②}{=} \chi(G_c)$$

= since G and hence G_c perfect

$\Longrightarrow$ z_P integer $\Longrightarrow$ M perfect //

We showed:

To understand perfect graphs it suffices to understand perfect matrices.

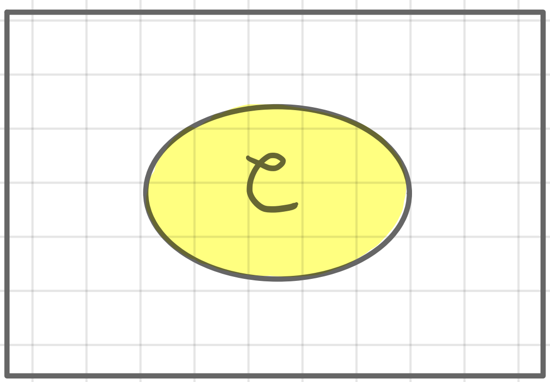
A characterization of perfect graphs/matrices

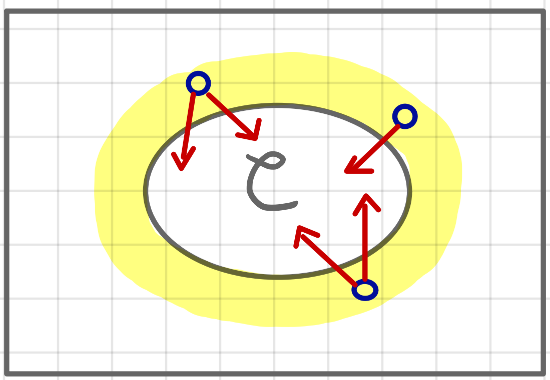
2 ways of characterizing a minor-closed class $\mathcal{C}$ of objects

① describe what is inside
structure theorem

② describe what is just outside
excluded minor theorem

world



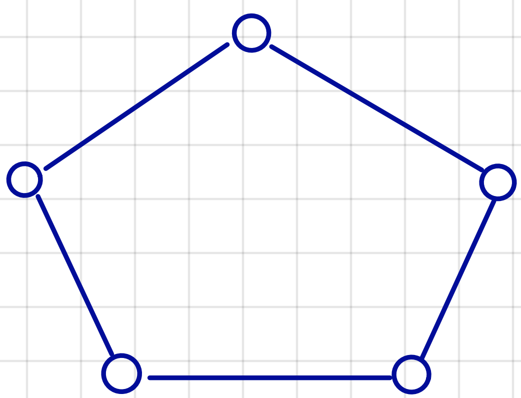


def: a graph is minimally imperfect if

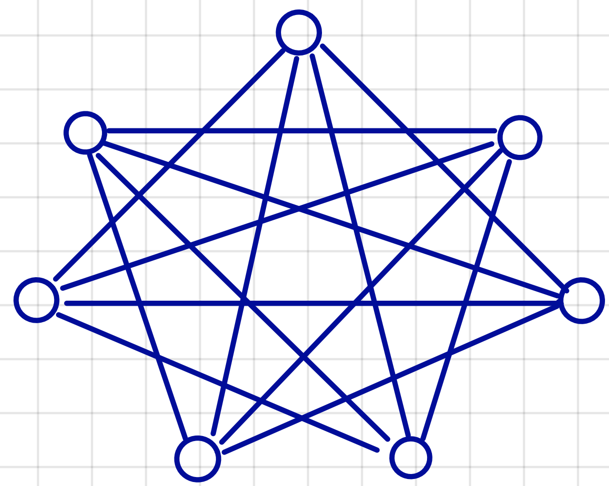
- it is not perfect
- proper induced subgraph are perfect

Examples?

def: an odd hole is an odd length ≥ 5 cordless cycle. An odd antihole is the complement of an odd hole.



odd hole



odd antihole

Strong perfect graph theorem
[Chudnovsky, Robertson, Seymour, Thomas]

The only minimally imperfect graphs are odd holes and odd antiholes.

$\Longrightarrow$

Perfect graph theorem [Lovász]

If a graph is perfect so is its complement.

What about perfect matrices?

def: a matrix is minimally imperfect if

- it is not perfect
- proper column submatrices are perfect

Th: restatement of Strong Perfect Graph th.

The only minimally imperfect matrices are

① clique matrices of odd holes

② " " " " antiholes

③ $$\left[\begin{array}{c} \begin{matrix} 0 & & & 1 \\ & 0 & & \\ & & 0 \ddots & \\ 1 & & & 0 \end{matrix} \\ \hline ? \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}}\right] \text{maximal rows} \\ \vphantom{?} \end{array}$$

Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

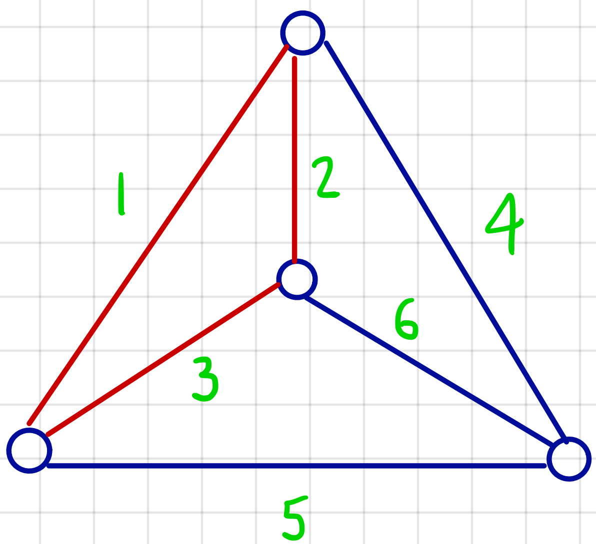
Recall the key definition:

A 0,1 matrix M is ideal if $\{x \geqslant 0 : Mx \geqslant 1\}$ integral.

Examples of ideal matrices

Odd cycles

def: M is an odd cycle matrix if its rows are the char. vectors of odd cycles of a graph.



$$M = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{bmatrix} \end{array}$$

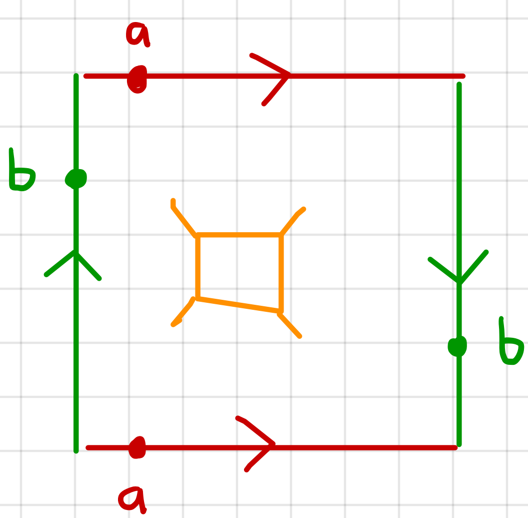
Q_6

Th [Barahona / Schrijver]

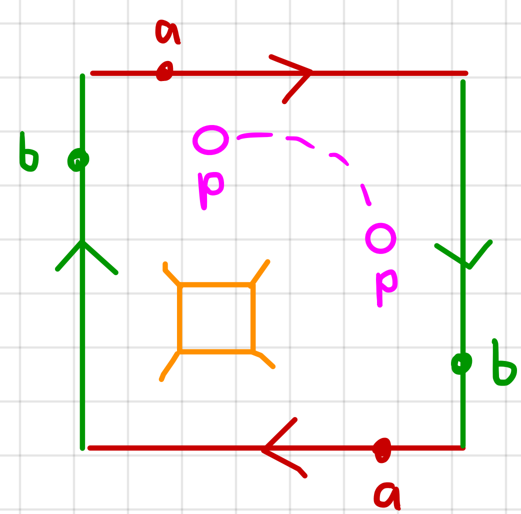
M odd cycle matrix of G. Then M is ideal if

1. G is planar <u>or</u>
2. G has EFE on Klein Bottle <u>or</u>
3. G has EFE on Pinched Projective Plane

even face embedding



Klein Bottle



Pinched Projective Plane

Question:

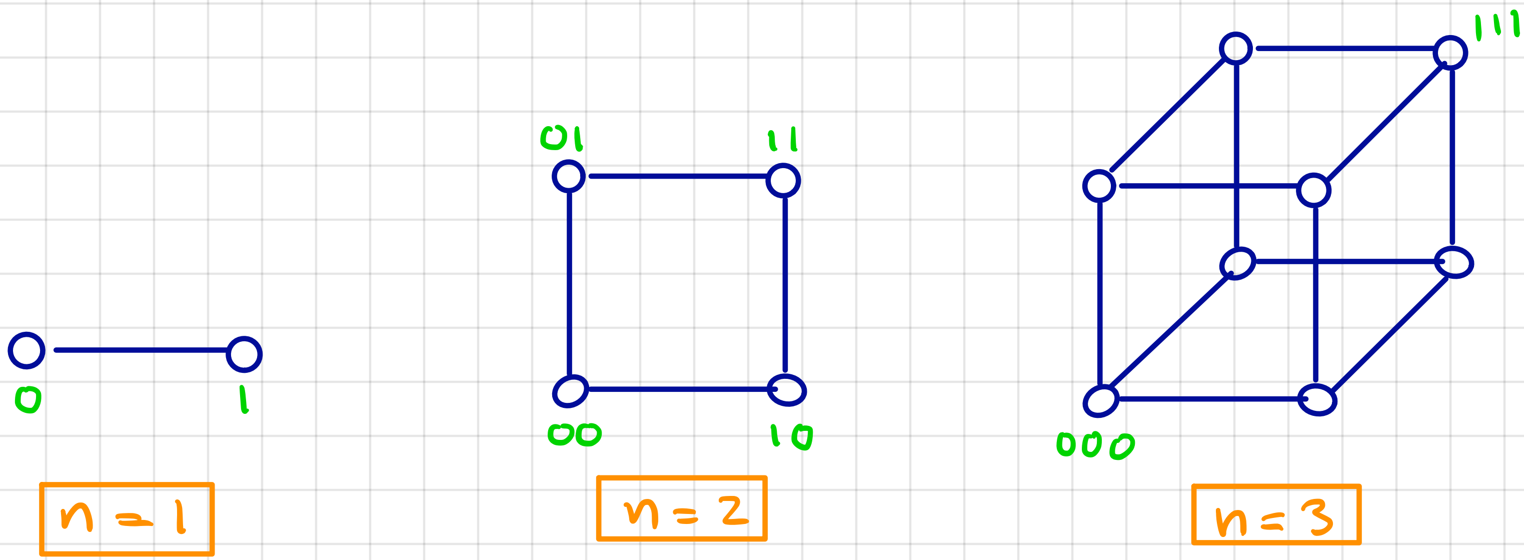
Do all ideal matrices arise from graph-like objects?

NO!

Let us see a completely geometric construction.

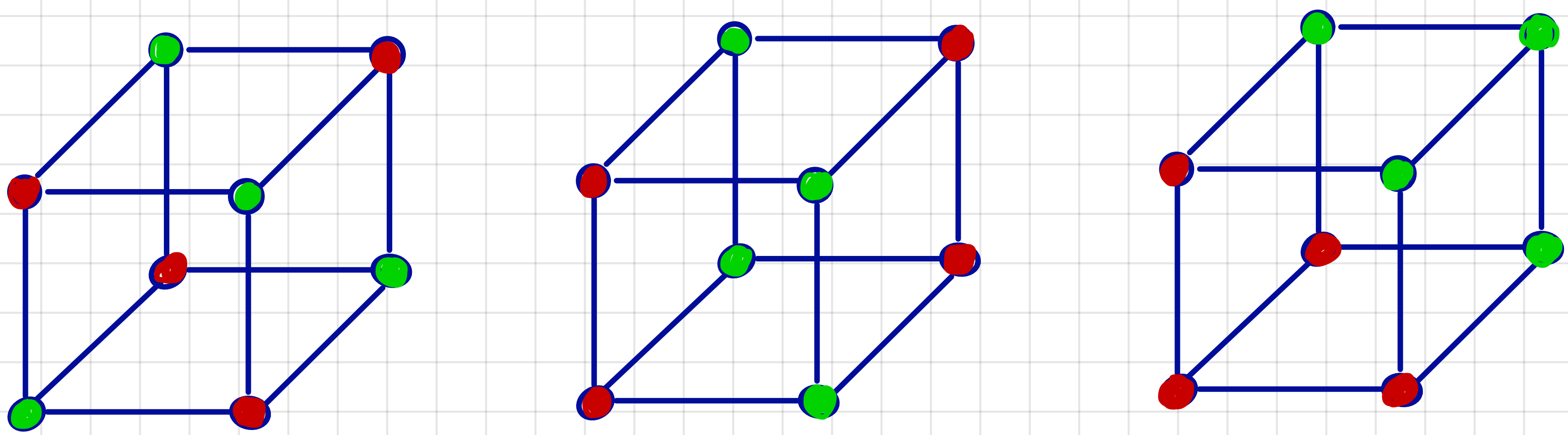
A geometric construction

def: Q_n is the n-dimensional hypercube



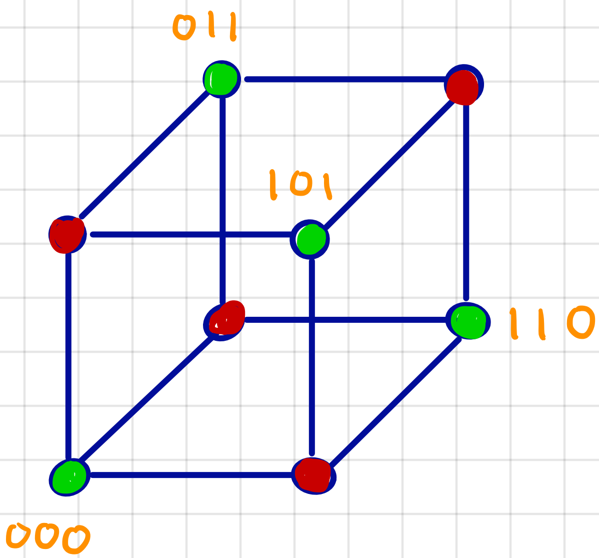
Recipe for building an ideal matrix

Step 1 Partion vertices of Q_n into sets R, G such that the components of Q_n induced by R are complete hypercubes.



ok ok not ok

Step 2: M is matrix with rows $[\bar{x}, 1-\bar{x}]$ for all $\bar{x} \in G$.



$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\nwarrow Q_6$

Pr: resulting matrix is always ideal.
[Angulo, Ahmed, Dey, Kaibel] + [Abdi, Pashkovich, Cornuéjols]

Ideal matrices also arise from

1. directed cuts
2. directed joins

[Lucchesi-Younger]

3. T-cuts
4. T-joins

[Edmonds, Johnson]

5. ...

Finding a structure theorem for ideal matrices appears out of reach

Operations preserving idealness

Clutters

For set covering polyhedron $\{x \geq 0 : Mx \geq 1\}$
wma all rows are minimal

$$x_1 + x_2 \geq 1 \implies x_1 + x_2 + x_3 \geq 1$$

def: family of sets $\mathcal{F}$ is clutter if
$\forall S, S' \in \mathcal{F}$, $S \subseteq S' \implies S = S'$

def: $M(\mathcal{F})$ = matrix with rows = char. vectors of $S \in \mathcal{F}$.

$$\mathcal{F} = \{135, 146, 245, 236\} \implies M(\mathcal{F}) = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & & 1 & & 1 & \\ 1 & & & 1 & & 1 \\ & 1 & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{bmatrix} \end{array}$$

Blocker

def: B is a cover of clutter $\mathcal{F}$ if
$B \cap S \neq \emptyset$ for all $S \in \mathcal{F}$

def: the set of all minimal covers of clutter $\mathcal{F}$ is the blocker $b\mathcal{F}$ of $\mathcal{F}$.

Example:

① Blocker of st-paths = st-cuts

② Blocker of st-cuts = st-paths

Pr [Isbell / Edmonds Fulkerson]

$b\,b\mathcal{F} = \mathcal{F}$

clutters come in pairs

Th [Lehman]

$\mathcal{F}$ is ideal $\iff$ $b\mathcal{F}$ ideal

ideal clutters come in pairs.

Counterpart of Perfect Graph Theorem !

Next outline proof ideas !

def: $P(\mathcal{F}) = \{x \geq 0 : M(\mathcal{F})x \geq 1\}$

$\mathcal{F}$ not ideal $\Longrightarrow$

$P(\mathcal{F})$ has fractional point $\bar{x}$ $\boxed{\Longrightarrow}$

$\mathrm{conv}[P(b\mathcal{F}) \cap \mathbb{Z}^n]$ has facet

$\alpha^T x \geq 1$ where α fractional $\Longrightarrow$

$\mathrm{conv}[P(b\mathcal{F}) \cap \mathbb{Z}^n] \subsetneq P(b\mathcal{F})$ $\Longrightarrow$

$P(b\mathcal{F})$ not integral $\Longrightarrow$

$b\mathcal{F}$ not ideal

$\blacksquare$

Minors

$\mathcal{F}$ clutter with e in ground set.

deletion: $\mathcal{F} \setminus e = \{S \in \mathcal{F} : e \notin S\}$

contraction: $\mathcal{F}/e$ = minimal sets in $\{S - e : S \in \mathcal{F}\}$

Example:

① $\mathcal{F}$ clutter of st-paths of G

- $\mathcal{F} \setminus e$ = clutter st-path of $\boxed{G \setminus e}$
- $\mathcal{F}/e$ = " " " $\boxed{G/e}$

② $\mathcal{F}$ clutter of st-cuts of G

- $\mathcal{F} \setminus e$ = clutter st-cuts of $\boxed{G/e}$
- $\mathcal{F}/e$ = " " $\boxed{G \setminus e}$

def: sequence of deletion + contractions

$\Longrightarrow$ minor

order of sequence does not matter

$\mathcal{F} \setminus I / J$ = clutter obtained from $\mathcal{F}$ by deleting I, contracting J.

st-paths / st-cuts suggest:

$$b(\mathcal{F} / I \setminus J) = b(\mathcal{F}) \setminus I / J$$

Always true!

Why are minors useful?

Pr: If $\mathcal{F}$ is ideal then so are minors

Pf:

Recall: $P(\mathcal{F}) = \{x \geq 0 : M(\mathcal{F})x \geq 1\}$

1. $P(\mathcal{F} \setminus e)$: set $x_e = 0$ in $P(\mathcal{F})$
2. $P(\mathcal{F} / e)$: project x_e in $P(\mathcal{F})$

Faces & projection of integral polyhedra are integral

$\square$

Excluded minors

def: $\mathcal{F}$ is minimally non-ideal (mni) if

- $\mathcal{F}$ non-ideal
- every proper minor of $\mathcal{F}$ ideal.

Exercise:

Show $\mathcal{F}$ mni $\Longleftrightarrow$ $b\mathcal{F}$ mni

mni clutters come in pairs

A menagerie of mni clutters

Deltas Δ_n

$$\begin{bmatrix} & 1 & 1 \\ 1 & 1 & \\ 1 & & 1 \end{bmatrix} \qquad \begin{bmatrix} & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & & & 1 \end{bmatrix} \qquad \begin{bmatrix} & 1 & 1 & 1 & 1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ 1 & & & 1 & \\ 1 & & & & 1 \end{bmatrix} \quad \cdots$$

$n=3 \qquad n=4 \qquad n=5$

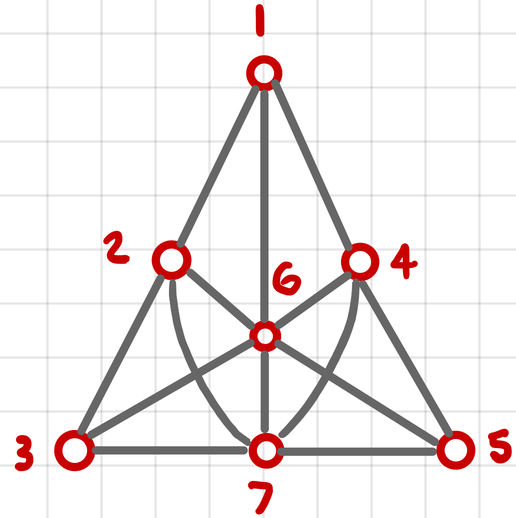
Odd holes C_2^n

$$\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ 1 & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ & & & 1 & 1 & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \\ 1 & & & & & & 1 \end{bmatrix} \quad \cdots$$

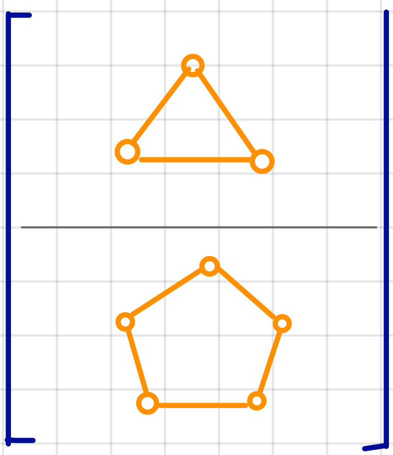
$n=3 \qquad n=5 \qquad n=7$

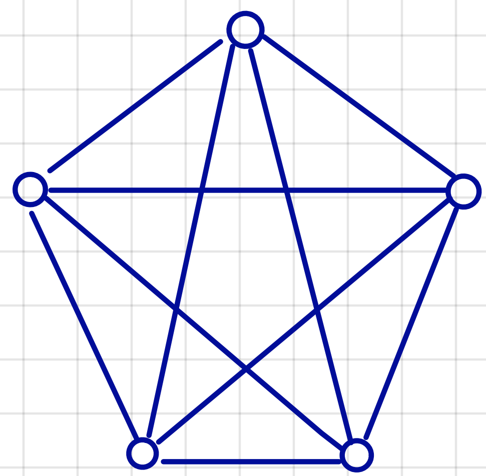
Lines of Fano L_7

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{array}$$
$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & & & & \\ 1 & & & 1 & 1 & & \\ 1 & & & & & 1 & 1 \\ & 1 & & 1 & & & 1 \\ & 1 & & & 1 & 1 & \\ & & 1 & 1 & & 1 & \\ & & 1 & & 1 & & 1 \end{array}\right]$$



Odd circuits of K_5 O_5





We also have:

$$\underbrace{b\Delta_n}_{=\Delta_n},\ bC_2^{\,n},\ \underbrace{bL_7}_{=L_7},\ bO_5$$

In fact there are lots more.

It is a zoo...

But an orderly one !!

Th [Lehman]

$M \neq \Delta_n$ mni, $P = \{x \geq 0 : Mx \geq 1\} \subseteq \mathbb{R}^n$

$\bar{x}$ fractional extreme point of P. Then

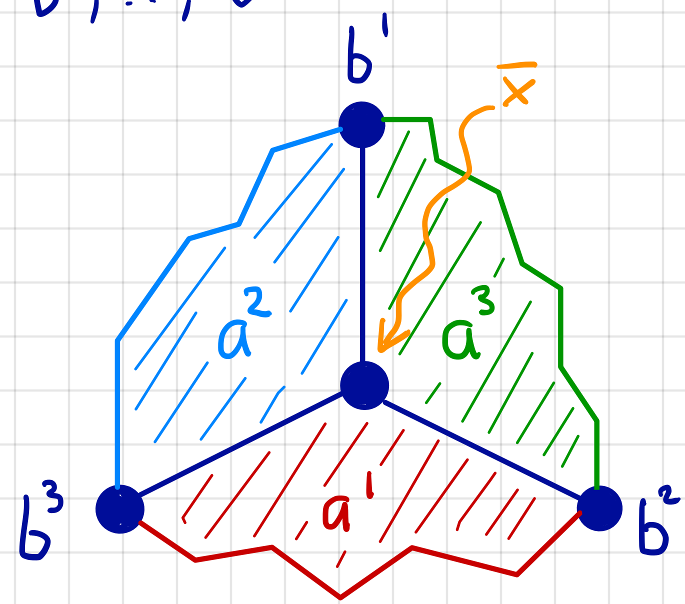
① $\bar{x} = \frac{1}{r} 1$ for some $r \geq 2$ no other fractional extreme point

② exactly n tight constraints $a^i x = 1$, $i = 1, \dots, n$ for $\bar{x}$

③ $\bar{x}$ has exactly n neighbors $b^1, \dots, b^n$

④ $A = \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}$ $B = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{bmatrix}$

$AB^T = J + dI$, $d \geq 1$



Example

$$M = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}$$

$\bar{x} = \left(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right)^T$

unique fractional extreme point

Then

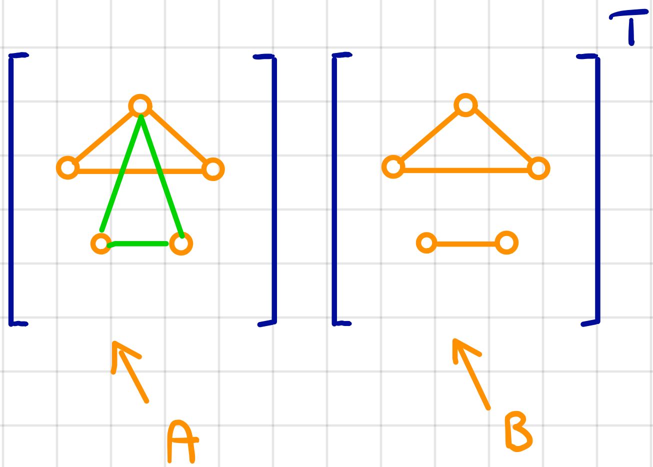
$$\underbrace{\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & & 1 & \\ & 1 & 1 & & 1 \\ 1 & & 1 & 1 & \\ & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \end{bmatrix}}_{B}{}^{T} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}}_{J+I}$$

Example

$$M = \begin{bmatrix} \text{odd} \\ \text{circuits} \\ K_5 \end{bmatrix} \qquad \bar{x} = (1/3, \dots, 1/3)^T$$

unique fractional extreme point

Then



$$= \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

Binary clutters

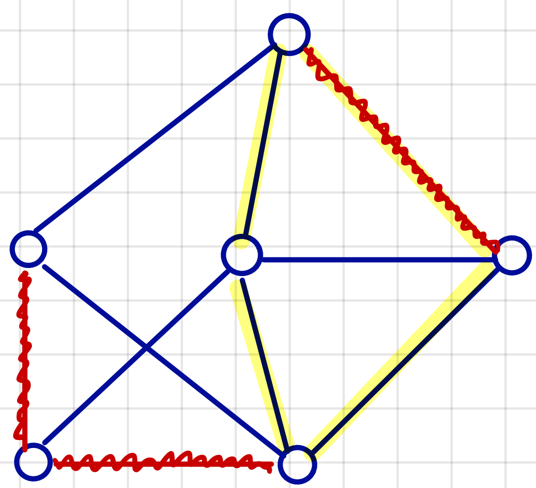
Signed graphs

def: signed graph is a pair (G, Σ).

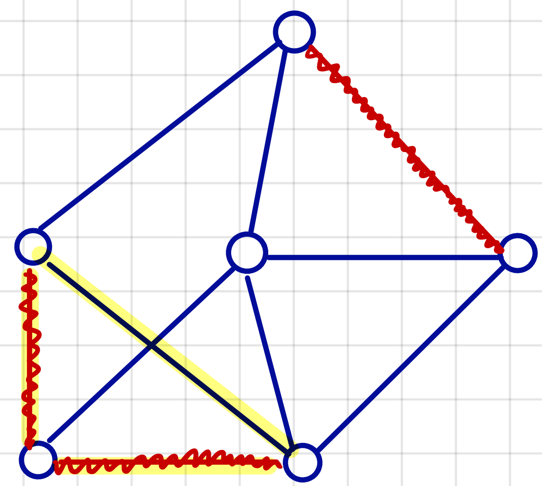
graph — signature = subset of edges

def: A circuit C of G is odd if $|C \cap \Sigma|$ odd.

Σ



odd circuit



even circuit

Binary clutters – definition

def: signed matroid is a pair (M, Σ)

binary matroid

signature = subset of elements

def: A circuit C of M is odd if $|C \cap \Sigma|$ odd.

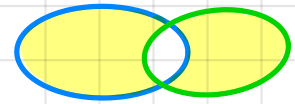
---

def: A clutter is binary if its sets are the odd circuits of a signed matroid

$\Longrightarrow$ odd circuits of a signed graph form a binary clutter.

case: M is graphic

Characterization

symmetric diff 

Pr [Lehman / Seymour]

The following are equivalent for clutter $\mathcal{F}$:

① $\mathcal{F}$ is binary

② $\forall S_1, S_2, S_3 \in \mathcal{F}$, $S_1 \triangle S_2 \triangle S_3 \supseteq S \in \mathcal{F}$

③ $\forall S \in \mathcal{F}$, $B \in b\mathcal{F}$, $|S \cap B|$ odd

Note ③ $\Longrightarrow$

Pr: If a clutter is binary so is its blocker.

Pr: If a clutter is binary so are its minors.

Examples

1. T-cuts
2. T-joins

blockers

3. odd circuits of signed graphs
4. signatures of signed graphs

blockers

5. st-T-cuts
6. odd st-walks

...

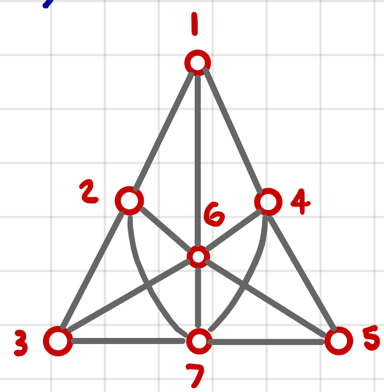
The Flowing Conjecture

The Flowing Conjecture [Seymour]

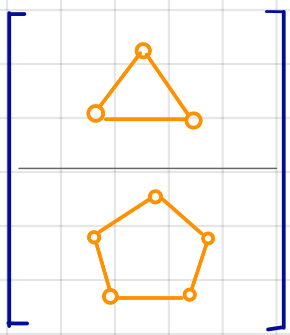
The only mni binary clutters are

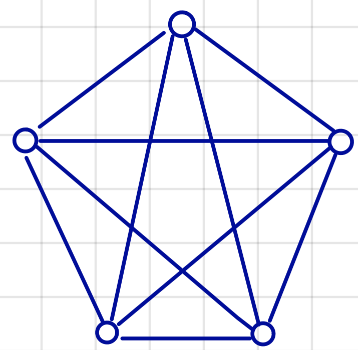
① L_7 (the line of Fano)

$$M(L_7) = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & & & & \\ 1 & & & 1 & 1 & & \\ 1 & & & & & 1 & 1 \\ & 1 & & 1 & & & 1 \\ & 1 & & & 1 & 1 & \\ & & 1 & 1 & & 1 & \\ & & 1 & & 1 & & 1 \end{bmatrix} \end{array}$$



② O_5 (the odd circuits of K_5)

$M(O_5) =$ 



③ bO_5

Restatement:

$\mathcal{F}$ mni binary clutter $\Longrightarrow$ $\mathcal{F}$ or $b\mathcal{F}$ is one of

① L_7 (line of Fano)

② O_5 (odd circuits of K_5)

Easier question?

Weak Flowing Conjecture:

$\mathcal{F}$ mni binary clutter $\Longrightarrow$

$\mathcal{F}$ or $b\mathcal{F}$ has a triangle

set of size 3

Th [Abdi, Guenin]

The Weak Flowing Conjecture implies the Flowing Conjecture.

Graphic case

$\mathcal{F}$ binary clutter $\Longleftrightarrow$
$\mathcal{F}$ = odd circuits of signed matroid (M, Σ)

Suppose M graphic. Then
$\mathcal{F}$ = odd circuits of signed graph (G, Σ)

Question: What does the Flowing Conj. say in that case?

"The only mni clutter of odd circuits is O_5".

odd circuits of K_5

Is it true? Yes

Th [Guenin]

The only mni clutter of odd circuits of a signed graph is O_5 – the odd circuits of (K_5, EK_5).

adding a few topological arguments $\Longrightarrow$

Th:

M odd cycle matrix of G. Then M is ideal if

1. G is planar <u>or</u>
2. G has EFE on Klein Bottle <u>or</u>
3. G has EFE on Pinched Projective Plane

even face embedding

We will see a proof next

Flowing Conjecture : proof of graphic case

Our goal is to prove :

Th [Guenin]
The only mni clutter of odd circuits of a signed graph is $\mathcal{O}_5$ – the odd circuits of (K_5, EK_5).

Preliminaries – signed graphs

Let (G, Σ) signed graph

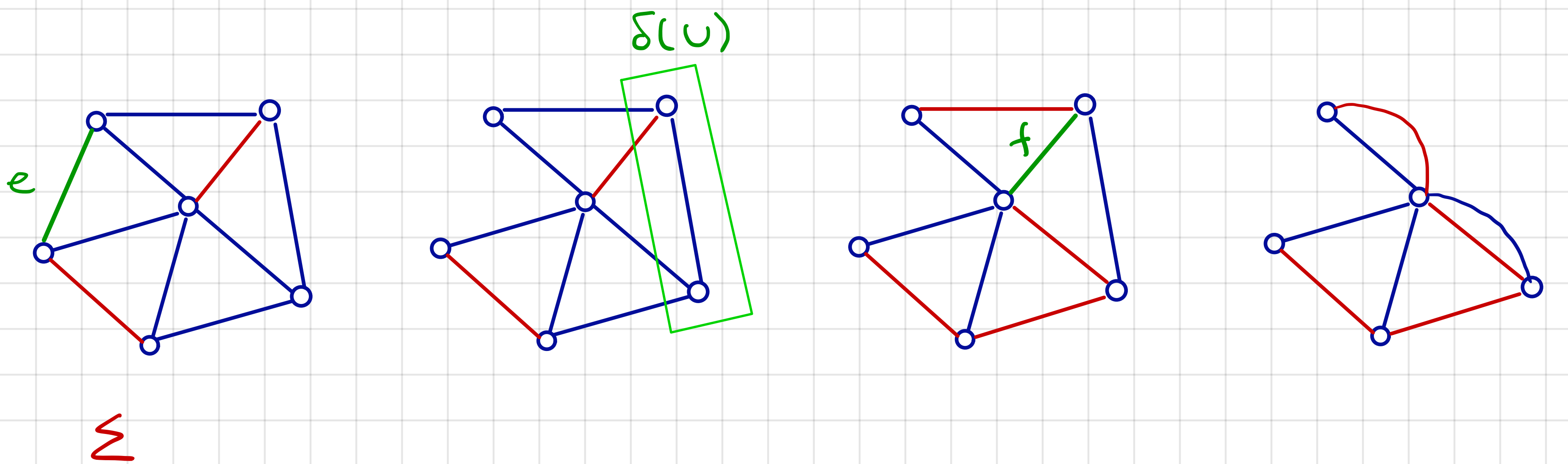
signature

def: minors

① delete edge e : $(G \setminus e, \Sigma - e)$

② contract edge $e \notin \Sigma$: $(G/e, \Sigma)$

③ resigning : $(G, \Sigma \,\triangle\, \delta(U))$ for some cut $\delta(U)$



Preliminaries – signed graphs

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3. resigning : $(G, \Sigma \triangle \delta(U))$ for some cut $\delta(U)$

Rem: minors for clutter of odd circuits

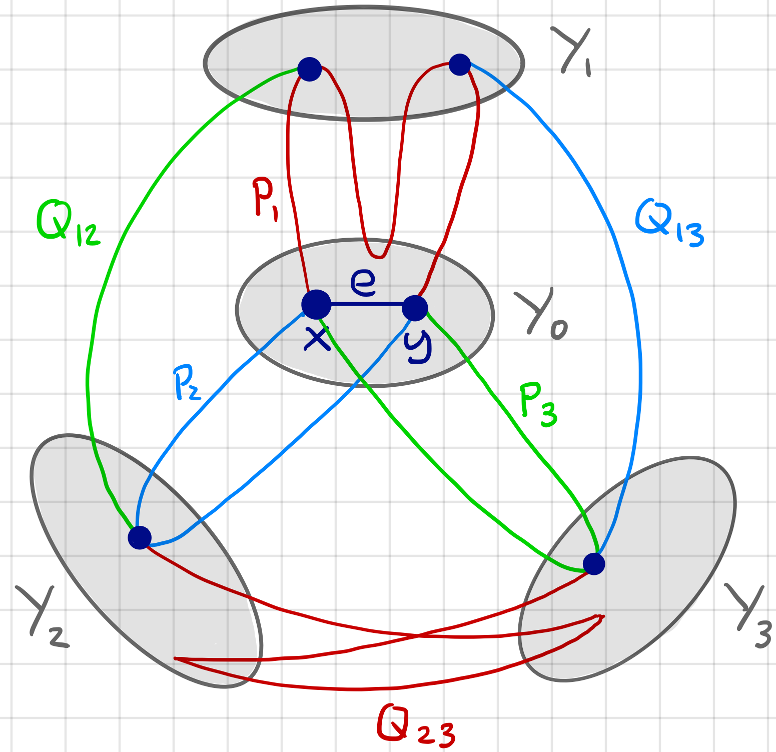
$\Longleftrightarrow$ minors for signed graphs

Whirlpool lemma: [Schrijver]

$G=(V,E)$ graph, $e=xy\in E$

Y_0, Y_1, Y_2, Y_3 partition V

P_1, P_2, P_3 internally disjoint paths



1. $x, y \in Y_0$
2. Y_0, Y_1, Y_2, Y_3 stable sets of $G\setminus e$
3. $VP_i \subseteq Y_0 \cup Y_i$, $i=1,2,3$
4. $\forall\, i,j \in \{1,2,3\}$, $i\neq j$, $\exists$ path Q_{ij} between VP_i & VP_j in $Y_i \cup Y_j$

$\Longrightarrow$

(G, EG) has minor (K_5, EK_5)

odd K_5

Restating Lehman's theorem for binary clutters

Th: $\mathcal{F}$ mni binary clutter, $\mathcal{B} := b\mathcal{F}$ & let n denote the nb of elements in groundset.

① $\mathcal{F}$ has minimum sets $A_1, \ldots, A_n$
② $\mathcal{B}$ " " " $B_1, \ldots, B_n$
③ $\exists\, d \geq 1$ such that $\forall i, j \in \{1, \ldots, n\}$

$$|A_i \cap B_j| = \begin{cases} d+1 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

def: then B_i is the mate of A_i

③ $\Longrightarrow$ unique mate

pf:

$\tilde{\mathcal{F}} \neq \Delta_n$ as Δ_n not binary.

$\bar{x}$ fractional extreme point of $P = \{x \geq 0 : Mx \geq 1\} \subseteq \mathbb{R}^n$

Lehman Th $\Longrightarrow$

> $\bar{x} = \frac{1}{r} 1$ for some $r \geq 2$
>
> exactly n tight constraints $a^i x = 1, i = 1, .., n$ for $\bar{x}$
>
> $\bar{x}$ has exactly n neighbors $b^1, .., b^n$
>
> $$\begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}^T = J + dI \quad (*)$$

For $i = 1, \dots, n$ let

A_i correspond to a^i & B_i correspond to b^i

$\bar{x} = \frac{1}{r} 1 \Longrightarrow$ ① & ② by symetry $\mathcal{F} \Leftrightarrow \mathcal{B}$

$(*) \Longrightarrow$ ③

$\square$

Properties of mni binary clutters

Kernel lemma:

$\mathcal{F}$ binary mni & $\mathcal{B} = b\mathcal{F}$.

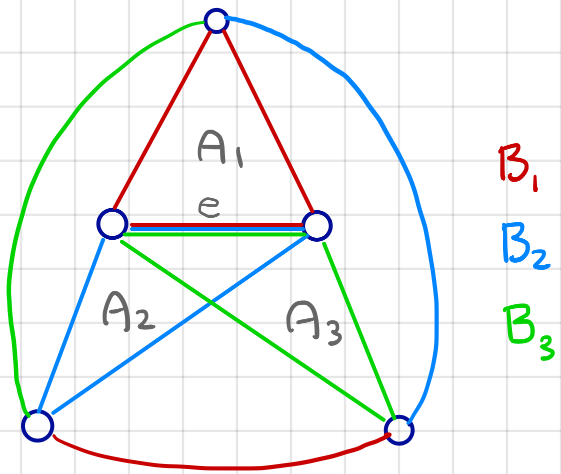
Then $\exists$ minimum sets A_1, A_2, A_3 of $\mathcal{F}$

$\exists$ " " B_1, B_2, B_3 " $\mathcal{B}$ st $\forall e$

① $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{e\}$

② $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{e\}$

③ B_i mate of A_i for $i = 1, 2, 3$



pf:

$a^i \to A^i$, $b^i \to B^i$, A^i, B^i mates

$$\begin{bmatrix} d+1 & & 1 \\ & \ddots & \\ 1 & & d+1 \end{bmatrix} = \begin{bmatrix} a^1 \\ \hline a^2 \\ \hline \vdots \\ \hline a^n \end{bmatrix} \begin{bmatrix} b^1 \\ \hline b^2 \\ \hline \vdots \\ \hline {}^n \end{bmatrix}^T =$$

$\nexists$ binary $\Rightarrow d+1 \geq 3$ (odd)

Bridges & Ryser

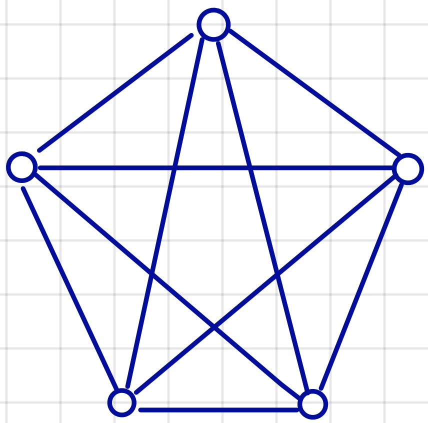
$$\begin{bmatrix} a^1 \\ \hline a^2 \\ \hline \vdots \\ \hline a^n \end{bmatrix}^T \begin{bmatrix} b^1 \\ \hline b^2 \\ \hline \vdots \\ \hline a^n \end{bmatrix} = \begin{array}{cc} & e \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ \vdots & & & \end{bmatrix}^T \end{array} \begin{array}{c} e \\ \begin{bmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ \vdots & & & \end{bmatrix} \end{array}$$

Inclusion lemma:

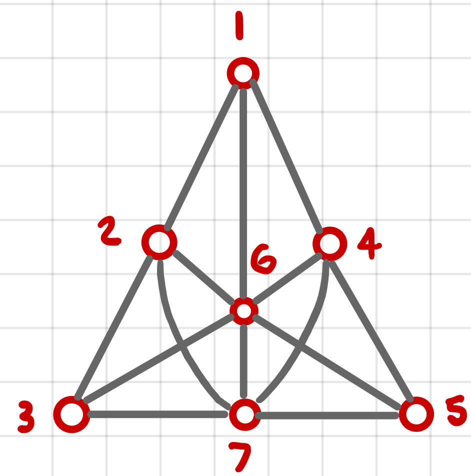
$\mathcal{F}$ binary mni & $A_1, A_2 \in \mathcal{F}$ minimum.

$A \subseteq A_1 \cup A_2$ & $A \in \mathcal{F}$

$\implies A = A_1$ or $A = A_2$



no $\triangle$ in union of 2 other $\triangle$s



no line in union of 2 other lines

pf:

$r := |A_1| = |A_2|$.

<u>Case 1</u> $|A| > r$

Let $A' := A_1 \triangle A_2 \triangle A$

Check: $|A'| \leq r-1$

$\mathcal{F}$ binary $\Longrightarrow$ A' contains set of $\mathcal{F}$ $\lightning$

<u>Case 2</u> $|A| = r$

Then $A = A_i$ for some i.

Let B_i be mate of A_i

$\mathcal{F}$ binary $\Longrightarrow |A_i \cap B_i| \geq 3$

$\Longrightarrow |A_1 \cap B_i| \geq 2$ or $|A_2 \cap B_i| \geq 2$

As mates unique, $A_i = A_1$ or $A_i = A_2$. $\square$

The proof of the graphic case

$\mathcal{F}$ mni clutter of odd circuits of (G, Σ)

To show: (G, Σ) has (K_5, EK_5) minor.

1. $\mathcal{B} := b\mathcal{F}$ is clutter of signatures

Pick $e = xy \in EG$. Kernel lemma $\Longrightarrow$

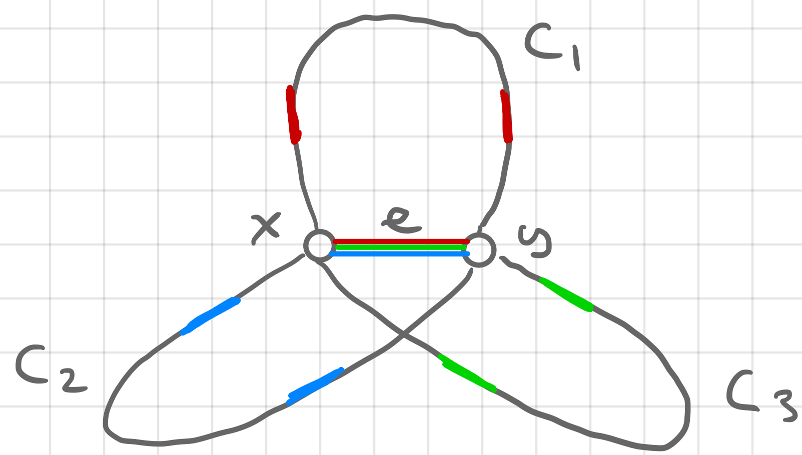
$\exists$ minimum sets A_1, A_2, A_3 of $\mathcal{F}$

$\exists$ " " B_1, B_2, B_3 " $\mathcal{B}$

$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{e\}$

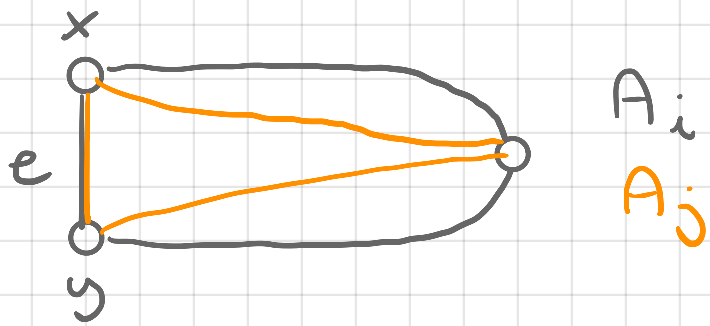
$B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{e\}$

B_i mate of A_i for $i = 1, 2, 3$



② $\forall i,j \in \{1,2,3\}$, $i \neq j$, A_i, A_j only share vertices x, y.

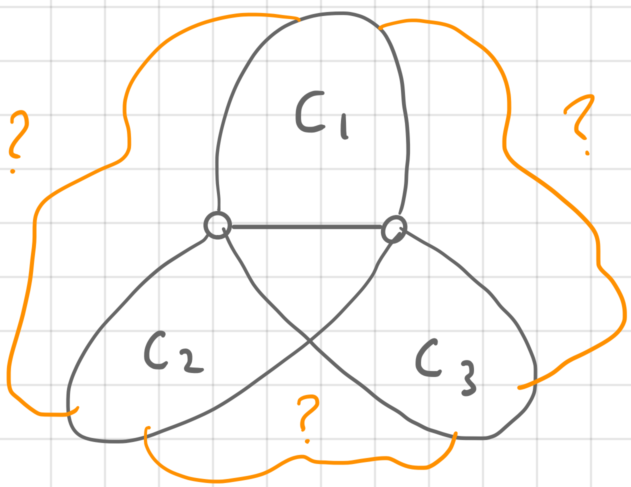
pf: otherwise



$\Longrightarrow$ $\exists$ odd circuit $A \subseteq A_i \cup A_j$

$\lightning$ inclusion lemma

$\boxed{}$



Time to consider blocker $\mathcal{B}$

B_1, B_2, B_3 signature $\Longrightarrow$

$$\boxed{\begin{aligned} B_1 \triangle B_2 &= \delta(U_{12}) \\ B_1 \triangle B_3 &= \delta(U_{13}) \\ B_2 \triangle B_3 &= \delta(U_{23}) \end{aligned}}$$

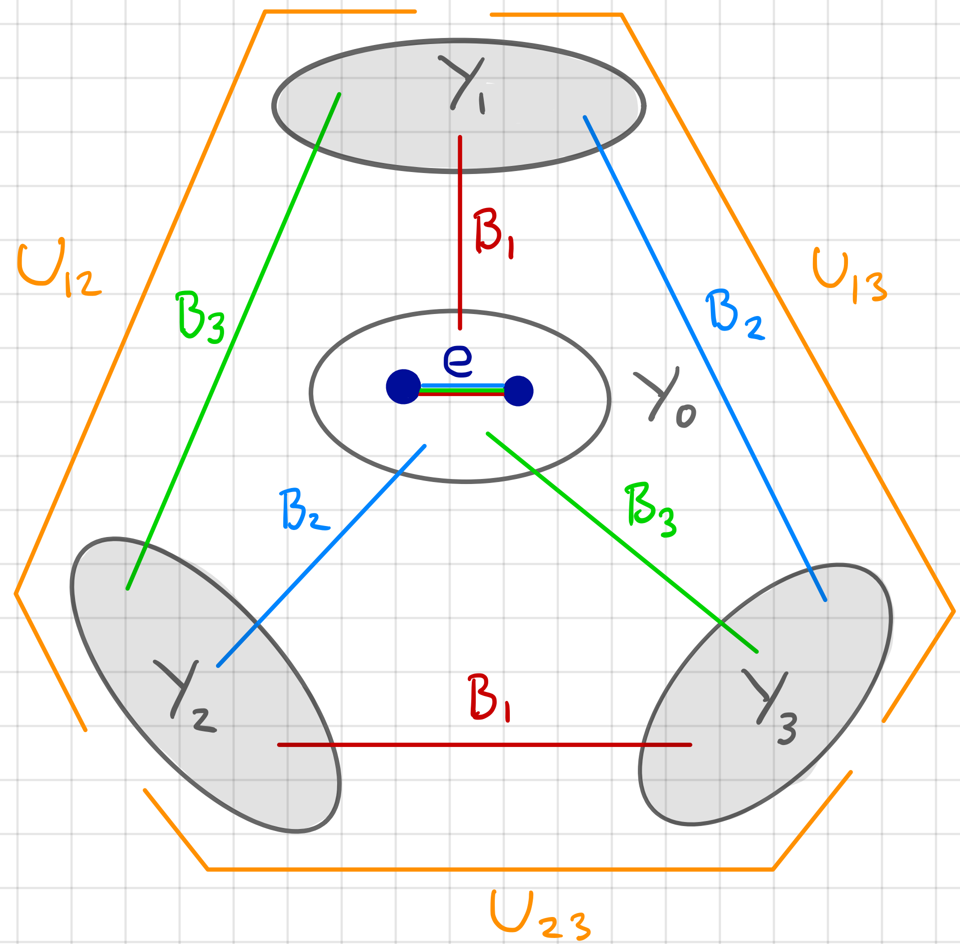
(Wma $x, y \notin U_{12} \cup U_{13} \cup U_{23}$)

$$\emptyset = \delta(U_{12}) \triangle \delta(U_{13}) \triangle \delta(U_{23})$$
$$= \delta(U_{12} \triangle U_{13} \triangle U_{23})$$

As G connected $\Longrightarrow$

$$U_{12} \triangle U_{13} \triangle U_{23} = \emptyset$$

Let
$$Y_1 = U_{12} \cap U_{13}$$
$$Y_2 = U_{12} \cap U_{23}$$
$$Y_3 = U_{13} \cap U_{23}$$
$$Y_0 = VG - (Y_1 \cup Y_2 \cup Y_3)$$

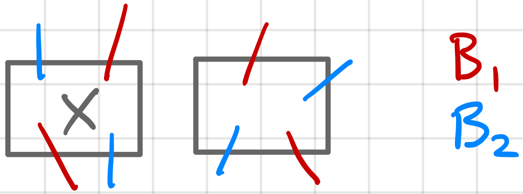


$$B_1 - e = \delta(U_{12}) \cap \delta(U_{13})$$

③ $G[U_{12}]$ connected

pf: Otherwise

$G[U_{12}]$



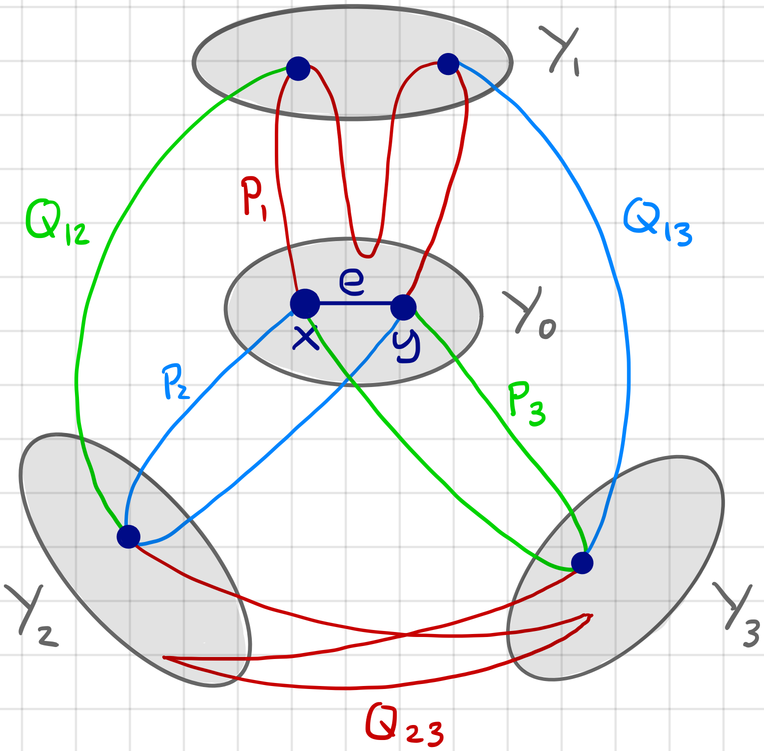
$\Longrightarrow \exists X \subseteq U_{12}$ with $\emptyset \subsetneq \delta(X) \subsetneq \delta(U_{12})$

Then $B' := B_1 \,\Delta\, \delta(X)$ is signature with

$B' \subseteq B_1 \cup B_2$ & $B' \neq B_1, B_2$ $\lightning$ inclusion lemma $\boxed{/\!/}$

Thus $\forall\, i,j \in \{1,2,3\},\ i \neq j$,

$\exists\, Q_{ij}$ between $Y_i \cap VP_i$ & $Y_j \cap VP_j$



- delete all edges not in $P_1 \cup P_2 \cup P_3 \cup Q_{12} \cup Q_{13} \cup Q_{23} \cup \{e\}$
- contract all remaining edges not in $B_1 \Delta B_2 \Delta B_2$

Whirlpool lemma $\Longrightarrow$ $\exists$ (K_5, EK_5) minor $\boxed{/\!/\!/}$

Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

Recall the key definition:

A 0,1 matrix is Mengerian if $Mx \geq 1,\ x \geq 0$ is TDI $\Longleftrightarrow$ $\forall w \in \mathbb{Z}_+^n$:

$$\min\{w^T x : Mx \geq 1, x \geq 0, x \text{ integer}\} = \max\{1^T y : M^T y \leq w, y \geq 0, y \text{ integer}\}$$

We say $\mathcal{F}$ is Mengerian if $M(\mathcal{F})$ Mengerian

Examples of Mengerian matrices/clutters

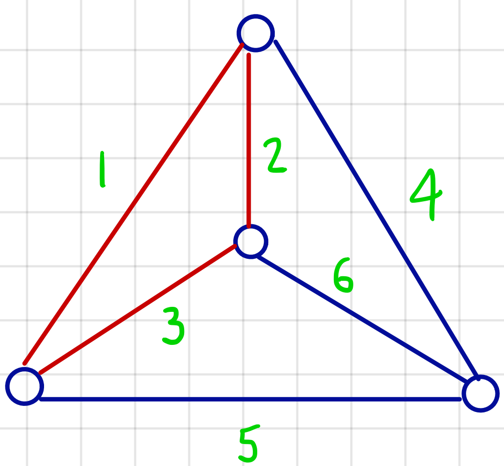
Odd cycles

Pr: Let $\mathcal{F}$ be clutter of odd circuit of signed graph (G, Σ). Then

① $\mathcal{F}$ Mengerian $\Longleftrightarrow$

② $\mathcal{F}$ has Q_6 minor $\Longleftrightarrow$

③ (G, Σ) has (K_4, EK_4) minor.



$$M(Q_6) = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{bmatrix} \end{array}$$

Directed cuts / directed joins

Consider a directed graph $\vec{G} = (V, E)$.

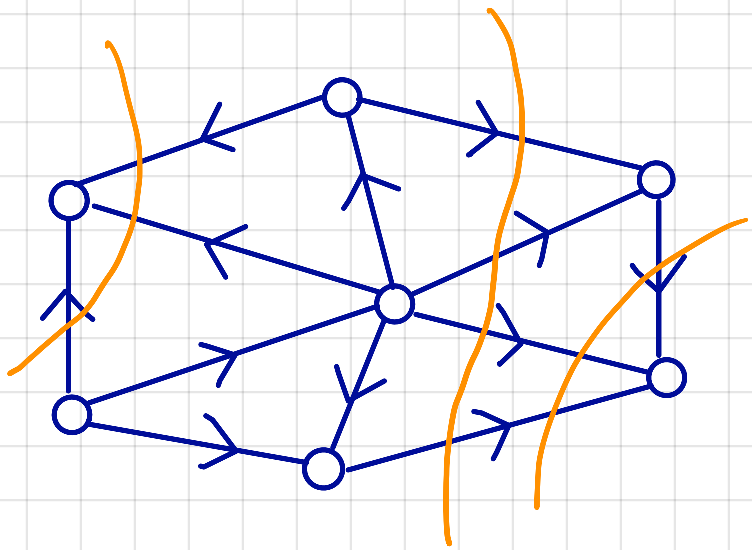
def: a dicut is a set of edges that form a cut in the underlying graph where all edges are directed from one shore to the other.

exist dipath between all pairs of vertices

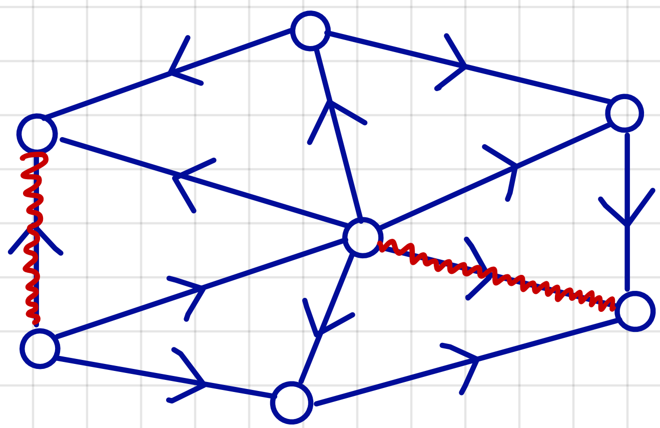
def: a dijoin is a set of edges whose contraction makes the digraph strongly connected.

Rem: dicuts & dijoins are blockers

Some dicuts



A dijoin



Th [Lucchesi-Younger]

The clutter of dicuts is Mengerian

def: $\mathcal{F}$ packs if for $M := M(\mathcal{F})$

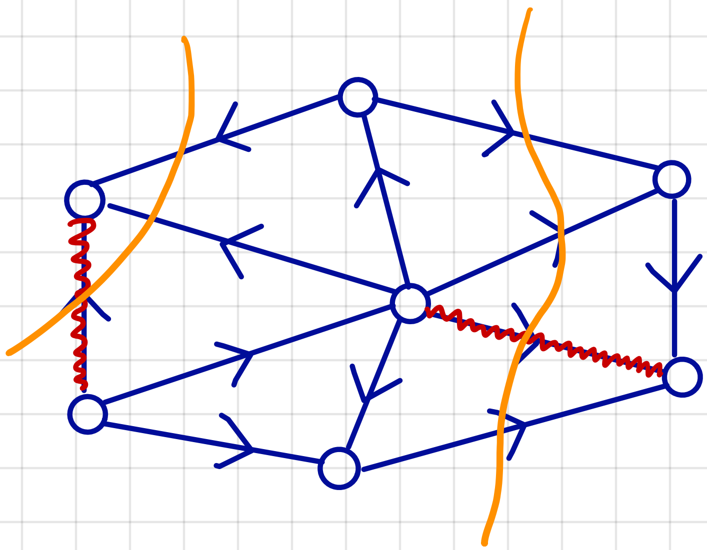
$$\min\{\mathbb{1}^T x : Mx \geq \mathbb{1}, x \geq 0, x \text{ integer}\} =$$
$$\max\{\mathbb{1}^T y : M^T y \leq \mathbb{1}, y \geq 0, y \text{ integer}\}$$

Rem: $\mathcal{F}$ Mengerian $\implies$ $\mathcal{F}$ packs

Thus clutter of dicuts packs:

$$\boxed{\begin{array}{l}\text{min size of dijoin} = \\ \text{max number of pairwise disjoint dicuts}\end{array}}$$

min size of dijoin =
max number of pairwise disjoint dicuts



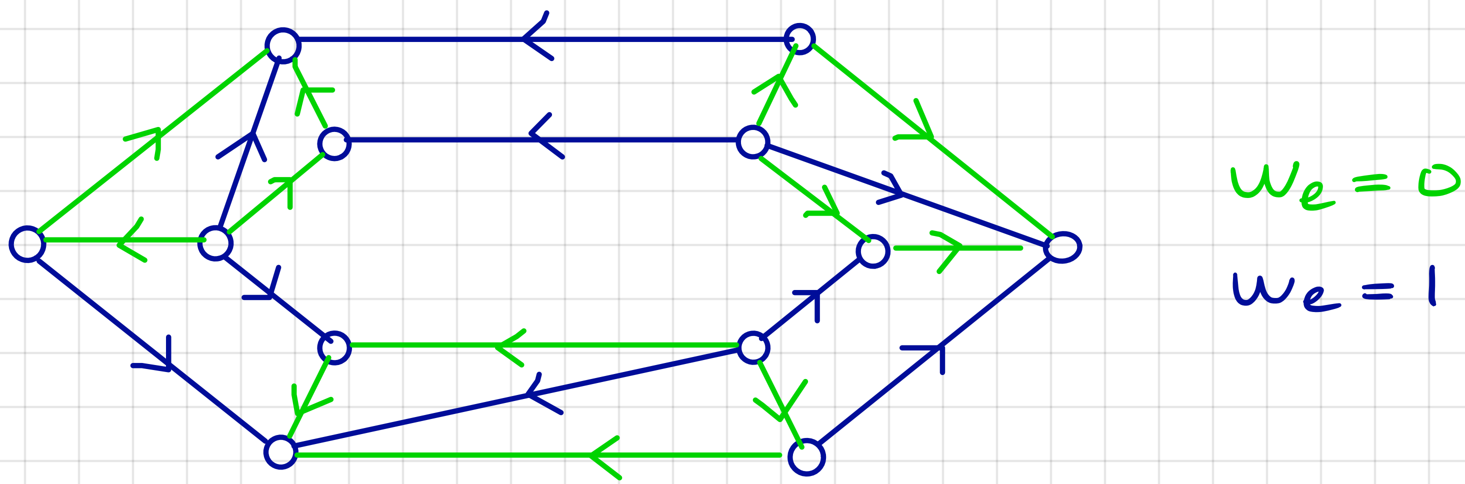
dicuts
dijoin

Th: The clutter of dicuts is Mengerian

Edmonds-Giles conjectured:

"The clutter of dijoins is Mengerian"

Not true [Schrijver]



$$2 = \min\{w^T x : Mx \geq 1, x \geq 0, x \text{ integer}\} >$$
$$= \max\{1^T y : M^T y \leq 1, y \geq 0, y \text{ integer}\} = 1$$

Edmonds & Giles conj. holds for special case:

Th [Schrijver]

Let $\vec{G}$ be digraph.

If $\exists$ dipath from every source to every sink then clutter of dijoins is Mengerian.

Maybe one can do better:

Conj. [Guenin, Williams]

Let $\vec{G}$ be digraph with source r & sink s.

If $\exists$ dipath from r to every sink &

$\exists$ dipath from every source to s then

clutter of dijoins is Mengerian

Unweighted version of Lucchesi-Younger th:

$\boxed{\text{min size of dijoin = max number of pairwise disjoint dicuts}}$

Can we swap role of dicuts/dijoins ?

Conj: [Woodal]

min size of dicut =

max number of pairwise disjoint dijoins

wide open (hard ?)

Replication & the packing property

Back to perfect graphs

Let G be graph & M stable-set matrix of G.

Recall, G perfect:

$\forall$ induced subgraph H of G: $\omega(H) = \chi(H)$

$\iff$

$$\boxed{\begin{array}{l} \forall \text{ column submatrix } N \text{ of } M: \\ \max\{\mathbb{1}^T x : Nx \leq \mathbb{1}, x \geq 0, x \text{ integer}\} = \\ \min\{\mathbb{1}^T y : N^T y \geq \mathbb{1}, y \geq 0, y \text{ integer}\} \end{array}}$$

Let us find analogue def. for <u>set covering</u>

$\forall$ column submatrix N of M:

$$\max\{\mathbf{1}^T x : Nx \leq \mathbf{1}, x \geq 0, x \text{ integer}\} =$$
$$\min\{\mathbf{1}^T y : N^T y \geq \mathbf{1}, y \geq 0, y \text{ integer}\}$$

def: clutter $\mathcal{F}$ has the packing property if $\forall$ minor H of $\mathcal{F}$, H packs

Packing property = analogue of perfect graphs

We proved

> **Pr:** G perfect graph, M stable matrix of G
> $\Longrightarrow$ $Mx \leq 1,\ x \geq 0$ TDI

The analogue for set-covering would be

Replication Conj [Conforti, Cornuejols]
The following are equivalent for clutter $\mathcal{F}$:

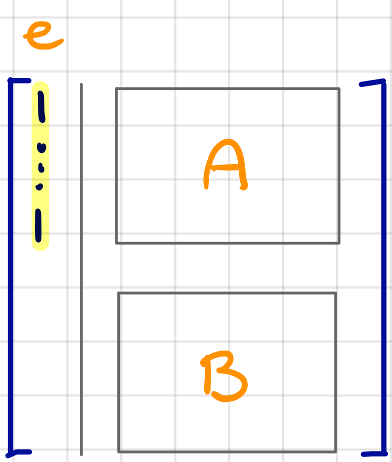
① $\mathcal{F}$ has the packing property
② $\mathcal{F}$ is Mengerian.

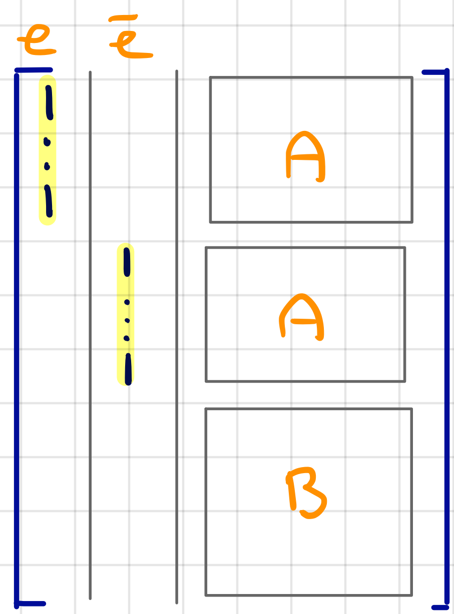
Note ② $\Rightarrow$ ① trivial.

Question: Why the name replication?

def: Let $\mathcal{F}$ clutter with groundset element e.
Then H is obtained from $\mathcal{F}$ by replicating e if

$$H = \{S : e \notin S \in \mathcal{F}\} \cup \{S, S - e \cup \bar{e} : e \in S \in \mathcal{F}\}$$

$M(\mathcal{F}) =$ 

$M(H) =$ 

(This is what duplication does for stable set matrices)

Replication conj – variant

If $\mathcal{F}$ has the packing property then so does any replication.

Exercise:

Show both version of replication conj. are equivalent

similar to use of duplication to prove TDI for set packing problem

The replication conj. predicts:

" Packing property $\Longrightarrow$ Mengerian property"

We proved:

" Mengerian property $\Longrightarrow$ idealness "

Thus we expect:

Pr:

If a clutter has the packing property then it is ideal.

This follows from Lehman's theorem

pf:

Spse $\tilde{\mathcal{F}}$ not ideal $\Longrightarrow$ $\tilde{\mathcal{F}}$ has mni minor $\mathcal{H}$.

If $\mathcal{H} \approx \Delta_n$ then $\mathcal{H}$ does not pack ✓

Otherwise by Lehman's th,

$\bar{x} = \frac{1}{r}\mathbb{1}$ fractional extreme point of $\{x \geq 0 \mid M(\mathcal{H})x \geq \mathbb{1}\}$

$$A = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \quad B = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}, \quad A^T B = J + dI \text{ for } d \geq 1$$

[Bridges, Ryser] $\Longrightarrow$ A r-regular, B s-regular

$$nrs = \mathbb{1}^T A B^T \mathbb{1} = \mathbb{1}^T (J + dI)\mathbb{1} = n(n+d)$$

$\Longrightarrow$ $n = rs - d$.

Thus $\mathbb{1}^T \bar{x} = \mathbb{1}^T \frac{1}{r} \mathbb{1} = \frac{n}{r} = \frac{rs-d}{r} = s - \frac{d}{r} \notin \mathbb{Z}$

$\Longrightarrow$ $\mathcal{H}$ does not pack ✓

$\square$

Excluded minors for the packing property

Question: why excluded minors for the packing property & not for Mengerian property?

Replication conj. suggest it is the same

def: a clutter $\mathcal{F}$ is minimally non-packing if

- $\mathcal{F}$ does not pack
- every proper minor of $\mathcal{F}$ packs

Clutters with packing property are ideal $\Longrightarrow$

A minimally non-packing clutter is

① minimally non-ideal <u>or</u>

② ideal

Minimally non-packing clutters that are mni

Lehman's th

$\bar{x} = \frac{1}{r}\mathbf{1}$ fractional extreme point of $\{x \geq 0 \mid M(\mathcal{H})x \geq \mathbf{1}\}$

$$A = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \quad B = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}, \quad A^T B = J + dI \text{ for } d \geq 1$$

def: a mni clutter is **thin** if $d = 1$

Pr [Cornuejols, Guenin, Margot]

Let $\mathcal{F}$ be a mni clutter

① $\mathcal{F}$ is minimally non-packing $\Longrightarrow$

② $\mathcal{F}$ is thin

Question: does the converse hold?

Minimally non-packing clutters that are ideal

A far-fetched conjecture ← but possibly true

$\tau = 2$ Conjecture [Cornuejols, Guenin, Margot]
If $\mathcal{F}$ is minimally non-packing and ideal then $\exists$ cover B of $\mathcal{F}$ with $|B| = 2$.

$$M(Q_6) = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{array}\right] \end{array}$$

$B = \{1, 2\}$ cover

Motivation for conjecture ?

Pr: $\tau = 2$ conj. $\Longrightarrow$ replication conj.

pf:

<u>To show</u>: replication preserves packing property

Otherwise, $\exists$ minimally non-packing clutter $\mathcal{F}$ with replicated element:

$$M(\mathcal{F}) = \begin{array}{c} \begin{array}{ccc} e & \bar{e} & \end{array} \\ \left[\begin{array}{c|c|c} 1 & & \\ \vdots & & A \\ 1 & & \\ \hline & 1 & \\ & \vdots & A \\ & 1 & \\ \hline & & B \end{array}\right] \end{array}$$

$\mathcal{F}/e$ packs $\Longrightarrow \exists\, S_1, S_2 \in \mathcal{F}$ st $S_1 \cap S_2 = \{e\}$

$\Longrightarrow S_1 - e \cup \bar{e} \in \mathcal{F}$ & $S_1 - e \cup \bar{e} \cap S_2 = \emptyset$

By $\tau = 2$ conj., $\mathcal{F}$ packs $\lightning$ $\square$

The $\tau = 2$ conjecture holds for binary clutters

Th: [Seymour]

The following are equivalent for a binary clutter $\mathcal{F}$:

① $\mathcal{F}$ is Mengerian

② $\mathcal{F}$ has a Q_6 minor

$$M(Q_6) = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 1 & & & \\ \hline 1 & & & 1 & 1 & \\ \hline & 1 & & 1 & & 1 \\ \hline & & 1 & & 1 & 1 \end{array}\right]$$

$\Longrightarrow$

The replication conj. holds for binary clutters.

Thank you for your attention

Keep safe !

I would like to thank Ahmad Abdi for numerous discussions on this topic.