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LINEAR SYSTEMS FOR CONSTRAINED MATCHING PROBLEMS*†

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Each polyhedron of full dimension has a unique (up to positive scalar multiples of the inequalities) minimal defining system and a unique minimal totally dual integral defining system with integer left hand sides. These two minimal systems are characterised for the convex hull of the simple b-matchings of a graph. These characterisations are then used to provide similar characterisations for the convex hull of matchings, b-matchings, and capacitated b-matchings. Each of these characterisations gives a "best possible" min-max relation for the corresponding combinatorial objects.

1. Introduction. A *matching* in a graph G is a subset of the edges such that each node of G is met by at most one edge in the subset. Fundamental results in the theory of matchings were proven by Tutte [34, 35, 36]. Tutte's results provide a min-max relation for the cardinality of a largest matching in a graph (see Berge [4]). In 1965, Edmonds [14] found a polynomial time algorithm for the weighted matching problem. A by-product of Edmonds' algorithm is a characterisation of a linear system that defines the convex hull of the (incidence vectors of the) matchings of a graph. Via the linear programming duality theorem, this result gives a min-max relation for weighted matchings.

Tutte [35, 36] and Edmonds and Johnson [17] have shown that, by means of a series of constructions, results on matchings imply results on considerably more general objects. In particular, Edmonds and Johnson [17] found descriptions of linear systems which define the convex hulls of these more general objects and hence min-max relations for these objects (see also Araoz, Cunningham, Edmonds, and Green-Krótki [2]).

Two ways to improve min-max results that are obtained by finding descriptions of linear systems which define certain convex hulls are to reduce the number of dual variables in the linear programming duality equation (that is, to find a smaller linear system that defines the given convex hull) and to restrict the dual variables to integer values. We discuss these methods below.

If P is a polyhedron of full dimension, then there exists a unique (up to positive scalar multiples of the inequalities) minimal linear system that defines P. So, for a generalisation of matchings whose convex hull is of full dimension, a description of the unique minimal defining system for the convex hull gives a "best possible" min-max relation for the generalised matchings. We characterise such a minimal system for the convex hull of the simple b-matchings of a graph and show that this result implies

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similar characterisations for matchings (Pulleyblank and Edmonds [28]), b-matchings (Pulleyblank [24]), and capacitated b-matchings.

The second way to improve min-max relations involves totally dual integral systems. Edmonds and Giles [16] defined a rational linear system $Ax \leq b$ to be a *totally dual integral* system if the linear program $\min\{yb: yA = w, \ y \geq 0\}$ has an integral optimal solution for each integer vector w for which the optimum exists. (For results on total dual integrality see Hoffman [20], Edmonds and Giles [16], Giles and Pulleyblank [19], Schrijver [29], and Cook, Lovász, and Schrijver [9].) Thus, integral min-max theorems can be obtained by finding totally dual integral defining systems for various convex hulls. These integral min-max theorems often have a nice combinatorial interpretation, since integer solutions to the dual linear programs often correspond to combinatorial objects, such as "covers" or "cuts" (for many examples of this see Schrijver [32]). Combinatorial min-max theorems, generalising the Tutte-Berge Theorem, obtained in this way for various generalisations of matchings can be found in Schrijver [31].

Min-max theorems arising from totally dual integral systems can be further strengthened by removing some of the inequalities to obtain a minimal totally dual integral defining system for the convex hull of the objects in question. Schrijver [29] has shown that for each polyhedron P of full dimension there exists a unique minimal totally dual integral system $Ax \leq b$ with A integral such that P is defined by $Ax \leq b$ (call $Ax \leq b$ the *Schrijver system* for P). So, for a generalisation of matchings whose convex hull is of full dimension, a second type of "best possible" min-max theorem can be obtained by finding the Schrijver system for the convex hull. We characterise such systems for the convex hull of capacitated b-matchings, simple b-matchings, b-matchings (Cook [6] and Pulleyblank [26]), triangle-free 2-matchings, and matchings (Cunningham and Marsh [13]).

Some terms and notation that will be used throughout the paper are given below.

Let G be an undirected graph (see Bondy and Murty [5] for standard terminology of graph theory). The node set of G is denoted by VG and the edge set by EG (we will assume that each edge has two distinct ends). For each node $v \in VG$, $\delta_G(v)$ denotes the set of edges of G which meet v, $d_G(v)$ denotes $|\delta_G(v)|$, and $N_G(v)$ denotes the set of nodes in $VG\text{-}\{v\}$ which are adjacent to v. For each $S \subseteq VG$, $\gamma_G(S)$ denotes the subset of edges of G having both ends in S, $\delta_G(S)$ denotes the subset of edges having exactly one end in S, and $G[S]$ denotes the subgraph of G induced by S. We write δ, d, γ, to denote δ_G, d_G, γ_G respectively.

Let P be a polyhedron. A linear system $Ax \leq b$ *defines* P if $P = \{x: Ax \leq b\}$. An inequality $\alpha x \leq \beta$ is *valid* for P if for each $\bar{x} \in P$ we have $\alpha\bar{x} \leq \beta$. Suppose that P is of full dimension. The inequality $\alpha x \leq \beta$ is *essential* for P if some positive scalar multiple of the inequality must be present in every defining system for P. A well-known result is that a valid inequality $\alpha x \leq \beta$ is essential for P if and only if it is *facet inducing* for P (that is, if and only if there exist dimension(P) affinely independent vectors in P for which the inequality holds as an equality). For an account of polyhedral theory see the papers of Bachem and Grötschel [3] and Pulleyblank [27] and the references cited in those two papers.

If $x = (x_i: i \in I)$ and $S \subseteq I$, where I is a finite set, then $x(S)$ denotes the sum $\Sigma\{x_i: i \in S\}$. If β is a number, then $\lfloor \beta \rfloor$ denotes the largest integer less than or equal to β.

2. Matchings and b-matchings. Let G be a graph. A matching M of G will be identified with its incidence vector $x = (x_e: e \in EG)$, where $x_e = 1$ if $e \in M$ and $x_e = 0$ if $e \in EG\text{-}M$. The fundamental result in the study of polyhedral aspects of matching theory was proven by Edmonds [14]:

THEOREM 2.1. *The convex hull of the matchings of G is defined by*

$$x_e \geqq 0 \quad \forall e \in EG, \tag{2.1}$$

$$x(\delta(v)) \leqq 1 \quad \forall v \in VG,$$

$$x(\gamma(S)) \leqq \lfloor |S|/2 \rfloor \quad \forall S \subseteq VG, \quad |S| \text{ odd}.$$

Edmonds proved this result by means of a polynomial time algorithm for the weighted matching problem. (The weighted matching problem is to maximise wx over all matchings x of G for a given weight vector $w = (w_e\colon e \in EG)$.) A short proof of this result can be found in Schrijver [30].

A matching M of G is *perfect* if each node in VG is met by an edge in M. The graph G is *hypomatchable* if for each $v \in VG$ the graph obtained by deleting v from G has a perfect matching. (Note that hypomatchable graphs are necessarily connected.) Let V' be the set of nodes $v \in VG$ such that either $|N(v)| \geqq 3$ *or* $|N(v)| = 2$ and $\gamma(N(v)) = \emptyset$ *or* $|N(v)| = 1$ and v is a node of a two node connected component of G.

Pulleyblank and Edmonds [28] found a description of the unique minimal defining system for $P(G)$, the convex hull of the matchings of G. Their result is as follows:

THEOREM 2.2. *The unique (up to positive scalar multiples of the inequalities) minimal defining system for $P(G)$ is*

$$x_e \geqq 0 \quad \forall e \in EG,$$

$$x(\delta_G(v)) \leqq 1 \quad \forall v \in V', \tag{2.2}$$

$$x(\gamma(S)) \leqq \lfloor |S|/2 \rfloor \quad \text{for each } S \subseteq VG,\ |S| \geqq 3,\ G[S] \text{ hypomatchable with no cutnode}.$$

A short proof of this result due to L. Lovász can be found in Cornuéjols and Pulleyblank [11]. The result follows from a more general theorem presented in §3.

Cunningham and Marsh [13] proved that the linear system (2.2) is totally dual integral, which immediately implies the following result.

THEOREM 2.3. *The Schrijver system for $P(G)$ is* (2.2).

This result also follows from a more general theorem given in §3.

A short proof of Theorem 2.3, which does not use the result of Pulleyblank and Edmonds [28], is given in Cook [8], where it is also shown that this theorem is related to a result of F. R. Giles on a type of separability for graphs. Let k be the maximum cardinality of a matching of G. Let E_1 and E_2 be nonempty subsets of EG with $E_1 \cup E_2 = EG$. Let k_i be the maximum cardinality of a matching of G contained in E_i, $i = 1, 2$. If $k_1 + k_2 = k$, then (E_1, E_2) is a *matching separation* of G. The graph G is *matching separable* if there exists a matching separation of G and *matching nonseparable* otherwise. The following result is due to F. R. Giles.

THEOREM 2.4. *A graph G is matching nonseparable if and only if either G is isomorphic to $K_{1,n}$ for some n or G is hypomatchable with no cutnode.*

To see the connection of this theorem to the theorem of Cunningham and Marsh, we state an easy general result on separability, the proof of which can be found in Cook [8]. Let E be a finite set and I a finite set of nonnegative integer vectors $a = (a_e\colon e \in E)$. The pair (E, I) is a *general independence system* if $0 \in I$ and for each $a \in I$

and nonnegative integral $b \leq a$ we have $b \in I$ (so (E, I) is an independence system if each $a \in I$ is 0-1 valued). The rank, $r(A)$, of a set $A \subseteq E$ is the maximum value of $x(A)$ over all vectors $x \in I$. A set $A \subseteq E$ is *closed* if for each $e \in E\text{-}A$ we have $r(A \cup \{e\}) > r(A)$. A *separation* of a set $A \subseteq E$ is a pair of nonempty subsets A_1, A_2 of A such that $A_1 \cup A_2 = A$ and $r(A_1) + r(A_2) = r(A)$. If there exists a separation of $A \subseteq E$ then A is *separable* (otherwise A is *nonseparable*). Let $C(I)$ denote the convex hull of I.

LEMMA 2.5. *Let (E, I) be a general independence system. Suppose that $r(\{e\}) \geq 1$ $\forall e \in E$ and that*

$$x(A) \leq r(A) \quad \forall A \subseteq E, A \neq \varnothing, \tag{2.3}$$

$$x_e \geq 0 \quad \forall e \in E,$$

is a totally dual integral defining system for $C(I)$. Then an inequality $x(A) \leq r(A)$ is in the Schrijver system for $C(I)$ if and only if $A \neq \varnothing$ is a closed nonseparable set.

This lemma combined with the Cunningham and Marsh result gives a proof of the characterisation of matching nonseparable graphs given in Theorem 2.4. Conversely, it is not difficult to show that Theorem 2.4, together with Edmonds' matching algorithm and the above lemma, yields a quick proof of the theorem of Cunningham and Marsh (see Cook [8]). We will use Lemma 2.5 in §§3, 4, and 5.

Let $b = (b_v\colon v \in VG)$ be a positive integer vector. A *b-matching* of G is a nonnegative integer vector $x = (x_e\colon e \in EG)$ such that $x(\delta(v)) \leq b_v$ for each $v \in VG$. A b-matching of G with $b_v = 1$ for each $v \in VG$ is a matching. A construction of Tutte [36] can be used to deduce results on b-matchings from results on matchings: Replace each node $v \in VG$ by the new nodes $v_1, v_2, \ldots, v_{b_v}$ and replace each edge $(u, v) \in EG$ by the new edges (u_i, v_j), $i = 1, \ldots, b_u$, $j = 1, \ldots, b_v$. A matching in the new graph corresponds to a b-matching in the original graph and vice versa. As presented in Schrijver [31], the total dual integrality of (2.1) for the new graph implies that the linear system

$$x_e \geq 0 \quad \forall e \in EG, \tag{2.4}$$

$$x(\delta(v)) \leq b_v \quad \forall v \in VG,$$

$$x(\gamma(S)) \leq \lfloor b(S)/2 \rfloor \quad \forall S \subseteq VG, |S| \geq 3.$$

is a totally dual integral defining system for $P(G, b)$, the convex hull of the b-matchings of G. This result is an easy consequence of Edmonds' [14] b-matching algorithm (see Pulleyblank [24, 25]) and has also been proven by Hoffman and Oppenheim [21] and Schrijver and Seymour [33].

The analogue of a hypomatchable graph for b-matchings is a b-critical graph. The graph G is *b-critical* if for each $v \in VG$ there exists a b-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v - 1$ and $\bar{x}(\delta(u)) = b_u$ for each $u \in VG\text{-}\{v\}$. (Again note that b-critical graphs are necessarily connected.) Let V'' be the set of nodes $v \in VG$ such that $b(N(v)) \geq b_v + 2$ *or* $b(N(v)) = b_v + 1$ and $\gamma(N(v)) = \varnothing$ *or* v belongs to a two node connected component of G and $b_v = b_u$ where u is the other node of the component. Pulleyblank [24] proved the following result.

THEOREM 2.6. *The unique (up to positive scalar multiples of the inequalities) minimal defining system for $P(G, b)$ is*

$$x_e \geqq 0 \quad \forall e \in EG, \tag{2.5}$$

$$x(\delta_G(v)) \leqq b_v \quad \forall v \in V'',$$

$$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \text{for each } S \subseteq VG,\ |S| \geqq 3,\ G[S]$$

b-critical with no cutnode v having $b_v = 1$.

This theorem generalises Theorem 2.2. In §4 we will indicate how this result follows from a more general result given in that section.

Unlike the matching case, (2.5) is not totally dual integral in general. This can be seen by considering a triangle with $b_v = 2$ for each node v and weight $w_e = 1$ for each edge e. Such a triangle is an example of a b-bicritical graph. A graph G is *b-bicritical* if it is connected and for each $v \in VG$ there exists a b-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v - 2$ and $\bar{x}(\delta(u)) = b_u$ for each $u \in VG\text{-}\{v\}$. Cook [6] and Pulleyblank [26] independently proved the following result.

THEOREM 2.7. *The Schrijver system for $P(G, b)$ is (2.5) together with the inequalities*

$$x(\gamma(S)) \leqq b(S)/2 \quad \text{for each } S \subseteq VG,\ |S| \geqq 3,\ G[S] \text{ b-bicritical and } b_v \geqq 2 \text{ for each node } v \in VG\text{-}S \text{ which is adjacent to a node in } S. \tag{2.6}$$

Since $b_v \geqq 2$ for each $v \in VG$ if G is b-bicritical, this theorem implies the result of Cunningham and Marsh on matching systems. We will also indicate later how this theorem follows from a result given in §4.

We close this section by presenting a fundamental theorem on b-matchings due to Tutte [34, 35]. This theorem will be used in §3. A b-matching $\bar{x}$ of G is perfect if $\bar{x}(\delta(v)) = b_v$ for each $v \in VG$. If $S \subseteq VG$, let

$$\mathscr{C}^0(S) = \{v \in VG\text{-}S\colon G[\{v\}] \text{ is a connected component of } G[VG\text{-}S]\} \tag{2.7}$$

and let

$$\mathscr{C}^1(S) = \{R \subseteq VG\text{-}S\colon |R| \geqq 2,\ b(R) \text{ is odd and } G[R] \text{ is a connected component of } G[VG\text{-}S]\}. \tag{2.8}$$

Tutte's b-matching theorem is as follows.

THEOREM 2.8. *A graph G has a perfect b-matching if and only if for each $S \subseteq VG$, $b(S) \geqq b(\mathscr{C}^0(S)) + |\mathscr{C}^1(S)|$.*

The total dual integrality of (2.4) can be used to prove this theorem by setting $w_e = 1$ for each $e \in EG$.

3. Simple b-matchings. We will now consider a constrained variation of b-matchings. Throughout this section, let G be a graph, possibly with multiple edges, and $b = (b_v\colon v \in VG)$ a positive integer vector. A *simple b-matching* of G is a subset M of EG such that each node $v \in VG$ meets at most b_v edges in M. A perfect simple

b-matching (that is, a simple b-matching which meets each node $v \in VG$ in exactly b_v edges) is often called a "b-factor". Again, we indentify a simple b-matching M with its incidence vector $x = (x_j \colon j \in EG)$.

Given a vector $w = (w_e \colon e \in EG)$ of edge weights, the simple b-matching problem is to maximize wx over all simple b-matchings of G. Tutte [36] described the following construction, which reduces a simple b-matching problem to a b-matching problem. For each edge $e = (u, v)$ of G (although G may have multiple edges, for simplicity edges will still be referred to as unordered pairs of nodes) add nodes u_e and v_e to VG and replace e by the edges (u, u_e), (u_e, v_e), (v_e, v). Also, for each $e \in EG$ let $b_{u_e} = b_{v_e} = 1$ and $w_{(u,u_e)} = w_{(u_e,v_e)} = w_{(v_e,v)} = w_e$. The maximum weight of a b-matching in the new graph is exactly $\Sigma\{w_e \colon e \in EG\}$ greater than the maximum weight of a simple b-matching of G. As presented in Araoz, Cunningham, Edmonds, and Green-Krótki [2] and Schrijver [31], this construction, together with the total dual integrality of (2.4), implies the following result, which is an easy consequence of a theorem of Edmonds and Johnson [17].

THEOREM 3.1. *A totally dual integral defining system for the convex hull of the simple b-matchings of G is*

$$0 \leqq x_e \leqq 1 \quad \forall e \in EG, \tag{3.1}$$

$$x(\delta(v)) \leqq b_v \quad \forall v \in VG,$$

$$x(\gamma(S)) + x(J) \leqq \lfloor (b(S) + |J|)/2 \rfloor \quad \forall S \subseteq VG,\ J \subseteq \delta(S).$$

If H is a subgraph of G, then for each $v \in VH$ let $b_v^H = \min\{b_v, d_H(v)\}$. The largest simple b-matching of G is of cardinality at most $\lfloor b^G(VG)/2 \rfloor$. Let $\mathscr{H}$ be the set of all connected subgraphs of G which have at least 3 nodes. Theorem 3.1 implies that

$$\begin{aligned} 0 \leqq x_e \leqq 1 \quad & \forall e \in EG, \\ x(\delta_G(v)) \leqq b_v \quad & \forall v \in VG, \\ x(EH) \leqq \lfloor b^H(VH)/2 \rfloor \quad & \forall H \in \mathscr{H}, \end{aligned} \tag{3.2}$$

is a totally dual integral defining system for $S(G, b)$, the convex hull of the simple b-matchings of G. By a series of results in this section, the unique minimal subset of these inequalities which defines $S(G, b)$ and the Schrijver system for $S(G, b)$ will be characterised.

We begin with a variation of Tutte's b-matching theorem. As presented in Schrijver [31], to determine if G has a perfect simple b-matching, the above transformation of Tutte can be applied to G to obtain a new graph G' and then Tutte's b-matching Theorem can be applied to G'. Suppose that $S \subseteq VG$ and $T \subseteq VG\text{-}S$. Let

$$Q(S, T) = \Sigma\{b_v - d_{G[VG\text{-}S]}(v) \colon v \in T\} \tag{3.3}$$

and let $G^T(VG\text{-}S)$ be the graph obtained from $G[VG\text{-}S]$ by taking each node $v \in T$ and splitting it into $d_{G[VG\text{-}S]}(v)$ nodes, each with $b_i = 1$ (that is, replace v by the nodes $v_1, \ldots, v_k$, where $k = d_{G[VG\text{-}S]}(v)$, and replace the edges $(u_1, v), (u_2, v), \ldots, (u_k, v)$ by the edges (u_i, v_i), $i = 1, \ldots, k$, and let $b_{v_i} = 1$, $i = 1, \ldots, k$). Let $\mathscr{D}_1(S, T)$ denote the set of odd connected components of $G^T(VG\text{-}S)$ which contain at least two nodes. (A connected component G_i of $G^T(VG\text{-}S)$ is odd if $b(VG_i)$ is odd.) Notice that each connected component of $G^T(VG\text{-}S)$ corresponds to a subgraph of $G[VG\text{-}S]$. Using Tutte's b-matching Theorem, the following result can be proven.

THEOREM 3.2. *There exists a perfect simple b-matching of G if and only if* $\forall S \subseteq VG$ *and* $\forall T \subseteq VG\text{-}S$: $b(S) \geq Q(S,T) + |\mathscr{D}_1(S,T)|$.

PROOF. Suppose that G has a perfect simple b-matching M and let $S \subseteq VG$ and $T \subseteq VG\text{-}S$. Let $M' = M \cap \gamma(VG\text{-}S)$. Since M' corresponds to a simple b-matching of $G^T(VG\text{-}S)$ of cardinality $|M'|$, we have $b(VG^T(VG\text{-}S)) - 2|M'| \geq |\mathscr{D}_1(S,T)|$. Since $b(VG^T(VG\text{-}S)) = b(VG\text{-}S) - Q(S,T)$, this implies that $b(VG\text{-}S) - 2|M'|$ must be at least $Q(S,T) + |\mathscr{D}_1(S,T)|$. Now since M is a perfect simple b-matching of G, $b(S) \geq b(VG\text{-}S) - 2|M'| \geq Q(S,T) + |\mathscr{D}_1(S,T)|$.

Conversely, suppose that G does not have a perfect simple b-matching. Let G' be the graph obtained from G by replacing each edge $e = (u,v) \in EG$ by the edges (u, u_e), (u_e, v_e), (v_e, v) and adding u_e and v_e to VG with $b_{u_e} = b_{v_e} = 1$. Since G does not have a perfect simple b-matching, G' does not have a perfect b-matching. So, by Tutte's b-matching theorem, there exists a set $X \subseteq VG'$ such that $b(X) < b(\mathscr{C}^0(X)) + |\mathscr{C}^1(X)|$. Let X be such a subset of VG' and let $S = \{v \in VG: v \in X\}$. It may be assumed that for each edge $e = (u,v)$ of G, if $u \in S$ and $v \notin S$ then $u_e \notin X$ and $v_e \in X$. It may also be assumed that for each edge $e = (u,v) \in EG$, if $u \in S$ and $v \in S$ then neither u_e nor v_e is in X. Furthermore, it may be assumed that for each edge $e = (u,v) \in EG$, if $u \notin S$ and $v \notin S$ then $u_e \in X$ only if $u_f \in X$ for each edge $f = (u,q)$ such that $q \in VG\text{-}S$. Let $T = \{v \in VG\text{-}S: v \in \mathscr{C}^0(X)\}$, that is, T is the set of nodes $v \in VG\text{-}S$ that are isolated in $G'[VG'\text{-}X]$. Since $b(X) < b(\mathscr{C}^0(X)) + |\mathscr{C}^1(X)|$, we have $b(S) < Q(S,T) + |\mathscr{D}_1(S,T)|$. ∎

We will use this theorem to prove some results on simple b-matching separability. A *simple b-separation* of G is a partition of EG into nonempty subsets E_1 and E_2 such that if k_i is the cardinality of a largest simple b-matching of G contained in E_i, for $i = 1,2$, then $k_1 + k_2$ is the cardinality of a largest simple b-matching of G. If G has a simple b-separation then G is *simple b-separable* (otherwise G is *simple b-nonseparable*). These definitions are analogous to those for matching separability given in the previous section.

As in the b-matching case, critical graphs play an important role here. If G is connected and $|VG| \geq 3$, then G is *simple b-critical* if for each $v \in VG$ there exists a matching of cardinality $\lfloor b^G(VG)/2 \rfloor$ which contains exactly $b_v^G - 1$ edges which meet v (that is, for each $v \in VG$ there exists a perfect simple b'-matching of G where $b'_v = b_v^G - 1$ and $b'_u = b_u^G$ for all $u \in VG\text{-}\{v\}$). The graph given in Figure 1 is an example of a simple b-critical graph with $b_v = 2$ for each node v. (Note that in this example $b_v^G = 1$ for all nodes of degree one and $b_v^G = 2$ for all other nodes.) We have that if G is simple b-critical then $b^G(VG)$ is odd. If G is connected, $|VG| \geq 3$, and $b^G(VG)$ is even, then G is *simple b-bicritical* if for each $v \in VG$ there exists a simple b-matching of G of cardinality $(b^G(VG)/2) - 1$ which contains exactly $b_v^G - 2$ edges

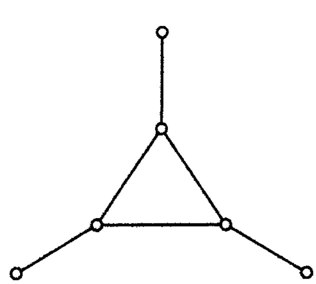

FIGURE 1

which meet v. The complete graph on four nodes with $b_v = 2$ for each node v is an example of a simple b-bicritical graph. Notice that this example of a simple b-bicritical graph has a perfect simple b^G-matching. We will use Theorem 3.2 to show that every simple b-bicritical graph has this property.

LEMMA 3.3. *If G is a simple b-bicritical graph then G has a perfect simple b^G-matching.*

PROOF. Suppose that G is simple b-bicritical. Let $S \subseteq VG$ and $T \subseteq VG\text{-}S$. by Theorem 3.2, it suffices to show that $b^G(S) \geq Q^G(S, T) + |\mathscr{D}_1^G(S, T)|$, where $Q^G(S, T)$ and $\mathscr{D}_1^G(S, T)$ are $Q(S, T)$ and $\mathscr{D}_1(S, T)$ with respect to b^G. Suppose that $S \neq \varnothing$ and let $v \in S$. Let $b'_v = b_v^G - 2$ and $b'_u = b_u^G$ for all $u \in VG - \{v\}$. Since G has a perfect simple b'-matching, $b'(S) \geq Q'(S, T) + |\mathscr{D}'_1(S, T)|$, where $Q'(S, T)$ and $\mathscr{D}'_1(S, T)$ are $Q(S, T)$ and $\mathscr{D}_1(S, T)$ with respect to b'. Now $b^G(S) = b'(S) + 2$, $Q'(S, T) = Q^G(S, T)$, and $\mathscr{D}'_1(S, T) = \mathscr{D}_1^G(S, T)$. So $b^G(S) \geq Q^G(S, T) + |\mathscr{D}_1^G(S, T)|$. Now suppose that $S = \varnothing$. If $T = VG$ then $Q^G(S, T) \leq 0$ and $|\mathscr{D}_1^G(S, T)| = 0$, which implies that $b^G(S) \geq Q^G(S, T) + |\mathscr{D}_1^G(S, T)|$. Suppose that $VG - T \neq \varnothing$ and let $v \in VG - T$. Define b' as above. Again, Theorem 3.2 implies that $b'(S) \geq Q'(S, T) + |\mathscr{D}'_1(S, T)|$. Since $b'(S) = b^G(S) = 0$, $Q'(S, T) = Q^G(S, T)$, and $\mathscr{D}'_1(S, T) = \mathscr{D}_1^G(S, T)$, we have $b^G(S) \geq Q^G(S, T) + |\mathscr{D}_1^G(S, T)|$. ∎

The following lemma gives the relationship between simple b-separability and the notions of criticality defined above.

LEMMA 3.4. *A connected graph G with $|VG| \geq 3$ is simple b-nonseparable only if it is isomorphic to $K_{1,n}$ for some n (with multiple edges allowed) or simple b-critical or simple b-bicritical.*

PROOF. Since (3.2) is a totally dual integral defining system for $S(G, b)$, Lemma 2.5 implies that G is simple b-nonseparable if and only if $x(EG) \leq r(EG)$ is in the Schrijver system for $S(G, b)$, where $r(EG)$ is the cardinality of a largest simple b-matching of G. Suppose that G is simple b-nonseparable and that G is not isomorphic to $K_{1,n}$ for some n. By the above comment, we must have $r(EG) = \lfloor b^G(VG)/2 \rfloor$, since $x(EG) \leq r(EG)$ must be present in the system (3.2). We will deal with the cases where $b^G(VG)$ is odd and where $b^G(VG)$ is even separately.

Case 1: $b^G(VG)$ is odd. We must show that G is simple b-critical. Suppose that $u \in VG$ is a node such that there does not exist a perfect simple b'-matching, where $b'_u = b_u^G - 1$ and $b'_v = b_v^G$ for each $v \in VG\text{-}\{u\}$. (If no such node exists then G is simple b-critical.) We will use Theorem 3.2 to show that G has a simple b-separation.

By Theorem 3.2, there exists $S \subseteq VG$ and $T \subseteq VG\text{-}S$ such that

$$b'(S) < Q'(S, T) + |\mathscr{D}'_1(S, T)| \tag{3.4}$$

where $Q'(S, T)$ and $\mathscr{D}'_1(S, T)$ are $Q(S, T)$ and $\mathscr{D}_1(S, T)$ with respect to b' and G. (Note that since $b'(VG)$ is even, either $S \neq \varnothing$ or $\mathscr{D}'_1(S, T) \neq \varnothing$ for any sets $S \subseteq VG$ and $T \subseteq VG\text{-}S$ for which (3.4) holds.) Let S and T be such subsets of VG (it may be assumed that if $v \in VG\text{-}S$ and $d_{G[VG\text{-}S]}(v) = 0$, then $v \in T$). The inequality (3.4) implies that

$$b^G(S) \leq Q^G(S, T) + |\mathscr{D}_1^G(S, T)| \tag{3.5}$$

where $Q^G(S, T)$ and $\mathscr{D}_1^G(S, T)$ are $Q(S, T)$ and $\mathscr{D}_1(S, T)$ with respect to b^G and G. Let $\mathscr{D}$ be the set of all connected components of $G^T(VG\text{-}S)$. Each connected component $\overline{H}$ of $G^T(VG\text{-}S)$ corresponds to a subgraph H of G. Note that if $\overline{H}$ is in $\mathscr{D}$ then $b^H(VH) \leq b^G(V\overline{H})$.

From the definition of $Q^G(S,T)$ and $\mathscr{D}$ the following equality holds

$$b^G(VG) = b^G(S) + Q^G(S,T) + b^G(\mathscr{D}) \tag{3.6}$$

where $b^G(\mathscr{D}) = \Sigma\{b^G(V\overline{H})\colon \overline{H} \in \mathscr{D}\}$. The inequality (3.5) implies that

$$b^G(VG) \geq b^G(VG) + b^G(S) - Q^G(S,T) - |\mathscr{D}_1^G(S,T)|. \tag{3.7}$$

Combining (3.6) and (3.7) gives

$$b^G(VG) \geq 2b^G(S) + b^G(\mathscr{D}) - |\mathscr{D}_1^G(S,T)|. \tag{3.8}$$

It follows that

$$\lfloor b^G(VG)/2 \rfloor \geq b^G(S) + (b^G(\mathscr{D})/2) - (|\mathscr{D}_1(S,T)|/2) \tag{3.9}$$

which is equivalent to

$$\lfloor b^G(VG)/2 \rfloor \geq b^G(S) + \Sigma\{\lfloor b^G(V\overline{H})/2 \rfloor : \overline{H} \in \mathscr{D}\}. \tag{3.10}$$

This inequality implies that

$$\lfloor b^G(VG)/2 \rfloor \geq b^G(S) + \Sigma\{\lfloor b^H(VH)/2 \rfloor : \overline{H} \in \mathscr{D}\}. \tag{3.11}$$

(Note that each edge in EG is either in $\delta_G(v)$ for some $v \in S$ or in EH for some $\overline{H} \in \mathscr{D}$.) If $S \neq \varnothing$, then (3.11) gives that $(\delta_G(v), EG\text{-}\delta_G(v))$ is a simple b-separation of G for any $v \in S$. If $S = \varnothing$, then $|\mathscr{D}| \geq 2$ (since $\mathscr{D}_1'(S,T) \neq \varnothing$) and, thus, (3.11) implies that $(EH, EG\text{-}EH)$ is a simple b-separation of G for any $\overline{H} \in \mathscr{D}$.

Case 2: $b^G(VG)$ is even. In this case we must show that G is simple b-bicritical. Suppose that $u \in VG$ is a node such that there does not exist a perfect simple b'-matching of G, where $b'_u = b^G_u - 2$ and $b'_v = b^G_v$ for each $v \in VG\text{-}\{u\}$. (If no such node exists then G is simple b-bicritical.) If $b^G_u = 1$, then $(\delta_G(u), EG\text{-}\delta_G(u))$ is a simple b-separation of G, since G has a perfect simple b^G-matching. So it may be assumed that $b'_u \geq 0$. Let $S \subseteq VG$ and $T \subseteq VG\text{-}S$ be sets such that

$$b'(S) < Q'(S,T) + |\mathscr{D}_1'(S,T)| \tag{3.12}$$

holds. This inequality implies that

$$b^G(S) \leq Q^G(S,T) + |\mathscr{D}_1^G(S,T)| + 1. \tag{3.13}$$

Now since $b^G(VG)$ is an even integer, (3.6) implies that $b^G(S) + Q^G(S,T) + |\mathscr{D}_1^G(S,T)|$ is an even integer. So (3.13) implies (3.5). Thus, a simple b-separation of G can be found as in Case 1. ∎

If G is a simple b-critical graph and G has a cutnode v with $b_v = 1$, then it is easy to see that G is simple b-separable. (Let G_1 be a connected component of $G[VG\text{-}\{v\}]$. Since G is simple b-critical and $b_v = 1$, $b^G(VG_1)$ is even and $b^G(VG\text{-}VG_1)$ is odd. Thus, $(\gamma(VG_1 \cup \{v\}), \gamma(VG\text{-}VG_1))$ is a simple b-separation of G.) However, unlike the matching case (see Theorem 2.4) and the b-matching case (see Theorem 2.7 and Cook [8]), it is not true that if G is simple b-critical and G has no cutnode v with $b_v = 1$ then G is simple b-nonseparable. Consider the graph H given in Figure 2 with $b_v = 2$ for each node v.

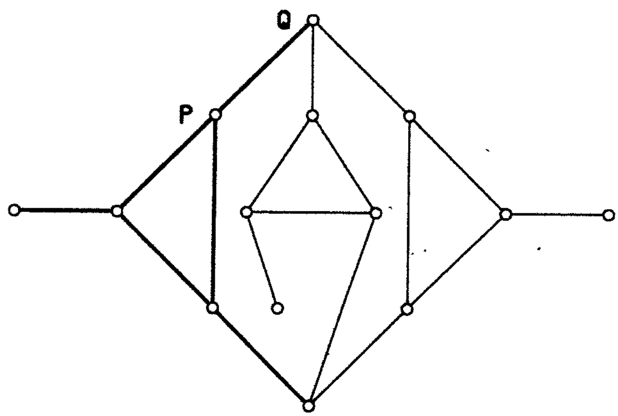

FIGURE 2

It is easy to check that H is simple b-critical (due to symmetry, H has essentially only four different types of nodes). However, H has a simple b-separation as indicated by the bold-faced edges in Figure 2. (Letting E_1 be the set of bold-faced edges, $(E_1, EH\text{-}E_1)$ is a simple b-separation of H.)

If G is a simple b-critical graph and M is a simple b-matching of G of cardinality $\lfloor b^G(VG)/2 \rfloor$, then M is a *near-perfect simple b-matching* of G. For each near-perfect simple b-matching of G there exists a unique node $v \in VG$ such that M is *deficient* at v (that is, M contains exactly $b_v^G - 1$ edges which meet v). An edge $e = (u, v)$ is *full* with respect to node u if e is contained in every near-perfect simple b-matching of G deficient at u. Notice that the edge (p, q) indicated in the graph given in Figure 2 is full with respect to p, when $b_v = 2$ for each node v of the graph.

THEOREM 3.5. *A simple b-critical graph G is simple b-nonseparable if and only if* $b_v^G = d_G(v)$ *for each node* $v \in VG$ *which meets an edge* $(u, v) \in EG$ *that is full with respect to u and the graph* $G^T(VG)$, *where* $T = \{v \in VG\colon b_v^G = d_G(v)\}$, *is connected with no cutnode q having* $b_q = 1$. (*Recall that* $G^T(VG)$ *is obtained from G by replacing each node* $v \in T$ *by the nodes* $v_1, \ldots, v_k$, *where* $k = d_G(v)$, *and replacing the edges* $(u_1, v), \ldots, (u_k, v)$ *by the edges* $(u_1, v_1), \ldots, (u_k, v_k)$ *and letting* $b_{v_i} = 1$, $i = 1, \ldots, k$.)

PROOF. Let G be a simple b-critical simple b-nonseparable graph. Suppose that edge $e = (u, v) \in EG$ is full with respect to u and that $b_v^G < d_G(v)$. Let G' be the graph obtained from G by adding a new node v' to VG with $b_{v'} = 1$ and replacing $e = (u, v)$ by the edge $e' = (u, v')$. Since $b_v^G < d_G(v)$, $b^{G'}(VG') = b^G(VG) + 1$, which is an even number. If M' is a perfect simple b-matching of G', then $e' \in M'$ and $M = M'\text{-}\{e'\}$ is a near-perfect simple b-matching of G deficient at u which does not contain the edge e. So G' has no perfect simple b-matching and hence, by Lemma 3.3, is not simple b-bicritical. So Lemma 3.4 implies that G' has a simple b-separation (E_1', E_2'). We may assume that $e' \in E_1'$ and $e' \notin E_2'$. Let $E_1 = (E_1'\text{-}\{e'\}) \cup \{e\}$ and $E_2 = E_2'$. Since the cardinality of a largest simple b-matching of G' is equal to the cardinality of a largest simple b-matching of G, (E_1, E_2) is a simple b-separation of G, a contradiction. So $b_v^G = d_G(v)$ for each node $v \in VG$ which meets an edge $(u, v) \in EG$ that is full with respect to u.

The graph G is simple b-nonseparable if and only if $G^T(VG)$, where $T = \{v \in VG\colon b_v^G = d_G(v)\}$, is simple b-nonseparable. Clearly, $G^T(VG)$ is connected. Since $b^{G^T(VG)}(VG^T(VG))$ is odd and $G^T(VG)$ is simple b-nonseparable, Lemma 3.4 implies that $G^T(VG)$ is simple b-critical. Thus, $G^T(VG)$ has no cutnode q with $b_q = 1$, since every simple b-critical simple b-nonseparable graph has this property.

Conversely, let G be a simple b-critical graph such that $b_v^G = d_G(v)$ for each node $v \in VG$ which meets an edge $(u, v) \in EG$ that is full with respect to u and such that the graph $G^T(VG)$, where $T = \{v \in VG: b_v^G = d_G(v)\}$, is connected with no cutnode q having $b_q = 1$. If $e = (u, v) \in EG$ is contained in every near-perfect simple b-matching of G then e is full with respect to u and full with respect to v and, hence, forms a connected component of $G^T(VG)$. Since $G^T(VG)$ is connected, no such edge exists. This implies that $G^T(VG)$ is simple b-critical and also that $G^T(VG)$ has no separation of the form $(\{e\}, EG^T(VG) - \{e\})$ for some $e \in EG^T(VG)$. From the condition on full edges of G, if $(u, v) \in EG^T(VG)$ is a full edge with respect to u in $G^T(VG)$ then $b_v = 1$. We will show that $G^T(VG)$ is simple b-nonseparable, which implies that G is also simple b-nonseparable.

To simplify the notation, let G' denote the graph $G^T(VG)$. Suppose that G' is simple b-separable. Let $E_1, \dots, E_k$ be nonempty subsets of EG' such that

(i) $E_1 \cup \cdots \cup E_k = EG'$, (3.14)

(ii) for $i = 1, \dots, k$ the graph G_i with edge set E_i and node set $\{v \in VG': v$ meets an edge in $E_i\}$ is simple b-nonseparable, and

(iii) if r_i is the cardinality of a largest simple b-matching of G_i, $i = 1, \dots, k$, then $r_1 + \cdots + r_k$ is the cardinality of a largest simple b-matching of G'.

Since G' is simple b-critical, G' does not have a simple b-separation of the form $(\delta_{G'}(v), EG'\text{-}\delta_{G'}(v))$ for some $v \in VG'$. Thus, for $j = 1, \dots, k$, G_j is not isomorphic to $K_{1,n}$ for some n. So Lemma 3.4 implies that for each $j \in \{1, \dots, k\}$, G_j is either simple b-critical or simple b-bicritical.

Claim 1: For each $j \in \{1, \dots, k\}$, $b_v^{G_j} = b_v$ for each $v \in VG_j$.

Suppose that the claim is not true for the graph G_1. Let $v \in VG_1$ be a node with $b_v^{G_1} < b_v$ and let $e = (u, v) \in EG_1$ be an edge which meets v. If G_1 is simple b-bicritical, then $(\{e\}, EG_1\text{-}\{e\})$ is a simple b-separation of G_1. So G_1 must be simple b-critical. Now, by (3.14)(iii), if M' is a near-perfect simple b-matching of G' then $M' \cap E_1$ is a near-perfect simple b-matching of G_1. So if M' is a near-perfect simple b-matching of G' deficient at u, then e must be contained in M'. So e is full with respect to u and $b_v = 1$, contrary to the assumption that $b_v^{G_1} < b_v$, which proves the claim.

So $b^{G_j}(VG_j) = b(VG_j)$ and $r_j = \lfloor b(VG_j)/2 \rfloor$ for each $j \in \{1, \dots, k\}$. Since G' is simple b-separable, k must be at least 2.

Claim 2: $b(VG_1) + \cdots + b(VG_k) \geq b(VG') + k$.

Since $b(VG')$ is odd, this claim implies that (3.14)(iii) is not satisfied, a contradiction. To prove the claim, define a graph H with nodes $t_1, \dots, t_k$ and an edge (t_i, t_j) for all $i \neq j$ such that $VG_i \cap VG_j \neq \emptyset$. Since G' is connected, the graph H is connected and $|EH| \geq k - 1$. Since $b(VG_1) + \cdots + b(VG_k) \geq b(VG') + |EH|$, we may assume that $|EH| = k - 1$. Let t_i be a node of degree 1 in H and let t_j be the node in VH that is adjacent to t_i. Since G' has no cutnode q with $b_q = 1$, $b(VG_i \cap VG_j) \geq 2$, which completes the proof of the claim and the proof of the theorem. ∎

REMARK 3.6. This theorem yields a polynomial time algorithm for testing whether or not a graph G is a simple b-critical simple b-nonseparable graph. Indeed, using Edmonds' [14, 15] blossom algorithm and a "scaling" argument, similar to the one used by Edmonds and Karp [18] to solve min-cost flow problems, Cunningham and Marsh (see Marsh [23]) developed a polynomial time algorithm for the b-matching problem. (A different polynomial time algorithm for the b-matching problem has been found by Anstee [1]. Anstee's algorithm uses a polynomial time min-cost flow algorithm as a subroutine and thus avoids a separate "scaling" argument.) Using the construction of Tutte mentioned earlier, this algorithm can be used to solve simple b-matching problems in polynomial time, which implies that it is possible to test whether or not G is simple b-critical in polynomial time. This simple b-matching algorithm can also be used to check that $b_v^G = d_G(v)$ for each node $v \in VG$ which meets an edge $(u, v) \in EG$ that is full with respect to u. It only remains to check that $G^T(VG)$ is connected with no cutnode q having $b_q = 1$, which can be done easily in polynomial time.

Using Theorem 3.5, we will show that simple b-critical simple b-nonseparable subgraphs of G produce a class of essential inequalities for $S(G, b)$. The proof technique used here is a generalisation of one used by L. Lovász to give a short proof of the corresponding result for matching polyhedra. (This short proof of L. Lovász can be found in Cornuéjols and Pulleyblank [11] and Lovász and Plummer [22a].)

LEMMA 3.7. *Let H be a subgraph of G with $|VH| \geqq 3$. If the inequality*

$$x(EH) \leqq \lfloor b^H(VH)/2 \rfloor \tag{3.15}$$

is not $x(\delta(v)) \leq b_v$ for some $v \in VG$, then it is essential for $S(G, b)$ if and only if it is simple b-critical and simple b-nonseparable and there does not exist an edge $e = (u, v) \in EG\text{-}EH$ with $u, v \in VH$ and $d_H(u) \geqq b_u$ and $d_H(v) \geqslant b_v$.

PROOF. Let H be a connected subgraph of G with $|VH| \geqq 3$ such that (3.15) is not $x(\delta_G(v)) \leq b_v$ for some $v \in VG$. Suppose that (3.15) is essential for $S(G, b)$. Clearly, H is a simple b-nonseparable graph and has a simple b-matching of cardinality $\lfloor b^H(VH)/2 \rfloor$. By Lemma 3.4, H is simple b-critical or simple b-bicritical. If H is simple b-bicritical, then (3.15) can be obtained by summing the valid inequalities $x(\delta_H(v)) \leq b_v^H$ for each $v \in VH$ and dividing the resulting inequality by 2. So H must be simple b-critical and simple b-nonseparable. If there exists an edge $e = (u, v) \in EG\text{-}EH$ with $u, v \in VH$ and $d_H(u) \geq b_u$ and $d_H(v) \geq b_v$, then $x(EH) + x_e \leq \lfloor b^H(VH)/2 \rfloor$ is a valid inequality for $S(G, b)$. So no such edge exists.

Conversely, suppose that H is simple b-critical and simple b-nonseparable and that there does not exist an edge as described in the statement of the lemma. Let $\mathcal{M}$ be the set of all simple b-matchings of G for which $x(EH) = \lfloor b^H(VH)/2 \rfloor$. Suppose that $ax \leqq \alpha$ is a valid inequality for $S(G, b)$ and that $a\bar{x} = \alpha$ for each $\bar{x} \in \mathcal{M}$ and that a is not the vector of all zeros. We will show that $a_e = 0$ for all $e \in EG\text{-}EH$ and that for some number $\lambda > 0$, $a_e = \lambda$ for each $e \in EH$. This implies that (3.15) is facet inducing for $S(G, b)$ and hence essential for $S(G, b)$ (see, for example, Pulleyblank [27]).

Suppose that $e = (u, v) \in EG\text{-}EH$. To show that $a_e = 0$ it suffices to show that there exists a simple b-matching $\overline{M} \in \mathcal{M}$ such that $e \in \overline{M}$ (since $\overline{M}\text{-}\{e\}$ is also a member of $\mathcal{M}$, which implies that $\alpha + a_e = \alpha$). If neither u nor v is in VH, then clearly such an $\overline{M}$ exists. If exactly one of u and v, say v, is in VH then let M be a near-perfect simple b-matching of H deficient at v and let $\overline{M} = M \cup \{e\}$. If both u and v are in VH, then at least one of u and v, say v, must have $d_H(v) < b_v$. In this case, let M be a near-perfect simple b-matching of H deficient at u and let $\overline{M} = M \cup \{e\}$.

Since $ax \leqq \alpha$ is valid for $S(G, b)$ and since each edge $e \in EH$ is in a near-perfect simple b-matching of H (as H is simple b-critical), we have that $a_e \geqq 0$ for each $e \in EH$. So $a_l > 0$ for some edge $l \in EH$. Suppose that it is not true that $a_e = a_l$ for each $e \in EH$.

Let H' denote the graph $H^T(VH)$, where $T = \{v \in VH\colon d_H(v) = b_v^H\}$. By Theorem 3.5, H' is simple b-critical and connected. So there exist a node $v \in VH'$ and edges $p, z \in \delta_{H'}(v)$ such that $a_p \neq a_z$. (We identify edges in EH' with their corresponding edges in EH.) Let $\beta = \min\{a_e\colon e \in \delta_{H'}(v)\}$. Let $J = \{e \in \delta_{H'}(v)\colon a_e = \beta\}$ and let $K = \delta_{H'}(v)\text{-}J$.

Claim. There exist an edge $e = (s, v) \in K$ and a near-perfect simple b-matching $\overline{M}$ of H' deficient at s such that $e \notin \overline{M}$ and $\overline{M} \cap J \neq \emptyset$.

Once the claim is shown, the proof will be complete, since if $f \in \overline{M} \cap J$ then both $\overline{M}$ and $(\overline{M}\text{-}\{f\}) \cup \{e\}$ are members of $\mathscr{M}$, which implies that $a_e = a_f$, a contradiction (since $a_f = \beta$ and $a_e > \beta$).

Case 1: $b_v = 1$. Let H'' be the graph obtained from H' by splitting the node v in the following way: Replace v by the nodes v', v'' with $b_{v'} = b_{v''} = 1$ and replace each edge $(t, v) \in J$ by (t, v') and each edge $(t, v) \in K$ by (t, v''). Suppose that H'' does not have a perfect simple $b^{H''}$-matching. By Lemma 3.4, H'' has a simple b-separation. Since the cardinality of a largest simple b-matching in H'' is equal to the cardinality of a largest simple b-matching in H', this implies that H' is simple b-separable and hence that H is simple b-separable, a contradiction. So there exists a perfect simple $b^{H''}$-matching M of H''. Letting e be the unique edge in M which meets the node v'' and letting $\overline{M} = M\text{-}\{e\}$, the conditions in the claim are satisfied. (Again, edges in EH'' are identified with the corresponding edges in EH').

Case 2: $b_v \geqq 2$. By Theorem 3.5, there is no edge $(u, v) \in \delta_{H'}(v)$ that is full with respect to u in H'. Let $f^1 = (u, v) \in J$ and let M^1 be a near-perfect simple b-matching of H' deficient at u such that $f^1 \notin M^1$. Choose edge $f^2 \in K$ and near-perfect simple b-matching M^2 of H' such that

(i) M^2 is deficient at q and $f^2 \notin M^2$ (where q is the end of f^2 which is not v) and (3.16)

(ii) of all edge, near-perfect simple b-matching of H' pairs which satisfy (i), $|M^1 \cap M^2|$ is as large as possible.

If $M^2 \cap J \neq \emptyset$, then $\overline{M} = M^2$ and $e = f^2$ satisfy the conditions of the claim. Similarly, if $M^1 \cap K \neq \emptyset$, then any $e \in M^1 \cap K$ and $\overline{M} = (M^1\text{-}\{e\}) \cup \{f^1\}$ satisfy the conditions of the claim. So we may assume that $M^1 \cap K = \emptyset$ and $M^2 \cap J = \emptyset$.

Since M^1 is deficient at u and M^2 at q, there exists a trail $T = ue_1v_1e_2v_2 \cdots e_{2k}q$ in H' such that $\{e_1, e_3, \dots, e_{2k-1}\} \subseteq M_2 - M_1$ and $\{e_2, e_4, \dots, e_{2k}\} \subseteq M_1 - M_2$. (A *trail* is defined as in Bondy and Murty [5], that is $e_i \in EH'$ for $i = 1, \dots, 2k$, $v_i \in VH'$ for $i = 1, \dots, 2k - 1$, $e_i \neq e_j$ for $i \neq j$, $e_1 = (u, v_1)$, $e_2 = (v_1, v_2), \dots, e_{2k} = (v_{2k-1}, q)$. To see that such a trail T exists, consider $M^1 \triangle M^2$, the symmetric difference of M^1 and M^2.) Let $ET = \{e_1, e_2, \dots, e_{2k}\}$.

Suppose that $|ET \cap J| \leqq 1$. Let $\overline{M} = (M^1\text{-}\{e_2, e_4, \dots, e_{2k}\}) \cup \{e_1, e_3, \dots, e_{2k-1}\}$. Since $\{e_1, e_3, \dots, e_{2k-1}\} \cap M^1 = \emptyset$ and $\{e_2, e_4, \dots, e_{2k}\} \subseteq M^1$, $\overline{M}$ is a near-perfect simple b-matching of H' deficient at q. Since $b_v \geqq 2$ and $M^1 \cap K = \emptyset$, we have $|M^1 \cap J| \geqq 2$ and hence $\overline{M} \cap J \neq \emptyset$. So $\overline{M}$ and $e = f^2$ satisfy the conditions of the claim.

Now suppose that $|ET \cap J| \geqq 2$. We must have $v_i = v$ and $v_j = v$ for some i, j in $\{1, 2, \dots, 2k - 1\}$, $i < j$. If $e_i \in K$ then $e_i \in M^2 - M^1$ and hence $M^3 = (M^1\text{-}\{e_2, e_4, \dots, e_{i-1}\}) \cup \{e_1, e_3, \dots, e_{i-2}\}$ is a near-perfect simple b-matching of H' deficient at v_{i-1} which does not contain $e_i = (v_{i-1}, v) \in K$. Since $|M^3 \cap M^1| > |M^2 \cap M^1|$, this contradicts the choice of the pair f^2, M^2. So we may assume that $e_i \in J$ and that $e_{i+1} \in K$ (since $M^2 \cap J = \emptyset$ and $M^1 \cap K = \emptyset$). By the same argument, we may also assume that $e_j \in J$ and $e_{j+1} \in K$. Notice that $ve_{i+1}v_{i+1}e_{i+2} \cdots e_jv$ is a trail from v to v with $\{e_{i+1}, e_{i+3}, \dots, e_{j-1}\} \subseteq M^2 - M^1$ and $\{e_{i+2}, e_{i+4}, \dots, e_j\} \subseteq M^1 - M^2$. So $M^3 = (M^2\text{-}\{e_{i+1}, e_{i+3}, \dots, e_{j-1}\}) \cup \{e_{i+2}, e_{i+4}, \dots, e_j\}$ is a near-perfect simple b-matching of H' deficient at q. Since $f^2 \in K$ and $|M^1 \cap K| = \emptyset$, f^2 is not in M^1 and hence not in $\{e_{i+2}, e_{i+4}, \dots, e_j\}$. So $f^2 \notin M^3$. However, $|M^3 \cap M^1| > |M^2 \cap M^1|$, a contradiction, which completes the proof of the lemma. ∎

REMARK 3.8. It is possible to prove a structural result for simple b-critical simple b-nonseparable graphs similar to Lovász' [22] result on ear-decompositions of hypomatchable graphs (see also Cornuéjols and Pulleyblank [12]). This structural result can be used to prove Lemma 3.7 by constructing $|EG|$ affinely independent simple b-matchings of G which satisfy (3.15) with equality. However, at present, our proof of these statements is tedious and has therefore been omitted in favor of the proof of Lemma 3.7 given above.

A consequence of Lemma 3.7 and Theorem 3.1 is that

(i) $0 \leq x_e \leq 1 \; \forall e \in EG$, (3.17)

(ii) $x(\delta_G(v)) \leq b_v \; \forall v \in VG$,

(iii) $x(EH) \leq \lfloor b^H(VH)/2 \rfloor$ for each simple b-critical simple b-nonseparable subgraph H of G,

is a defining system for $S(G, b)$. Thus, to prove that an inequality is essential for $S(G, b)$ it suffices to exhibit a vector $\bar{x}$ which does not satisfy the inequality but does satisfy each of the other inequalities in (3.17). This technique will be used to characterise which inequalities in (3.17)(i) and (3.17)(ii) are essential for $S(G, b)$.

LEMMA 3.9. *For each $e \in EG$, $x_e \geq 0$ is an essential inequality for $S(G, b)$.*

PROOF. Let $e \in EG$ and let $x_e = -1$ and $x_f = 0$ for each $f \in EG\text{-}\{e\}$. The vector x does not satisfy $x_e \geq 0$ but it does satisfy each of the other inequalities in (3.17). ∎

LEMMA 3.10. *Let $e \in EG$. The inequality $x_e \leq 1$ is essential for $S(G, b)$ if and only if e does not meet a node v with $b_v = 1$ and $d_G(v) > 1$.*

PROOF. If e meets a node v with $b_v = 1$ and $d_G(v) > 1$, then $x(\delta(v)) \leq 1$ implies the inequality $x_e \leq 1$. Suppose that e does not meet such a node v. Let $x_e = 2$ and $x_f = 0$ for each $f \in EG\text{-}\{e\}$. The vector x does not satisfy $x_e \leq 1$ but it does satisfy each of the other inequalities in (3.17). ∎

LEMMA 3.11. *Let $v \in VG$ and let $b'_u = \min\{b_u |\delta(u) \cap \delta(v)|\}$ for each node $u \in N(v)$. The inequality $x(\delta(v)) \leq b_v$ is essential for $S(G, b)$ if and only if one of the following conditions holds*:

(i) *$b'(N(v)) = b_v$ and* (ii) *v belongs to a connected component of G which contains exactly two nodes and* (iii) *if $d_G(v) \leq b_v$ then $b_v = 1$.* (3.18)

(i) *$b'(N(v)) = b_v + 1$ and* (ii) *there is no edge $(v_1, v_2) \in \gamma(N(v))$ such that $b_{v_1} = b'_{v_1}$ and $b_{v_2} = b'_{v_2}$.* (3.19)

$$b'(N(v)) \geq b_v + 2. \tag{3.20}$$

PROOF. Suppose that $b'(N(v)) \leq b_v$ and that (3.18) does not hold. The inequality $x(\delta(v)) \leq b_v$ is implied by the inequalities $x_e \leq 1$ for each $e \in \delta(v)$ and $x(\delta(u)) \leq b_u$ for each $u \in N(v)$. Now suppose that (3.18) holds. Let $x_e = 1$ for each $e \in \delta(v)$ (if $d_G(v) = 1$ let $x_e = 2$ for $e \in \delta(v)$) and let $x_e = 0$ for each $e \in EG\text{-}\delta(v)$. The vector x does not satisfy $x(\delta(v)) \leq b_v$ but it does satisfy each of the other inequalities in (3.17). So $x(\delta(v)) \leq b_v$ is essential for $S(G, b)$.

Suppose that $b'(N(v)) = b_v + 1$ and that (3.19) does not hold. Let $e' = (v_1, v_2) \in \gamma(N(v))$ be an edge such that $b_{v_1} = b'_{v_1}$ and $b_{v_2} = b'_{v_2}$. The inequality $x(\delta(v)) + x_{e'} \leq b_v$ is valid for $S(G, b)$ and implies $x(\delta(v)) \leq b_v$. Now suppose that (3.19) holds. For each $u \in N(v)$ select b'_u edges from $\delta(u) \cap \delta(v)$ and let J be the collection of these edges. Let $x_e = 1$ for each $e \in J$ and $x_e = 0$ for each $e \in EG\text{-}J$. The vector x does not satisfy $x(\delta(v)) \leq b_v$ but it does satisfy each of the other inequalities in (3.17). So the inequality $x(\delta(v)) \leq b_v$ is essential for $S(G, b)$.

Suppose that $b'(N(v)) \geq b_v + 2$. Let J be a collection of $b_v + 2$ edges in $\delta(v)$ such that $|J \cap \delta(u)| \leq b'_u$ for each $u \in N(v)$. Let $\bar{x}_e = b_v/(b_v + 1)$ for each $e \in J$ and

$\bar{x}_e = 0$ for each $e \in EG\text{-}J$. The vector $\bar{x}$ does not satisfy $x(\delta(v)) \leq b_v$ but it does satisfy each of the other inequalities in (3.17)(i) and (3.17)(ii). Let $\alpha x \leq \beta$ be an inequality in (3.17)(iii). Let S be a proper subset of J. If $|S| \leq b_v$, then $\bar{x}(S) \leq |S|$ and S is a simple b-matching of G. If $|S| = b_v + 1$, then $\bar{x}(S) = b_v$ and S contains a simple b-matching of cardinality b_v. So if $\alpha_e \neq 1$ for some $e \in J$ then $\bar{x}$ satisfies $\alpha x \leq \beta$. On the other hand, if $\alpha_e = 1$ for each $e \in J$ then β must be at least $\lfloor (b_v + (b_v + 2))/2 \rfloor = b_v + 1$. Since $\bar{x}(J) < b_v + 1$, if $\alpha_e = 1$ for each $e \in J$ then $\bar{x}$ satisfies $\alpha x \leq \beta$. So $\bar{x}$ satisfies all other inequalities in (3.17), which implies that $x(\delta(v)) \leq b_v$ is essential for $S(G, b)$. ∎

Using these lemmas, the minimal defining system for $S(G, b)$ can be described.

THEOREM 3.12. *The unique (up to positive scalar multiples of the inequalities) minimal defining system for the convex hull of the simple b-matchings of G is*

(i) $x_e \geq 0\ \forall e \in EG$. (3.21)

(ii) $x_e \leq 1$ *for each* $e \in EG$ *such that* e *does not meet a node* v *with* $b_v = 1$ *and* $d_G(v) > 1$.

(iii) $x(\delta(v)) \leq b_v$ *for each* $v \in VG$ *for which either* (3.18), (3.19), *or* (3.20) *holds.*

(iv) $x(EH) \leq \lfloor b^H(VH)/2 \rfloor$ *for each simple b-critical simple b-nonseparable subgraph* H *of* G *such that there does not exist an edge* $e = (u, v) \in EG\text{-}EH$ *with* $u, v \in VH$ *and* $d_H(u) \geq b_u$ *and* $d_H(v) \geq b_v$.

PROOF. This follows from Lemma 3.7, Lemma 3.9, Lemma 3.10, Lemma 3.11 and the fact that linear system (3.2) is a defining system for $S(G, b)$.

By setting $b_v = 1$ for each $v \in VG$, this theorem, together with the characterisation of simple b-critical simple b-nonseparable graphs given in Theorem 3.5, implies the characterisation, of Pulleyblack and Edmonds [28], of the minimal defining system for the convex hull of the matchings of G given in Theorem 2.2.

REMARK 3.13. In virtue of the characterisation of simple b-critical simple b-nonseparable graphs given in Theorem 3.5, it is possible to test in polynomial time for a given graph G and inequality $\alpha x \leq \beta$ whether or not $\alpha x \leq \beta$ is in the system (3.21) (see Remark 3.6).

Linear system (3.21) is not, in general, totally dual integral. (Consider the example of a simple b-bicritical graph given earlier, the complete graph on four nodes with $b_v = 2$ for each node v, with each edge receiving weight 1.) However, the following result shows that if G is connected then either (3.21) is totally dual integral or, if not, the addition of a single valid inequality to the system makes it totally dual integral.

THEOREM 3.14. *The Schrijver system for the convex hull of the simple b-matchings of G is* (3.21) *together with*

$$x(EH) \leq b^H(VH)/2 \text{ for each connected component } H \text{ of } G \text{ such that } H \text{ is a simple } b\text{-bicritical simple } b\text{-nonseparable graph.} \tag{3.22}$$

PROOF. Since linear system (3.2) is a totally dual integral defining system for $S(G, b)$, Lemma 2.5 implies that the Schrijver system for $S(G, b)$ is $x_e \geq 0$ for each $e \in EG$ and $x(J) \leq r(J)$ for each $J \subseteq EG$, $J \neq \emptyset$ which is closed and nonseparable for the general independence system (EG, I), where I is the set of simple b-matchings of G and $r(J)$ is the cardinality of a largest simple b-matching contained in J.

A set $\{e\}$ for some $e \in EG$ is closed if and only if e does not meet a node v with $b_v = 1$ and $d_G(v) > 1$. So the inequality $x_e \leq 1$ is in the Schrijver system for $S(G, b)$ if and only if it is in (3.21)(ii).

A set $\delta(v)$ for some $v \in VG$ contains a simple b-matching of size b_v if and only if $b'(N(v)) \geq b_v$, where $b'_u = \min\{b_u, |\delta(u) \cap \delta(v)|\}$ for each $u \in N(v)$. So the inequality $x(\delta(v)) \leq b_v$ is in the Schrijver system for $S(G, b)$ if and only if $b'(N(v)) \geq b_v$ and $\delta(v)$ is closed and nonseparable. This implies that $x(\delta(v)) \leq b_v$ is in the Schrijver system for $S(G, b)$ if and only if it is in (3.21)(iii).

By the form of system (3.2), any inequality in the Schrijver system for $S(G, b)$ that is not of the form $x_e \leq 1$ or $x_e \geq 0$ for some $e \in EG$ nor of the form $x(\delta(v)) \leq b_v$ for some $v \in VG$ must be of the form

$$x(EH) \leq \lfloor b^H(VH)/2 \rfloor \tag{3.23}$$

for some connected subgraph H of G with $|VH| \geq 3$.

Let H be a connected subgraph of G with $|VH| \geq 3$. The inequality (3.23) is in the Schrijver system for $S(G, b)$ if and only if EH is closed and nonseparable and H has a simple b-matching of cardinality $\lfloor b^H(VH)/2 \rfloor$. So, by Lemma 3.3 and Lemma 3.4, if (3.23) is not of the form $x(\delta(v)) \leq b_v$ for some $v \in VG$ then it is in the Schrijver system for $S(G, b)$ if and only if EH is closed and either H is simple b-critical simple b-nonseparable or H is simple b-bicritical simple b-nonseparable.

If H is a simple b-critical simple b-nonseparable graph then EH is closed if and only if there does not exist an edge $e = (u, v) \in EG - EH$ with $u, v \in VH$ and $d_H(u) \geq b_u$ and $d_H(v) \geq b_v$. It follows that if $b^H(VH)$ is odd and (3.23) is not of the form $x(\delta(v)) \leq b_v$ for some $v \in VG$ then (3.23) is in the Schrijver system for $S(G, b)$ if and only if it is in (3.21)(iv).

Suppose that h is simple b-bicritical and simple b-nonseparable. By Lemma 3.3, there exists a perfect simple b^H-matching of H. It follows that $d_H(v) > b_v$ for each $v \in VH$ (since if $d_H(v) \leq b_v$ then $(\{e\}, EH\text{-}\{e\})$ is a simple b-separation of H for each $e \in \delta_H(v)$). If H is not a connected component of G then there exists an edge $e \in EG\text{-}EH$ which meets a node in VH. If e is such an edge then $r(EH \cup \{e\}) = r(EH)$. So EH is closed if and only if H is a connected component of G. Thus, if H is a connected subgraph of G with $|VH| \geq 3$ such that $b^H(VH)$ is even and (3.23) is not of the form $x(\delta(v)) \leq b_v$ for some $v \in VG$, then (3.23) is in the Schrijver system of $S(G, b)$ if and only if it is in (3.22). ∎

Notice that Theorem 3.12 is not used in the above proof.

This theorem can be used to obtain the characterisation of the Schrijver system for the convex hull of the matchings of G, given in Theorem 2.3, due to Cunningham and Marsh [13], by setting $b_v = 1$ for each $v \in VG$. (Any simple b-bicritical graph must have $b_v \geq 2$ for each node v, so if $b_v = 1$ for each node v then G has no simple b-bicritical subgraph.)

We complete our discussion of simple b-matchings by characterising simple b-bicritical simple b-nonseparable graphs. For each edge $e = (u, v) \in EG$, let $G_{e,v}$ denote the graph obtained from G by replacing v by the nodes v', v'' with $b_{v'} = 1$ and $b_{v''} = b_v$, replacing $e = (u, v)$ by the edge (u, v'), and replacing each edge $f = (t, v)$ in $\delta_G(v)\text{-}\{e\}$ by (t, v''). (The edges in $G_{e,v}$ will be identified with their corresponding edges in G.)

THEOREM 3.15. *A simple b-bicritical graph G is simple b-nonseparable if and only if for each $e = (u, v) \in EG$ the graphs $G_{e,u}$ and $G_{e,v}$ are simple b-critical simple b-nonseparable graphs.*

PROOF. Suppose that G is a simple b-bicritical simple b-nonseparable graph. Let $e = (u, v)$ be an edge of G. If $b^{G_{e,v}}(VG_{e,v}) = b^G(VG)$, then $d_G(v) \leq b_v$ and thus $(\{f\}, EG\text{-}\{f\})$ is a simple b-separation of G for each $f \in \delta_G(v)$ (since G has a perfect simple b^G-matching). So $b^{G_{e,v}}(VG_{e,v}) = b^G(VG) + 1$, which implies that $b^{G_{e,v}}(VG_{e,v})$ is odd. This implies that the cardinality of a largest simple b-matching of $G_{e,v}$ is equal to the cardinality of a largest simple b-matching of G. Therefore, any simple b-separation of $G_{e,v}$ would be a simple b-separation of G. Thus, $G_{e,v}$ is simple b-nonseparable and, by Lemma 3.4, also simple b-critical.

Now suppose that G is simple b-bicritical and that for each $e = (u, v) \in EG$ the graphs $G_{e,u}$ and $G_{e,v}$ are simple b-critical and simple b-nonseparable. Let $E_1, \dots, E_k$ be subsets of EG which satisfy the conditions given in (3.14), where $G' = G$. Since $G_{e,u}$ is simple b-critical, no set E_i, $i \in \{1, \dots, k\}$, can be of the form $\{e\}$ for some $e = (u, v) \in EG$. Also, since G is simple b-bicritical, no set E_i, $i \in \{1, \dots, k\}$, can be of the form $\delta(v)$ for some $v \in VG$. So, by Lemma 3.4, for $i = 1, \dots, k$, the graph G_i (as defined in (3.14)) is simple b-critical or simple b-bicritical. Thus,

$$\lfloor b^{G_1}(VG_1)/2 \rfloor + \cdots + \lfloor b^{G_k}(VG_k)/2 \rfloor \leq \lfloor b^{G}(VG)/2 \rfloor.$$

If for some $i \in \{1, \dots, k\}$ and $v \in VG_i$ we have $d_{G_i}(v) = b_v^{G_i}$, then $(E_i, EG\text{-}E_i)$ is a simple b-separation of $G_{e,v}$ for any $e \in \delta_{G_i}(v)$. So for each $i \in \{1, \dots, k\}$, we have $b^{G_i}(VG_i) = b(VG_i)$. If $k \geq 2$ then for each $i \in \{1, \dots, k\}$ there exists a $j \in \{1, \dots, k\}$, $j \neq i$, such that $VG_i \cap VG_j \neq \emptyset$. Since $b_v \geq 2$ for each $v \in VG$, if $k \geq 3$ then this implies that $b(VG_1) + \cdots + b(VG_k) \geq b(VG) + k + 1$ and hence that (3.24) is not satisfied. So k is at most 2. Suppose that $k = 2$. By (3.24), we must have that $b(VG_1 \cap VG_2) = 2$. Thus $VG_1 \cap VG_2 = \{v\}$ for some $v \in VG$ and $b_v = 2$. Now since G is simple b-bicritical, G has a simple b-matching $\bar{x}$ such that $\bar{x}(\delta(v)) = 0$ and $\bar{x}(\delta(u)) = b_u$ for each $u \in VG\text{-}\{v\}$. So $b(VG_1)$ and $b(VG_2)$ are even numbers, which implies that (3.24) is not satisfied. So $k = 1$, which implies that G is simple b-nonseparable. ■

REMARK 3.16. Using this characterisation of simple b-bicritical simple b-nonseparable graphs and the characterisation of simple b-critical simple b-nonseparable graphs given in Theorem 3.5, it is possible to test in polynomial time whether or not a given graph G is a simple b-bicritical simple b-nonseparable graph (see Remark 3.6). Thus, it is possible to test in polynomial time for a given graph G and inequality $\alpha x \leq \beta$ whether or not $\alpha x \leq \beta$ is in (3.21) or (3.22). Also, Lemma 2.5, Theorem 3.5, Theorem 3.14, and Theorem 3.15 together imply that it is possible to test in polynomial time whether or not a given graph G is simple b-nonseparable.

4. Capacitated b-matchings. Capacitated b-matchings are a generalisation of simple b-matchings and b-matchings. Let G be a graph, possibly with multiple edges, $b = (b_v: v \in VG)$ a positive integer vector, and $c = (c_e: e \in EG)$ a positive integer vector of edge capacities. A *c-capacitated b-matching* of G is a b-matching x such that $x_e \leq c_e$ for each $e \in EG$. We abbreviate "c-capacitated b-matching" by "(b, c)-matching". If $c_e = 1$ for each $e \in EG$ then a (b, c)-matching of G is a simple b-matching of G. If $\beta = \max\{b_v: v \in VG\}$ and $c_e = \beta$ for each $e \in EG$ then x is a (b, c)-matching of G if and only if x is a b-matching of G.

A (b, c)-matching problem can be reduced to a simple b-matching problem by replacing each edge $e \in EG$ with c_e edges, each of which has the same end nodes as e. However, from an algorithmic point of view it is better to use Tutte's construction for reducing simple b-matching problems to b-matching problems to reduce (b, c)-matching problems directly to b-matching problems as follows: Suppose that $w = (w_e: e \in EG)$ is a vector of edge weights. For each edge $e = (u, v)$ of G add nodes u_e and v_e to VG and replace e by the edges (u, u_e), (u_e, v_e), (v_e, v). For each $e \in EG$ let $b_{u_e} = b_{v_e} = c_e$ and $w_{(u,u_e)} = w_{(u_e,v_e)} = w_{(v_e,v)} = w_e$. The maximum weight of a b-matching in the new graph is exactly $\Sigma\{w_e c_e: e \in EG\}$ greater than the maximum weight of a (b, c)-matching of G. Again, as presented in Araoz, Cunningham, Edmonds, and Green-Krótki [2] and Schrijver [31], Tutte's construction and the total dual integrality of (2.4) together imply the following result, which follows from a theorem of Edmonds and Johnson [17].

THEOREM 4.1. *A totally dual integral defining system for the convex hull of the (b, c)-matchings of G is*

$$0 \leqq x_e \leqq c_e \quad \forall e \in EG, \tag{4.1}$$
$$x(\delta(v)) \leqq b_v \quad \forall v \in VG,$$
$$x(\gamma(S)) + x(J) \leqq \lfloor (b(S) + c(J))/2 \rfloor \quad \forall S \subseteq VG, J \subseteq \delta(S).$$

If H is a subgraph of G, let $b_v^{(H,c)} = \min\{b_v, c(\delta_H(v))\}$ for each $v \in VH$. Again, the size of a largest (b, c)-matching of G (that is, the maximum value of $x(EG)$ over all (b,c)-matchings of G) is at most $\lfloor b^{(G,c)}(VG)/2 \rfloor$. So, since for each $S \subseteq VG$, $J \subseteq \delta(S)$ the set of edges $\gamma(S) \cup J$ is the edge set of a subgraph of G, Theorem 4.1 implies that

$$0 \leqq x_e \leqq c_e \quad \forall e \in EG, \tag{4.2}$$
$$x(\delta_G(v)) \leqq b_v \quad \forall v \in VG,$$
$$x(EH) \leqq \lfloor b^{(H,c)}(VH)/2 \rfloor \quad \text{for each connected subgraph } H \text{ of } G \text{ with } |VH| \geqq 3,$$

is a totally dual integral defining system for $P(G, b, c)$, the convex hull of the (b, c)-matchings of G. We will use this result and the results on simple b-matchings given in the previous section to describe the minimal defining system and the Schrijver system for $P(G, b, c)$.

We generalise the notions of simple b-matching criticality, bicriticality, and separability as follows. If G is connected then G is (b, c)-*critical* if $|VG| \geqq 3$ and for each $v \in VG$ there exists a (b, c)-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v^{(G,c)} - 1$ and $\bar{x}(\delta(u)) = b_u^{(G,c)}$ for each $u \in VG$-$\{v\}$. If G is connected and $|VG| \geqq 3$ then G is (b, c)-*bicritical* if for each $v \in VG$ there exists a (b, c)-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v^{(G,c)} - 2$ and $\bar{x}(\delta(u)) = b_u^{(G,c)}$ for each $u \in VG$-$\{v\}$. A (b, c)-*separation* of G is a pair (E_1, E_2) such that $E_1, E_2 \subseteq EG$, $E_1 \cup E_2 = EG$, $E_1 \neq \varnothing \neq E_2$, and if, for $i = 1,2$, k_i is the size of largest (b, c)-matching of G such that $x_e = 0$ for all $e \in EG$-E_i, then $k_1 + k_2$ is the size of a largest (b, c)-matching of G. A (b, c)-matching x of G is perfect if $x(\delta(v)) = b_v$ for each $v \in VG$.

Throughout the remainder of this section, let G' denote the graph obtained from G by replacing each edge $e \in EG$ with edges $e_1, \ldots, e_{c_e}$, each of which has the same end nodes as e. Each subgraph H of G corresponds to a subgraph H' of G' (that is $VH' = VH$ and $EH' = \cup\{\{e_1, \ldots, e_{c_e}\}: e \in EH\}$).

LEMMA 4.2. *If G is (b, c)-bicritical then G has a perfect $(b^{(G,c)}, c)$-matching.*

PROOF. Let G be a (b,c)-bicritical graph. By Lemma 3.3, the graph G' has a perfect simple $b^{G'}$-matching x', since G' is simple b-bicritical. Letting $\bar{x}_e = \Sigma\{x'_{e_i}: i = 1, \ldots, c_e\}$ for each $e \in EG$ we have a perfect $(b^{(G,c)}, c)$-matching of G. ∎

LEMMA 4.3. *A connected graph G with $|VG| \geqq 3$ is (b, c)-nonseparable only if G is isomorphic to $K_{1,n}$ for some n (with multiple edges allowed) or G is (b, c)-critical or G is (b, c)-bicritical.*

PROOF. Let G be a graph with $|VG| \geqq 3$ that is not isomorphic to $K_{1,n}$ for some n. Suppose that G is neither (b, c)-critical nor (b, c)-bicritical. It follows from this assumption that G' is not isomorphic to $K_{1,n}$ for some n nor simple b-critical nor simple b-bicritical. In the proof of Lemma 3.4, a simple b-separation (E_1', E_2') for such a graph is constructed. This simple b-separation has the property that, for $i = 1,2$, if

$e_j \in E_i'$ for some $e \in EG$ and $j \in \{1, \dots, c_e\}$, then $e_k \in E_i'$ for each $k \in \{1, \dots, c_e\}$. So (E_1, E_2) is a (b, c)-separation of G, where, for $i = 1, 2$, $E_i = \cup\{\{e_1, \dots, e_{c_e}\}: e \in E_i'\}$. ■

Using these lemmas we can describe the Schrijver system for $P(G, b, c)$. Let $V^2 \subseteq VG$ be the set of nodes v for which one of the following conditions holds, where $b_u' = \min\{b_u, \Sigma\{c_e: e \in \delta(u) \cap \delta(v)\}\}$ for each node $u \in N(v)$.

$b'(N(v)) = b_v$ and v is in a two-node connected component of G and if $\Sigma\{c_e: e \in \delta(v)\} = b_v$ then $d_G(v) = 1$. (4.3)

$b'(N(v)) = b_v + 1$ and there does not exist an edge $e = (v_1, v_2) \in \gamma(N(v))$ with $b_{v_1}' = b_{v_1}$ and $b_{v_2}' = b_{v_2}$. (4.4)

$$b'(N(v)) \geqq b_v + 2. \tag{4.5}$$

A subgraph H of G is *edge maximal* if there does not exist an edge $(u, v) \in EG\text{-}EH$ with $u, v \in VH$ and with $b_u^{(H,c)} = b_u$ and $b_v^{(H,c)} = b_v$.

THEOREM 4.4. *The Schrijver system for the convex hull of the (b, c)-matchings of G is*
(i) $x_e \geqq 0\ \forall e \in EG$, (4.6)
(ii) $x_e \leqq c_e$ *for each* $e \in EG$ *such that* e *does not meet a node* $v \in VG$ *with* $b_v < c_e$ *nor one with* $b_v = c_e$ *and* $d_G(v) \geqq 2$,
(iii) $x(\delta_G(v)) \leqq b_v\ \forall v \in V^2$,
(iv) $x(EH) \leqq \lfloor b^{(H,c)}(VH)/2 \rfloor$ *for each edge maximal, (b, c)-critical, (b, c)-nonseparable subgraph H of G.*
(v) $x(EH) \leqq b^{(H,c)}(VH)/2$ *for each edge maximal, (b, c)-bicritical, (b, c)-nonseparable subgraph H of G such that there does not exist an edge $(u, v) \in EG\text{-}EH$ with $u \in VH$, $v \notin VH$ and either $c_{(u,v)} = 1$ or $b_v = 1$.*

PROOF. Let (EG, I) be the general independence system with I the set of (b, c)-matchings of G. It follows from Lemma 2.5 and the fact that (4.2) is a totally dual integral defining system for $P(G, b, c)$ that the Schrijver system for $P(G, b, c)$ is $x \geqq 0$ and $x(J) \leqq r(J)$ for all $J \subseteq EG$, $J \neq \varnothing$, J closed and nonseparable for (EG, I), where $r(J)$ is the size of a largest (b, c)-matching of G such that $x_e = 0$ for each $e \in EG\text{-}J$. Using Lemma 4.2 and Lemma 4.3, it is straightforward to check, as in the proof of Theorem 3.14, that this system is identical to (4.6). (Note that if H is an edge maximal, (b, c)-bicritical subgraph of G, then EH is closed if and only if there does not exist an edge satisfying the conditions given in (4.6)(v).) ■

If $c_e = 1$ for each $e \in EG$, then $P(G, b, c)$ is equal to the convex hull of the simple b-matchings of G and linear system (4.6) is identical to the linear system consisting of (3.21) and (3.22). So Theorem 4.4 implies Theorem 3.14. If $\beta = 2 \cdot \max\{b_v: v \in VG\}$ and $c_e = \beta$ for each $e \in EG$, then $P(G, b, c)$ is equal to the convex hull of the b-matchings of G and it is straightforward to check that linear system (4.6) is identical to the linear system consisting of (2.5) and (2.6). So Theorem 4.4 can also be used to prove Theorem 2.7.

It is clear that not every inequality in (4.6) is essential for $P(G, b, c)$, since if H is a (b, c)-bicritical subgraph of G then $x(EH) \leqq b^{(H,c)}/2$ is implied by the valid inequalities $x(\delta_H(v)) \leqq b_v^{(H,c)}$ for each $v \in VH$. It will be shown that removing the inequalities (4.6)(v) from (4.6) gives the minimal defining system for $P(G, b, c)$. We begin by characterising (b, c)-critical (b, c)-nonseparable graphs.

LEMMA 4.5. *A (b, c)-critical graph G is (b, c)-nonseparable if and only if G' is a simple b-critical simple b-nonseparable graph.*

PROOF. Let G be a (b, c)-critical graph. Clearly, G' is simple b-critical. If (E_1, E_2) is a (b, c)-separation of G, then (E_1', E_2') is a simple b-separation of G', where $E_i' = \cup\{\{e_1, \ldots, e_{c_e}\}: e \in E_i\}$ for $i = 1, 2$. So if G' is simple b-nonseparable then G is (b, c)-nonseparable.

Suppose that G' is simple b-separable. Let $E_1, \ldots, E_k$ be subsets of EG' which satisfy the conditions given in (3.14). If for each $i \in \{1, \ldots, k\}$ there exists an edge $e \in EG'$ such that $E_i = \{e\}$, then $(F, EG\text{-}F)$ is a (b, c)-separation of G where $F = \{f_1, \ldots, f_{c_f}\}$ for any $f \in EG$. So it can be assumed that $|E_1| \geq 2$. It may also be assumed that E_t is closed with respect to the general independence system (EG', I) for each $t \in \{1, \ldots, k\}$, where I is the set of simple b-matchings of G'. If E_1 is equal to $\delta_{G'}(v)$ for some $v \in VG'$, then $(\delta_G(v), EG\text{-}\delta_G(v))$ is a (b, c)-separation of G. So it may be assumed that E_1 is not of the form $\delta_{G'}(v)$ for some $v \in VG'$. So G_1 is either simple b-critical or simple b-bicritical and $r(E_1) = \lfloor b^{G_1}(VG_1)/2 \rfloor$. Suppose that G_1 is simple b-bicritical. Since G_1 is simple b-nonseparable, $d_{G_1}(v) > b_v$ for each $v \in VG_1$. So if $e_t \in E_1$ for some $e \in EG$ and $t \in \{1, \ldots, c_e\}$, then $e_s \in E_1$ for each $s \in \{1, \ldots, c_e\}$. This implies that $(\gamma_G(VG_1), EG\text{-}\gamma_G(VG_1))$ is a (b, c)-separation of G. So it may be assumed that G_1 is simple b-critical and that for some $e \in EG$ and $i, j \in \{1, \ldots, c_e\}$ we have $e_i \in E_1$, $e_j \notin E_1$, and $e_i \notin E_t$ for each $t \neq 1$. Let $u, v \in VG_1$ be the ends of e_i. Since E_1 is closed and $e_j \notin E_1$, we must have $d_{G_1}(v) < b_v$ or $d_{G_1}(u) < b_u$. So we may assume that $d_{G_1}(v) < b_v$. Since G_1 is simple b-nonseparable, we must have $b_u^{G_1} = b_u$. Let q be a node in $VG_1\text{-}\{u, v\}$ such that $b_q^{G_1} = b_q$ (such a node must exist since a simple b-critical graph cannot be isomorphic to $K_{1,n}$ for some n). Let M be a near-perfect simple b-matching of G' deficient at q. Since $(E_1, EG'\text{-}E_1)$ is a simple b-separation of G', we must have that $M \cap E_1$ is a near-perfect simple b-matching of G_1. So $|\delta_{G_1}(u) \cap M| = b_u$, which implies that $e_j \notin M$. Thus, $(\{e_j\}, EG'\text{-}\{e_j\})$ is not a simple b-separation of G' and there does not exist a t such that $E_t = \{e_j\}$. So we may assume that $e_j \in E_2$ and that $|E_2| \geq 2$. By the above arguments we may also assume that G_2 is simple b-critical with either $d_{G_2}(v) < b_v$ or $d_{G_2}(u) < b_u$ (since $e_i \notin E_2$). If $d_{G_2}(v) < b_v$ then $b_u^{G_2} = b_u$, which implies that $|M \cap \delta_{G_2}(u)| < b_u^{G_2}$ and $|M \cap \delta_{G_2}(v)| < b_v^{G_2}$ (since $b_v^{G_2} = d_{G_2}(v)$ and $e_j \notin M$). So if $d_{G_2}(v) < b_v$ then $(E_2, EG'\text{-}E_2)$ is not a simple b-separation of G', which contradicts the choice of $E_1, \ldots, E_k$. So it may be assumed that $d_{G_2}(u) < b_u$. This implies that $b_v^{G_2} = b_v$. Now since $M \cap E_1$ is a near-perfect simple b-matching of G_1 deficient at q, we have $e_i \in M$. So, once again, we have $|M \cap \delta_{G_2}(v)| < b_v^{G_2}$ and $|M \cap \delta_{G_2}(u)| < b_u^{G_2}$, a contradiction. So we must have that G is (b, c)-separable. ∎

This lemma and Theorem 3.5 together imply the following result, where (b, c)-full edges and near-perfect (b, c)-matchings of (b, c)-critical graphs are defined in a way analogous to the simple b-matching case and $G^{(T, c)}(VG)$ for $T \subseteq VG$ is the graph obtained from G by replacing each node $v \in T$ by the nodes $v_1, \ldots, v_k$, where $k = d_G(v)$, and replacing the edges $e_1 = (u_1, v), \ldots, e_k = (u_k, v)$ by the edges $(u_1, v_1), \ldots, (u_k, v_k)$, and letting $b_{v_i} = c_{e_i}$ for $i = 1, \ldots, k$.

THEOREM 4.6. *A (b, c)-critical graph G is (b, c)-nonseparable if and only if $b_v^{(G, c)} = d_G(v)$ for each node $v \in VG$ which meets an edge $(u, v) \in EG$ that is (b, c)-full with respect to u and the graph $G^{(T, c)}(VG)$, where $T = \{v \in VG: b_v^{(G, c)} = d_G(v)\}$, is connected with no cutnode q having $b_q = 1$.*

REMARK 4.7. Using the construction of Tutte to reduce a (b, c)-matching problem directly to a b-matching problem, (b, c)-matching problems can be solved in polynomial time (see Remark 3.6). So the above theorem provides a method to test in polynomial time whether or not a graph G is a (b, c)-critical (b, c)-nonseparable graph.

The following lemma allows us to take advantage of the fact that we know, via Theorem 3.12, the essential inequalities for the convex hull of the simple b-matchings of G'.

LEMMA 4.8. *Let H be a subgraph of G. If $x(EH') \leqq \alpha$ is an essential inequality for $S(G', b)$ then $x(EH) \leqq \alpha$ is an essential inequality for $P(G, b, c)$.*

PROOF. Suppose that $x(EH') \leqq \alpha$ is essential, and hence facet inducing, for $S(G', b)$. Let $\mathcal{M}'$ be a collection of $|EG'|$ affinely independent simple b-matchings of G' for which $x(EH') \leqq \alpha$ holds as an equality. For each simple b-matching x' in $\mathcal{M}'$ let $\bar{x}$ be the (b, c)-matching of G obtained by setting $\bar{x}_e = \Sigma\{x'_{e_i}: i = 1, \ldots, c_e\}$ for each $e \in EG$. Consider the set of (b, c)-matchings $\mathcal{M} = \{\bar{x}: x' \in \mathcal{M}'\}$. Each (b, c)-matching in $\mathcal{M}$ satisfies $x(EH) \leqq \alpha$ with equality. Furthermore, since $\mathcal{M}'$ is an affinely independent set of vectors, $\mathcal{M}$ contains $|EG|$ affinely independent vectors. So $x(EH) \leqq \alpha$ is facet inducing, and hence essential, for $P(G, b, c)$. ■

THEOREM 4.9. *The unique (up to positive scalar multiples of the inequalities) minimal defining system for the convex hull of the (b, c)-matchings of G is* (4.6)(i), (ii), (iii), (iv).

PROOF. Since linear system (4.6) defines $P(G, b, c)$ and since we have already observed that the inequalities (4.6)(v) are not essential for $P(G, b, c)$, it suffices to prove that each inequality in (4.6)(i), (ii), (iii), (iv) is essential for $P(G, b, c)$.

Let $e \in EG$ and let $x_e = -1$ and $x_f = 0$ for each $f \in EG\text{-}\{e\}$. The vector x does not satisfy $x_e \geqq 0$ but it does satisfy each of the other inequalities in (4.6). So each inequality in (4.6)(i) is essential for $P(G, b, c)$. Suppose that e does not meet a node $v \in VG$ with $b_v < c_e$ or one with $b_v = c_e$ and $d_G(v) \geqq 2$. Let $\bar{x}_e = c_e + 1$ and $\bar{x}_f = 0$ for each $f \in EG\text{-}\{e\}$. The vector $\bar{x}$ does not satisfy $x_e \leqq c_e$, but it does satisfy each of the other inequalities in (4.6). So each inequality in (4.6)(ii) is essential for $P(G, b, c)$.

Let $v \in VG$ and suppose that either (4.3), (4.4), or (4.5) holds for v (that is, $v \in V^2$). If either (4.4) or (4.5) holds for v, then Lemma 3.11 and Lemma 4.8 together imply that $x(\delta_G(v)) \leqq b_v$ is essential for $P(G, b, c)$. Suppose that (4.3) holds for v. If $\Sigma\{c_e: e \in \delta_G(v)\} > b_v$, let $x_e = c_e$ for each $e \in \delta_G(v)$ and let $x_e = 0$ for each $e \in EG\text{-}\delta_G(v)$. If $\Sigma\{c_e: e \in \delta_G(v)\} = b_v$, let $x_e = c_e + 1$ for the edge e which meets v and $x_f = 0$ for each $f \in EG\text{-}\{e\}$. In either case, x satisfies each inequality in (4.6) other than $x(\delta_G(v)) \leqq b_v$. So each inequality in (4.6)(iii) is essential for $P(G, b, c)$.

Let H be a (b, c)-critical (b, c)-nonseparable subgraph of G such that there does not exist an edge $(u, v) \in EG\text{-}EH$ with $u, v \in VH$ and with $b_u^{(H,c)} = b_u$ and $b_v^{(H,c)} = b_v$ (that is, H is edge maximal). By Lemma 4.5, H' is a simple b-critical simple b-nonseparable subgraph of G'. So, Lemma 3.7 and Lemma 4.8 together imply that $x(EH) \leqq \lfloor b^{(H,c)}(VH)/2 \rfloor$ is essential for $P(G, b, c)$. So each inequality in (4.6)(iv) is essential for $P(G, b, c)$. ■

Theorem 3.12 can be obtained from the above result by setting $c_e = 1$ for each $e \in EG$. Letting $c_e = \beta$ for each $e \in EG$ where $\beta = 2 \cdot \max\{b_v: v \in VG\}$ and using the characterisation of (b, c)-critical (b, c)-nonseparable graphs given in Theorem 4.6, one obtains Theorem 2.6 (which is due to Pulleyblank [24]).

REMARK 4.10. Using the characterisation of (b, c)-critical (b, c)-nonseparable graphs given in Theorem 4.6 and a polynomial time (b, c)-matching algorithm (see Remark 3.6 and Remark 4.7), it is possible to test in polynomial time for a given graph G and inequality $\alpha x \leqq \beta$ whether or not $\alpha x \leqq \beta$ is in (4.6)(i), (ii), (iii), (iv). Note however that, unlike the simple b-matching case, we have not found an algorithm to check in polynomial time for a given graph G and inequality $\alpha x \leqq \beta$ whether or not $\alpha x \leqq \beta$ is in the Schrijver system for $P(G, b, c)$ (that is, whether or not it is in system (4.6)). The problem is in testing whether or not a (b, c)-bicritical graph is (b, c)-nonseparable. (The analogue of Lemma 4.5 for (b, c)-bicritical graphs is not true, as can be seen by considering the graph with nodes v_1, v_2, v_3, v_4 and edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_1)$, $e_4 = (v_1, v_4)$ with $b_{v_1} = b_{v_2} = b_{v_3} = 4$, $b_{v_4} = 2$, $c_{e_1} = c_{e_2} = c_{e_3} = 4$ and $c_{e_4} = 2$.)

5. Triangle-free 2-matchings. A *2-matching* of a graph G is a b-matching with $b_v = 2$ for each $v \in VG$. Motivated by the fact that the 2-matching problem is a relaxation of the travelling salesman problem, Cornuéjols and Pulleyblank [10] considered a constrained variation of 2-matchings which they named triangle-free 2-matchings. A 2-matching x is a *triangle-free 2-matching* if $x(T) \leqq 2$ for each triple of edges $T = \{e_1, e_2, e_3\} \subseteq EG$ which form the edges of a triangle of G. (A triangle is a circuit of length 3.) Cornuéjols and Pulleyblank found a polynomial time algorithm for solving the triangle-free 2-matching problem. A consequence of their algorithm is the following result:

THEOREM 5.1. *A totally dual integral defining system for the convex hull of the triangle-free 2-matchings of G is*

$$\begin{aligned} x_e &\geqq 0 \quad \forall e \in EG, \\ x(\delta(v)) &\leqq 2 \quad \forall v \in VG, \\ x(ET) &\leqq 2 \quad \forall \text{ triangle } T \text{ of } G, \\ x(\gamma(S)) &\leqq |S| \quad \forall S \subseteq VG, |S| \geqq 4. \end{aligned} \tag{5.1}$$

The following result of Cornuéjols and Pulleyblank [10] follows easily from this theorem.

THEOREM 5.2. *The unique (up to positive scalar multiples of the inequalities) minimal defining system for the convex hull of the triangle-free 2-matchings of G is*

$$\begin{aligned} x_e &\geqq 0 \quad \forall e \in EG, \\ x(\delta(v)) &\leqq 2 \quad \textit{for each } v \in VG \textit{ such that either } d_G(v) \geqq 3 \textit{ or } d_G(v) = 2 \\ &\qquad \textit{and } v \textit{ is not a node of a triangle or } d_G(v) = 1 \textit{ and } v \textit{ is} \\ &\qquad \textit{in a two-node connected component of } G, \\ x(ET) &\leqq 2 \quad \forall \textit{ triangle } T \textit{ of } G. \end{aligned} \tag{5.2}$$

If G is a circuit of length 5, then (5.2) is not totally dual integral. Such a circuit is an example of a triangle-free-bicritical graph. If x is a triangle-free 2-matching such that $x(\delta(v)) = 2$ for each $v \in VG$, then x is a perfect triangle-free 2-matching. If G is connected and $|VG| \geqq 4$, then G is *triangle-free-bicritical* if for each $v \in VG$ the graph obtained by deleting v from G has a perfect triangle-free 2-matching. A triangle T of a connected graph G is a *pendent triangle* of G if T contains a cutnode of G and T contains two nodes v_1, v_2 with $d_G(v_1) = d_G(v_2) = 2$. Using Lemma 2.5 and Theorem 5.1, the following result can be proven (for details see Cook [7]).

THEOREM 5.3. *The Schrijver system for the convex hull of the triangle-free 2-matchings of G is* (5.2) *together with*

$$x(\gamma(S)) \leqq |S| \quad \textit{for each } S \subseteq VG \textit{ such that } G[S] \textit{ is triangle-free-bicritical and contains no triangle } T \textit{ which is a pendent triangle of } G[S]. \tag{5.3}$$

REMARK 5.4. Using the triangle-free 2-matching algorithm of Cornuéjols and Pulleyblank [10], it is possible to test in polynomial time for a given graph G and inequality $\alpha x \leq \beta$ whether or not $\alpha x \leq \beta$ is in (5.2), (5.3).

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