Mathematical Programming 34 (1986) 48-61
North-Holland

ON BOX TOTALLY DUAL INTEGRAL POLYHEDRA

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Received 10 December 1984
Revised manuscript received 18 April 1985

Edmonds and Giles introduced the class of box totally dual integral polyhedra as a generalization of submodular flow polyhedra. In this paper a geometric characterization of these polyhedra is given. This geometric result is used to show that each TDI defining system for a box TDI polyhedron is in fact a box TDI system, that the class of box TDI polyhedra is in co-NP and is closed under taking projections and dominants, that the class of box perfect graphs is in co-NP, and a result of Edmonds and Giles which is related to the facets of box TDI polyhdera.

Key words: Total Dual Integrality, Hilbert Basis, Polyhedra.

1. Introduction

In recent years, Edmonds and Giles [11], Hoffman and Schwartz [28], Frank [14, 15], Frank and Tardos [16], Gröflin and Hoffman [22], and others have introduced classes of polyhedra which generalize in different ways the class of polymatroids, as defined by Edmonds [10]. Each of these classes is defined by a class of combinatorially described linear systems $Ax \leq b$ which have the property that both sides of the linear programming duality equation

$$\min\{yb\colon yA = w, y \geq 0\} = \max\{wx\colon Ax \leq b\} \tag{1.1}$$

can be achieved by integral vectors, whenever w is integral and the optima exist. The combinatorial min-max theorems obtained from these polyhedra via (1.1) generalize many of the well known min-max theorems of graph theory and matroid theory. (An excellent survey of these min-max results is given in Schrijver [37].)

In each case, the fact that (1.1) can be achieved by integral vectors was established by showing that $Ax \leq b$ is totally dual integral and then applying a theorem of Hoffman [27] and Edmonds and Giles [11]. (A rational linear system $Ax \leq b$ is a *totally dual integral system* (TDI system) if for each integral vector w for which the optima in (1.1) exist, the minimum can be achieved by an integral vector. The theorem of Hoffman [27] and Edmonds and Giles [11] is that if $Ax \leq b$ is a TDI system and b is an integral vector, then the maximum in (1.1) can be achieved by an integral vector for each vector w for which the optima exist. For a general

Supported by a grant from the Alexander von Humboldt-Stiftung.

framework for proving min–max theorems via TDI systems see Schrijver [36].) In fact, in each case the class of linear systems $Ax \leqslant b$ was shown to have the stronger property that for every choice of rational vectors l and u the linear system $Ax \leqslant b$, $l \leqslant x \leqslant u$ is a TDI system, that is that the linear system $Ax \leqslant b$ is a *box totally dual integral system* (box TDI system). Giles and Pulleyblank [20] proved that every rational polyhedron can be defined by a TDI system, but it is easy to see that not every rational polyhedron can be defined by a box TDI system. Thus, the fact that each polyhedron in the above classes is a box TDI polyhedron (a nonempty polyhedron is a *box TDI polyhedron* if it can be defined by a box TDI linear system) is a nontrivial property of these classes; so the structure of box TDI polyhedra may provide insights into the above min–max results. Edmonds and Giles [11, 12, 13] made a study of box TDI polyhedra, obtaining some interesting results. In this paper a further study of these polyhedra is made. In Section 2, a geometric characterization of box TDI polyhedra is proven and is used to show that each TDI defining system for a box TDI polyhedron is in fact a box TDI system. In Section 3, this geometric characterization is used to prove a "duplication" result of Edmonds and Giles [13] which implies that every box TDI polyhedron can be defined by a linear system $Ax \leqslant b$ where A is 0, 1, -1 valued. It is also shown in that section that the class of box TDI polyhedra is closed under taking projections and dominants. In Section 4, an algorithmic result on the recognition of box TDI polyhedra is given, an application of which is a proof that the class of box perfect graphs (as defined by Cameron [5]) is in co-NP.

It should be noted that throughout the paper all linear systems, linear spaces, and polyhedra are assumed to be rational. For basic results in the theory of polyhedra the reader is referred to Bachem and Grötschel [1], Pulleyblank [33], Rockafellar [34], and the forthcoming book of Schrijver [38].

2. A geometric characterization

When one studies total dual integrality it is often easier to work with the closely related concept of Hilbert bases. A finite set of vectors $h_1, \ldots, h_k$ is a *Hilbert basis* if each integral vector in the convex cone $\{\lambda_1 h_1 + \cdots + \lambda_k h_k \colon \lambda_i \geqslant 0, i = 1, \ldots, k\}$ generated by $h_1, \ldots, h_k$ can be written as $\gamma_1 h_1 + \cdots + \gamma_k h_k$ for some nonnegative integral γ_i, $i = 1, \ldots, k$. (Note that the vectors $h_1, \ldots, h_k$ are not necessarily integral.) If $S \subseteq \{x \colon Ax \leqslant b\}$ then an inequality $a_i x \leqslant b_i$ in the system $Ax \leqslant b$ is an *active inequality* for S in $Ax \leqslant b$ if $a_i \bar{x} = b_i$ for each $\bar{x} \in S$. A row a_i of A is an *active row* for S in $Ax \leqslant b$ if $a_i x \leqslant b_i$ is an active inequality for S in $Ax \leqslant b$. The following proposition can be proven easily, using complementary slackness.

Proposition 2.1. *A linear system $Ax \leqslant b$ is a TDI system if and only if for each face F of $\{x \colon Ax \leqslant b\}$ the set of active rows for F in $Ax \leqslant b$ is a Hilbert basis.* □

This proposition was used (together with a theorem of Hilbert which implies that every polyhedral convex cone can be generated by a finite integral Hilbert basis) by Giles and Pulleyblank [20] to show that every polyhedron can be defined by a TDI system with integral left-hand sides and by Schrijver [35] to show that for each full dimensional polyhedron P, there exists a unique minimal TDI system $Ax \leq b$ such that A is integral and $P = \{x: Ax \leq b\}$.

To obtain an analogue of Proposition 2.1 for box TDI systems, we define a finite set of integral vectors $H \subseteq \mathbb{Q}^n$ to be a *box Hilbert basis* if for each $M \subseteq \{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$ the set $H \cup M$ is a Hilbert basis, where e_i denotes the ith unit vector in $\mathbb{Q}^n$ (that is, for $i = 1, \ldots, n$, e_i denotes the vector with a 1 in the ith component and a 0 in each of the other components). Not every Hilbert basis is a box Hilbert basis, for example if $h_1^T = (1, 2)$ and $h_2^T = (1, 3)$ then $\{h_1, h_2\}$ is a Hilbert basis since $[h_1 : h_2]$ is a unimodular matrix, but $\{h_1, h_2\}$ is not a box Hilbert basis since $\{h_1, h_2, -e_1\}$ is not a Hilbert basis.

Proposition 2.2. *A linear system $Ax \leq b$ is a box* TDI *system if and only if for each face F of $\{x: Ax \leq b\}$ the set of active rows for F in $Ax \leq b$ is a box Hilbert basis.*

Proof. Suppose that $Ax \leq b$ is a box TDI system. Let F be a face of $\{x: Ax \leq b\}$ and let $\bar{x} \in F$ be a vector such that an inequality $a_i x \leq b_i$ in $Ax \leq b$ is active for F if and only if $a_i \bar{x} = b_i$. (Such an $\bar{x}$ can be found as follows: For each inequality $a_i x \leq b_i$ in $Ax \leq b$ that is not active for F let $x^i \in F$ be a vector such that $a_i x^i < b_i$. Taking a convex combination of the vectors x^i with all multipliers positive gives such a vector.) Let H denote the set of active rows for F in $Ax \leq b$ and let $M \subseteq \{e_1, \ldots, e_n - e_1, \ldots, -e_n\}$. We can choose l and u such that the set of active rows of $\{\bar{x}\}$ in $Ax \leq b$, $-x \leq -l$, $x \leq u$ is $H \cup M$. This implies that there exists a face F' of $\{x: Ax \leq b, l \leq x \leq u\}$ which has $H \cup M$ as its set of active rows in $Ax \leq b$, $-x \leq -l$, $x \leq u$. So, by Proposition 2.1, $H \cup M$ is a Hilbert basis.

Conversely, suppose that for each face of $\{x: Ax \leq b\}$ the set of active rows of $Ax \leq b$ is a box Hilbert basis. Let l and u be vectors and let F' be a face of the polyhedron $\{x: Ax \leq b, l \leq x \leq u\}$. Now let $A'x \leq b'$ be the set of inequalities in $Ax \leq b$ that are active for F' and let $F = \{x: Ax \leq b, A'x = b'\}$. The set F is a face of $\{x: Ax \leq b\}$ whose active inequalities in $Ax \leq b$ are precisely the inequalities $A'x \leq b'$. So the active rows of F' in $Ax \leq b$, $-x \leq -l$, $x \leq u$ are the active rows of F in $Ax \leq b$ together with a subset of $\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$. So the set of active rows of F' in $Ax \leq b$, $-x \leq -l$, $x \leq u$ is a Hilbert basis, which implies, by Proposition 2.1, that $Ax \leq b$, $l \leq x \leq u$ is a TDI system. □

This proposition is useful, since those Hilbert bases that are box Hilbert bases can be characterized geometrically in the following way. For an n-component vector x let $\mathcal{I}(x) = \{i: 1 \leq i \leq n, x_i \text{ is integral}\}$. Let C be a convex cone and $x = (x_1, \ldots, x_n) \in C$. Let (U, L) be a partition of $\{1, \ldots, n\} \setminus \mathcal{I}(x)$, that is $U \cap L = \emptyset$ and $U \cup L = \{1, \ldots, n\} \setminus \mathcal{I}(x)$. The vector x has the *box property in C with respect to* (U, L) if there exists an integral vector $x' \in C$ such that $x'_i = x_i$ for all $i \in \mathcal{I}(x)$, $x'_i \leq \lceil x_i \rceil$ for all $i \in U$ and $x'_i \geq \lfloor x_i \rfloor$ for all $i \in L$ (where $\lceil x_i \rceil$ denotes the least integer greater than or equal to x_i and $\lfloor x_i \rfloor$ denotes the greatest integer lesser than or equal to x_i). The vector x has the *box property* in C if for each partition (U, L) of $\{1, \ldots, n\} \backslash \mathcal{I}(x)$, x has the box property in C with respect to (U, L). Finally, the cone C has the *box property* if each $x \in C$ has the box property.

Lemma 2.3. *Let $H = \{h_1, \ldots, h_p\}$ be a Hilbert basis and let C be the cone generated by H. Then H is a box Hilbert basis if and only if C has the box property.*

Proof. Supose that C has the box property and that H is not a box Hilbert basis. Choose $K \subseteq \{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$ and integral vector w such that w is in the cone generated by $H \cup K$, w cannot be expressed as a nonnegative integral combination of vectors in $H \cup K$ and of all such K and w, $|K|$ is minimum. Let $k_1, \ldots, k_q$ be the elements of K. Note that by the assumption that $|K|$ is minimum we have that for each $i \in \{1, \ldots, n\}$ at most one of e_i and $-e_i$ is in K.

Since w is contained in the cone generated by $H \cup K$, there exist $\lambda_i \geq 0$, $i = 1, \ldots, p+q$ such that $w = \lambda_1 h_1 + \cdots + \lambda_p h_p + \lambda_{p+1} k_1 + \cdots + \lambda_{p+q} k_q$. We may assume that $0 < \lambda_i < 1$ for each $i \in \{p+1, \ldots, p+q\}$ since $|K|$ is minimum and since we may replace w by $w - (\lfloor \lambda_i \rfloor k_{i-p})$. Let $w' = \lambda_1 h_1 + \cdots + \lambda_p h_p$. If neither e_i nor $-e_i$ is in K then w'_i is integral, that is $i \in \mathcal{I}(w')$. If $e_i \in K$ then $w'_i \leq w_i$ which implies that $w_i = \lceil w'_i \rceil$, since $0 < \lambda_j < 1$ for each $j \in \{p+1, \ldots, p+q\}$. Similarly, if $-e_i \in K$ then $w_i = \lfloor w'_i \rfloor$. Since $w' \in C$ and since C has the box property, there exists an integral $c \in C$ such that $c_i = w_i$ if $\{e_i, -e_i\} \cap K = \emptyset$, $c_i \leq w_i$ if $e_i \in K$, and $c_i \geq w_i$ if $-e_i \in K$. Since H is a Hilbert basis, c can be expressed as a nonnegative integral combination of vectors in H. But w can be expressed as the sum of c and a nonnegative integral combination of the vectors in K, a contradiction.

Conversely, suppose that H is a box Hilbert basis. Let $w \in C$ and let (U, L) be a partition of $\{1, \ldots, n\} \backslash \mathcal{I}(w)$. Let $K = \{e_i: i \in U\} \cup \{-e_i: i \in L\}$. The vector w' defined as $w'_i = w_i$ for $i \in \mathcal{I}(w)$, $w'_i = \lceil w_i \rceil$ for $i \in U$ and $w'_i = \lfloor w_i \rfloor$ for $i \in L$ is in the cone generated by $H \cup K$. Since $H \cup K$ is a Hilbert basis, there exist nonnegative integers λ_i, $i = 1, \ldots, p+q$ such that $\lambda_1 h_1 + \cdots + \lambda_p h_p + \lambda_{p+1} k_1 + \cdots + \lambda_{p+q} k_q = w'$ (where $k_1, \ldots, k_q$ are the elements of K). Let $w'' = \lambda_1 h_1 + \cdots + \lambda_p h_p$. Since $\lambda_{p+1} k_1 + \cdots + \lambda_{p+q} k_q$ is integral and since w' is integral, w'' is also integral. Furthermore, $w''_i \leq w'_i$ for $i \in U$ and $w''_i \geq w'_i$ for $i \in L$. So w has the box property with respect to (U, L). □

The following result is a geometric characterization of box TDI polyhedra.

Theorem 2.4. *A polyhedron P is a box TDI polyhedron if and only if for each face F of P the cone of all vectors w such that* $\max\{wx: x \in P\}$ *is achieved by each vector in F has the box property.*

Proof. Suppose that P is a box TDI polyhedron. Let $Ax \leq b$ be a box TDI system such that $P = \{x: Ax \leq b\}$ and let F be a face of P. It is well known (and follows easily from the complementary slackness theorem) that the cone generated by the set of active rows for F in $Ax \leq b$ is equal to the set of all vectors w such that $\max\{wx: x \in P\}$ is achieved by each vector in F. By Proposition 2.2, the set of active rows for F in $Ax \leq b$ is a box Hilbert basis and hence generates a cone with the box property, by Lemma 2.3.

Conversely, suppose that for each face of a polyhedron P the cone of all vectors w such that $\max\{wx: x \in P\}$ is achieved by each vector in F has the box property. By the result of Giles and Pulleyblank [20] mentioned earlier, there exists a TDI system $Ax \leq b$ such that $P = \{x: Ax \leq b\}$. Let F be a face of P. By Proposition 2.1, the set of active rows for F in $Ax \leq b$ is a Hilbert basis. Thus, by Lemma 2.3 and the above assumption, the set of active rows for F in $Ax \leq b$ is a box Hilbert basis. So, by Proposition 2.2, $Ax \leq b$ is a box TDI system and hence P is a box TDI polyhedron. □

It follows from Lemma 2.3 that if H is a box Hilbert basis then any other Hilbert basis for the cone generated by H is also a box Hilbert basis. Hence, Theorem 2.4 implies the following result.

Corollary 2.5. *If $Ax \leq b$ is a box* TDI *system then each* TDI *system $Mx \leq d$ such that $\{x: Mx \leq d\} = \{x: Ax \leq b\}$ is also a box TDI system.* □

3. Defining systems

If P is a box TDI polyhedron of full dimension, then Theorem 2.4 implies that each facet-inducing inequality, $\alpha x \leq \beta$, for P can be scaled so that it has 0, 1, -1 values on the left hand side. (Since $F = \{x: \alpha x = \beta\} \cap P$ is a facet of P, the cone of all vectors w such that $\max\{wx: x \in P\}$ is achieved by each vector in F is just the ray $\{\lambda\alpha: \lambda \in \mathbb{Q}, \lambda \geq 0\}$ and the only such rays that have the box property are those which contain a 0, 1, -1 valued point.) Thus, each box TDI polyhedron of full dimension can be defined by a linear system which has 0, 1, -1 valued left hand sides. Edmonds and Giles [11, 13] proved this result in general.

Theorem 3.1. *If P is a box TDI polyhedron then there exists a linear system $Ax \leq b$ such that $P = \{x: Ax \leq b\}$ and A is* 0, 1, -1 *valued.* □

The proof of Edmonds and Giles uses the following 'duplication' result, for which a new proof, based on the results of the previous section, is given below. (See Schrijver [37] for other operations that preserve the fact that a system is a box TDI system.)

Theorem 3.2. *Let $Ax \leq b$ be a box TDI system, where A is an $r \times n$ matrix, and let A_n denote the nth column of A. Then $Ax + A_n x_{n+1} \leq b$ is also a box TDI system, where x_{n+1} is a new variable.*

Proof. Let $H = \{h_1, \dots, h_m\}$ be a box Hilbert basis in $\mathbb{Q}^n$. For each vector $h_i = \{h_{i1}, \dots, h_{in}) \in H$ let $\bar{h}_i$ denote the $(n+1)$-component vector $(h_{i1}, \dots, h_{in}, h_{in})$. By Proposition 2.2, it suffices to prove that $\bar{H} = (\bar{h}_1, \dots, \bar{h}_m)$ is also a box Hilbert basis. Furthermore, since it is clear that $\bar{H}$ is a Hilbert basis, Lemma 2.3 implies that it suffices to show that the cone $\bar{C}$ generated by $\bar{H}$ has the box property.

Let $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n, \bar{w}_{n+1}) \in \bar{C}$ and let w denote the n-component vector $(\bar{w}_1, \dots, \bar{w}_m)$. Notice that $w \in C$, the cone generated by H. Since $\bar{w}_n = \bar{w}_{n+1}$, we have that either $\{n, n+1\} \subseteq \mathcal{I}(\bar{w})$ or $\{n, n+1\} \cap \mathcal{I}(\bar{w}) = \emptyset$. If $\{n, n+1\} \subseteq \mathcal{I}(\bar{w})$, then since w has the box property in C, $\bar{w}$ has the box property in $\bar{C}$. So we may assume that $\{n, n+1\} \cap \mathcal{I}(\bar{w}) = \emptyset$.

Let (U, L) be a partition of $\{1, \dots, n+1\} \backslash \mathcal{I}(\bar{w})$. If $\{n, n+1\} \subseteq U$ or $\{n, n+1\} \subseteq L$, then $\bar{w}$ has the box property in $\bar{C}$ with respect to (U, L) since w has the box property in C with respect to $(U \backslash \{n+1\}, L \backslash \{n+1\})$. So we may assume, by symmetry, that $n \in U$ and $n+1 \in L$. Let w' be an integral vector in C such that

$$\begin{aligned} &\text{(i)} \quad w'_i = \bar{w}_i \quad \text{for each } i \in \mathcal{I}(\bar{w}), \\ &\text{(ii)} \quad w'_i \leq \lceil \bar{w}_i \rceil \text{ for each } i \in U, \\ &\text{(iii)} \quad w'_i \geq \lfloor \bar{w}_i \rfloor \text{ for each } i \in L \backslash \{n+1\}. \end{aligned} \tag{3.1}$$

If $w'_n \geq \lfloor \bar{w}_n \rfloor$, then, since $\bar{w}' = (w'_1, \dots, w'_n, w'_n)$ is an integral vector in $\bar{C}$, we have that $\bar{w}$ has the box property in $\bar{C}$ with respect to (U, L). So we may assume

$$w'_n < \lfloor \bar{w}_n \rfloor < \bar{w}_n. \tag{3.2}$$

Now since each convex combination of w' and w is a vector in C which satisfies the conditions given in (3.1), the inequalities in (3.2) imply that there exists a vector $w^2 \in C$ which satisfies the conditions in (3.1) and is such that $w^2_n = \lfloor \bar{w}_n \rfloor$. Now since w^2 has the box property in C, there exists an integral vector $w^3 \in C$ such that the conditions in (3.1) are satisfied and $w^3_n = \lfloor \bar{w}_n \rfloor$. The integral vector $\bar{w}^3 = (w^3_1, \dots, w^3_n, w^3_n)$ in $\bar{C}$ proves that $\bar{w}$ has the box property in $\bar{C}$ with respect to (U, L). □

The nice proof by Edmonds and Giles [13] of Theorem 3.1 goes as follows: Let $Ax \leq b$ be a box TDI system and $P = \{x\colon Ax \leq b\}$. A vector x^* is in P if and only if $\max\{-1x' + 1x''\colon Ax + Ax' + Ax'' \leq b, x = x^*, x' \geq 0, x'' \leq 0\} \geq 0$, which, by the linear programming duality theorem and the fact that $Ax + Ax' + Ax'' \leq b$ is again a box TDI system, is true if and only if $\pi A x^* \leq \pi b$ for all integral $\pi \geq 0$ such that πA is integral and $-1 \leq \pi A \leq 1$.

Remark 3.3. *It should be noted that Theorem* 3.1 *does not imply that if P is a box TDI polyhedron then there exists a box TDI system* $Ax \leqslant b$ *such that* $P = \{x: Ax \leqslant b\}$ *and A is* 0, 1, −1 *valued. Indeed, this stronger statement is not true, as evidenced by examples of Edmonds and Giles* [12] *and Schrijver* [38]. □

Several results related to the above theorems of Edmonds and Giles will now be cosidered.

For a set $K \subseteq \mathbb{Q}^{n+m}$, the *projection* of K onto the first n coordinates is the set $\{x \in \mathbb{Q}^n: \exists z \in \mathbb{Q}^m$ such that $(x, z) \in K\}$. It is well known that the projection of a polyhedron is again a polyhedron (see Bachem and Grötschel [1], for example). Furthermore, it is easy to see that if all vertices of a polyhedron are integral then all vertices of each projection of that polyhedron are also integral, a result which has been used to prove combinatorial theorems in Balas and Pulleyblank [2] and Cameron [5]. The above proof technique of Edmonds and Giles can be used to show that if P is a box TDI polyhedron and Q is a projection of P, then Q can be defined by a linear system with 0, 1, −1 valued left hand sides. In fact, the class of box TDI polyhedra is closed under taking projections.

Theorem 3.4. *Each projection of a box TDI polyhedron onto a subset of its coordinates is again a box TDI polyhedron.*

Proof. Let P be a box TDI polyhedron in the space $\mathbb{Q}^{n+m}$, let P' be the projection of P onto the first n coordinates, let F' be a face of P' and let C' be the cone for F', that is, C' is the set of all vectors $w \in \mathbb{Q}^n$ such that $\max\{wx: x \in P'\}$ is achieved by each vector x in F'. Let $c \in C'$ be a vector such that $\max\{cx: x \in P'\}$ is achieved only by the set of vectors in F'. Now let F be the maximal face of P such that $\max\{cx: (x, z) \in P\}$ is achieved by each vector $(x, z) \in F$, that is, let $F = \{(x, z) \in P: cx = \beta\}$ where $\beta = \max\{cx: (x, z) \in P\}$. It follows that F' is the projection of F onto the first n coordinates. Let C be the cone of all vectors $(u, v) \in \mathbb{Q}^{n+m}$ such that $\max\{ux + vx: (x, z) \in P\}$ is achieved by each vector (x, z) in F. Each $w \in C'$ has the box property in C' since the $n+m$ component vector $(w, 0)$ has the box property in C. □

The *dominant* of a polyhedron $P \subseteq \mathbb{Q}^n$ is the polyhedron $\{x \in \mathbb{Q}^n: x \geqslant z$ for some $z \in P\}$. The problem of finding properties of the dominant of a polyhedron arises in the study of blocking pairs of polyhedra (see Fulkerson [17], Cunningham [8], and McDiarmid [32]). In general, linear systems which define the dominant of a polyhedron P may necessarily have a complicated structure, even though the inequalities defining P are of a simple form. Indeed, Cunningham and Green-Krotki [9] have shown that for any positive integer n there exists a perfect matching polyhedron having dominant $P \subseteq \mathbb{Q}^m$ and an inequality $\alpha_1 x_1 + \cdots + \alpha_m x_m \geqslant \beta$ such that $\alpha_1 x_1 + \cdots + \alpha_m x_m \geqslant \beta$ is a facet-inducing inequality for P and $\{1, \ldots, n\} \subseteq \{\alpha_1, \ldots \alpha_m\}$. This problem, however, does not occur with box TDI polyhedra.

Edmonds and Giles [11, 13] have shown that if P is a box TDI polyhedron then the dominant of P can be defined by a linear system $Ax \geqslant b$ where A is a 0, 1 valued matrix. Again, the stronger statement that the class of box TDI polyhedra is closed under taking dominants is true. The following lemma will be used in the proof of this stronger result.

Lemma 3.5. *If C is a cone with the box property then $C \cap \{x: x \leqslant 0\}$ also has the box property.*

Proof. Let C be a cone with the box property and suppose $C \cap \{x: x \leqslant 0\}$ does not have the box property. Choose $w \in C \cap \{x: x \leqslant 0\}$ and partition (U, L) of $\{1, \ldots, n\} \backslash \mathscr{I}(w)$ such that

(i) w does not have the box property with respect to (U, L) in $C \cap \{x: x \leqslant 0\}$ and

(ii) of all vectors $v \in C \cap \{x: x \leqslant 0\}$ and partitions of $\{1, \ldots, n\} \backslash \mathscr{I}(v)$ satisfying (i), $|L|$ is minimum.

Suppose $L = \emptyset$. Since C has the box property, there exists a vector $\bar{w} \in C$ such that $\bar{w}_i = w_i$ for all $i \in \mathscr{I}(w)$ and $\bar{w}_i \leqslant \lceil w_i \rceil$ for all $i \in \{1, \ldots, n\} \backslash \mathscr{I}(w)$. Now since $w \leqslant 0$, we have $\bar{w} \leqslant 0$. So $\bar{w} \in C \cap \{x: x \leqslant 0\}$, a contradiction.

Suppose $L \neq \emptyset$. There exists an integral vector $\bar{w} \in C$ such that $\bar{w}_i = w_i$ for all $i \in \mathscr{I}(w)$, $\bar{w}_i \leqslant \lceil w_i \rceil$ for all $i \in U$, and $\bar{w}_i \geqslant \lfloor w_i \rfloor$ for all $i \in L$. By assumption, $\bar{w} \notin C \cap \{x: x \leqslant 0\}$. So for some $t \in L$, $\bar{w}_t > 0$. Thus, since $w \leqslant 0$, there exists a vector v which is a convex combination of w and $\bar{w}$ such that $v \leqslant 0$ and $v_r = 0$ for some $r \in L$. Now $v \in C \cap \{x: x \leqslant 0\}$ and $v_i = w_i$ for all $i \in \mathscr{I}(w)$, $v_i \leqslant \lceil w_i \rceil$ for all $i \in U$ and $v_i \geqslant \lfloor w_i \rfloor$ for all $i \in L$. So, by assumption (i), v does not have the box property in $C \cap \{x: x \leqslant 0\}$ with respect to $(U \backslash \mathscr{I}(v), L \backslash \mathscr{I}(v))$. But $|L \backslash \mathscr{I}(v)| < |L|$, since $r \in L$, a contradiction to assumption (ii). □

Theorem 3.6. *If P is a box TDI polyhedron then the dominant of P is also a box TDI polyhedron.*

Proof. Let P be a box TDI polyhedron and let P^* be the dominant of P. Let F^* be a face of P^* and let $F = F^* \cap P$. There exists a hyperplane H such that $H \cap P^* = F^*$. We have $F^* \cap P = H \cap P^* \cap P = H \cap P$, so F is a face of P. Let C be the cone for F in P, that is C is the set of all vectors w such that $\max\{wx: x \in P\}$ is achieved by each vector in F, and let C^* be the cone for F^* in P^*. By Theorem 2.4, C has the box property. We must show that C^* also has the box property.

Suppose $w \in C^*$. We have that $w \leqslant 0$, since otherwise $\max\{wx: x \in P^*\}$ does not exist. Also, $w \in C$, since $\max\{wx: x \in P^*\} = \max\{wx: x \in P\}$. Let $\mathscr{T} = \{i: 1 \leqslant i \leqslant n$ and there exist points $x \in F^*$ and $y \in P$ such that $x \geqslant y$ and $x_i > y_i\}$. For each $i \in \mathscr{T}$ we have $w_i = 0$, since if $x \in F^*$ and $y \in P$ with $x \geqslant y$ then $wx = wy$, which implies that $w_j = 0$ for each j such that $x_j > y_j$. So w is contained in the cone

$$C' = C \cap \{x: x \leqslant 0\} \cap \{x: x_i = 0 \text{ for all } i \in \mathscr{T}\}.$$

Now suppose $w \in C'$. Let $\beta = \max\{wx: x \in P\}$. Since $w \leqslant 0$, $\max\{wx: x \in P^*\} = \beta$. Suppose there exists a vector $y \in F^*$ such that $wy < \beta$. There exists a vector $z \in P$ such that $y \geqslant z$. It follows that $z \in F$ and, hence, that $wz = \beta$. But $w_i = 0$ for each $i \in \{1, \ldots, n\}$ such that $y_i > z_i$, which implies $wy = wz$, a contradiction. So $wy' = \beta$ for each $y' \in F^*$. Thus $w \in C^*$.

Since C has the box property, Lemma 3.5 implies that $C \cap \{x: x \leqslant 0\}$ has the box property. Thus, $C' = C^*$ has the box property. □

For an example of the application of this result see Edmonds and Giles [11].

Remark 3.7. *One can similarly prove that if P is a box TDI polyhedron then $\{x: x \leqslant z$ for some $z \in P\}$ is also a box TDI polyhedron. Polyhedra of this type arise in the study of antiblocking pairs of polyhedra (see Cunningham and Green-Krotki* [9], *Fulkerson* [17] *and McDiarmid* [32]). □

4. An algorithmic characterization

Consider the following geometric characterization of polymatroids given by Edmonds [10]: A compact nonempty set $P \subseteq \{x \in \mathbb{Q}^n: x \geqslant 0\}$ is a polymatroid if and only if

(i) $x \in P$ and $0 \leqslant y \leqslant x$ imply that $y \in P$ and

(ii) for each nonnegative $a \in \mathbb{Q}^n$, every maximal vector $y \in P \cap \{x: x \leqslant a\}$ has the same coordinate sum

(where maximal is with respect to the ordinary $\leqslant$ ordering). An attractive feature of this geometric characterization is that it gives an easy way to prove that a polyhedron is not a polymatroid. It implies that if we are given a polymatroid P by an optization oracle (see Grötschel, Lovász, and Schrijver [23, 26] for definitions regarding oracles), then if P is not a polymatroid then there exists a short proof of this fact. It is shown below that Theorem 2.4 implies that a similar statement can be made for box TDI polyhedra.

Grötschel, Lovász, and Schrijver [26] define a *well-described polyhedron* to be a triple $(P; n, \rho)$ where $P \subseteq \mathbb{Q}^n$ is a polyhedron which can be defined by a finite system of linear inequalities such that the input size (in binary notation) of each inequality is at most ρ.

Lemma 4.1. *Let $(C; n, \rho)$ be a well-described convex cone given by a strong separation oracle. If C does not have the box property then there exists an oracle-polynomial-time proof of this fact.*

Proof. Suppose that C does not have the box property. Let $w \in C$ be a vector which does not have the box property in C. Since C is a well-described polyhedron, there exists a set G of integral vectors which generates C such that the size of each vector

$g \in G$ is polynomial in n and ρ (see Grötschel, Lovász, and Schrijver [23, 26]). By Carathéodory's Theorem, w may be expressed as $\lambda_1 g_1 + \cdots + \lambda_t g_t$ where t is the dimension of C and for $i = 1, \ldots, t$: $g_i \in G$ and $\lambda_i \in \mathbb{Q}$ is nonnegative. Since w does not have the box property in C, $(\lambda_1 - \lfloor \lambda_1 \rfloor) g_1 + \cdots + (\lambda_t - \lfloor \lambda_t \rfloor) g_t$ does not have the box property in C. So we may assume that $0 \leq \lambda_i < 1$ for $i = 1, \ldots, t$.

Let (U, L) be a partition of $\{1, \ldots, n\} \backslash \mathcal{I}(w)$ such that w does not have the box property in C with respect to (U, L). Since the size of each g_i, $i = 1, \ldots, t$, is polynomial in n and ρ, the size of the vector w^1 defined as $w_i^1 = w_i$ for $i \in \mathcal{I}(w)$, $w_i^1 = \lceil w_i \rceil$ for $i \in U$, and $w_i^1 = \lfloor w_i \rfloor$ for $i \in L$ is also polynomial in n and ρ. Let w^2 be a vector in $B = C \cap \{x \colon x_i = w_i^1$ for $i \in \mathcal{I}(w)$, $x_i \leq w_i^1$ for $i \in U$, $x_i \geq w_i^1$ for $i \in L\}$ such that $|\mathcal{I}(w^2)|$ is maximum. Consider the mixed integer (feasibility) program: $x \in B$, x_i is integral for all $i \in \mathcal{I}(w^2)$. It follows from the results of von zur Gathen and Sieveking [19], on bounds for integral solutions to linear inequality systems, that there exists a solution vector w^3 to this program, where the size of w^3 is polynomial in n and ρ. (One way to see this is to note that, using Farkas' Lemma, the projection of B onto the $\mathcal{I}(w^2)$ coordinates can be described by a system of inequalities each of which has input size bounded above by a polynomial in n and ρ.)

Let $D = \{x \colon x_i = w^3$ for all $i \in \mathcal{I}(w^3)$, $x_i \leq \lceil w_i^3 \rceil$ for all $i \in U \backslash \mathcal{I}(w^3)$, $x_i \geq \lfloor w_i^3 \rfloor$ for all $i \in L \backslash \mathcal{I}(w^3)\}$. By the choice of w and (U, L), there does not exist an integral vector in $C \cap D$. So either $U \backslash \mathcal{I}(w^3) \neq \emptyset$ or $L \backslash \mathcal{I}(w^3) \neq \emptyset$. Suppose that $U \backslash \mathcal{I}(w^3) \neq \emptyset$ and let $i \in U \backslash \mathcal{I}(w^3)$. If there exists a vector v in $C \cap D$ with v_i integral then there also exists a vector v' in $C \cap D$ with $v_i' = \lceil w_i^3 \rceil$ or $v_i' = \lceil w_i^3 \rceil - 1$ (since any convex combination of v and w^3 is also in $C \cap D$ and if $v_i \neq \lceil w_i^3 \rceil$ then, since v_i is integral, there exists a convex combination, v', of v and w^3 such that $v_i' = \lceil w_i^3 \rceil - 1$). Using the results of Grötschel, Lovász, and Schrijver [26], we can check in oracle-polynomial time that $C \cap D \cap \{x \colon x_i = \lceil w_i^3 \rceil\}$ and $C \cap D \cap \{x \colon x_i = \lceil w_i^3 \rceil - 1\}$ are empty. (Since $|\mathcal{I}(w^2)|$ is maximum, each of these polyhedra is indeed empty.) This proves that w^3 does not have the box property with respect to $(U \backslash \mathcal{I}(w^3), L \backslash \mathcal{I}(w^3))$. A similar argument can be used in the case where $L \backslash \mathcal{I}(w^3) \neq \emptyset$. □

Theorem 4.2. *Let $(P; n, \rho)$ be a well described polyhedron given by a strong optimization oracle. If P is not a box TDI polyhedron then there exists an oracle-polynomial-time proof of this fact.*

Proof. Suppose P is not a box TDI polyhedron. By Theorem 2.4, there exists a face F of P such that the cone C of all vectors w such that $\max\{wx \colon x \in P\}$ is achieved by each vector in F, does not have the box property. We may describe F as $P \cap \{x \colon \alpha x = \beta\}$, where $\alpha x = \beta$ is an equation of size polynomial in n and ρ. (To see that such an equation exists, let $A^1 x \leq b^1$, $A^2 x \leq b^2$ be a defining system for P such that each inequality has size at most ρ and such that $F = \{x \colon A^1 x \leq b^1, A^2 x = b^2\}$. Let $k = \operatorname{rank}(A^2)$ and let $A^3 x = b^3$ be a subsystem of k equations from $A^2 x = b^2$ such that $\operatorname{rank}(A^3) = k$. Letting $\alpha x = \beta$ be the sum of the k equations in $A^3 x = b^3$, we have an appropriate equation.) Thus, we can optimize over F in oracle-polynomial time (see Grötschel, Lovász, and Schrijver [23, 26]).

If $Ax \leq b$ is a defining system for P then C is generated by the active rows for F in $Ax \leq b$. So C is generated by a finite set of vectors, each of which is of size at most ρ. This implies (see Grötschel, Lovász, and Schrijver [26]) that C can be defined by a linear system such that the size of each inequality in the system is polynomial in n and ρ. So, by Lemma 4.1, it suffices to show that we can solve the strong separation problem for C in oracle-polynomial time. This can be done in the following way, as observed by L. Lovász. Suppose $w \in \mathbb{Q}^n$. Let $\lambda_1 = \max\{wx\colon x \in P\}$ and $\lambda_2 = \min\{wx\colon x \in F\}$. If $\lambda_1 = \lambda_2$, then $w \in C$. If $\lambda_1 > \lambda_2$, then we have found vectors v^1, $v^2 \in P$ such that $v^2 \in F$ and $v^1 w > v^2 w$, that is $(v^1 - v^2)w > 0$. Let $\bar{x} \in C$. We have $v^2\bar{x} = \max\{y\bar{x}\colon y \in P\}$, since $v^2 \in F$ and $\bar{x} \in C$. So $v^1\bar{x} \leq v^2\bar{x}$, that is $(v^1 - v^2)\bar{x} \leq 0$. Thus, $\{x\colon (v^1 - v^2)x = 0\}$ is a separating hyperplane for w and C. □

If we are given a linear system $Ax \leq b$ as an explicit list of inequalities, then, using the algorithm of Khachiyan [29], we can optimize over the polytope $\{x\colon Ax \leq b\}$ in polynomial time. Thus, the above theorem implies the following result.

Corollary 4.3. *The class of linear systems $Ax \leq b$ which define box TDI polyhedra is in co-NP.* □

(See Garey and Johnson [18] for definitions and results on the classes NP and co-NP.)

Remark 4.4. A related result, which follows from the algorithmic work of Chandrasekaran [6], is that the class of box TDI linear systems $Ax \leq b$, with A integral, is in co-NP (see Cook, Lovász, and Schrijver [7]). This result, however, does not imply Corollary 4.3, since the size of any TDI defining system $Mx \leq d$, with M integral, may necessarily be exponential in the size of $Ax \leq b$ and, furthermore, one would have to certify that $Mx \leq d$ is in fact a TDI system (it is an open problem to decide whether or not the class of TDI linear systems is in NP).

Another related result, which follows from Proposition 2.2 and the results of Cook, Lovász, and Schrijver [7], is that in fixed dimension one can test in polynomial time whether or not $Ax \leq b$, where A is integral, is a box TDI system. □

To give an application of the above results, the class of perfect graphs will now be considered. A graph G is defined to be a *perfect graph* if for each induced subgraph H of G the chromatic number of H is equal to the cardinality of a maximum clique of H (see Berge [3]). There are many interesting results in the theory of perfect graphs, a survey of which can be found in Lovász [31] (see also Berge and Chvátal [4]). Cameron [5] defined a graph to be *box perfect* if the linear system

$$\sum\{x_v\colon v\in K\}\leqslant 1 \quad \text{for each (not necessarily maximal) clique } K \text{ of } G,$$
$$x_v\geqslant 1 \quad \text{for each node } v \text{ of } G \tag{4.1}$$

is a box TDI system. This definition is motivated, in part, by the result of Fulkerson [17] and Lovász [30] that a graph G is perfect if and only if (4.1) is a TDI system. Thus, box perfect graphs are a subset of perfect graphs. In fact, it is easy to see that a box perfect graph is k-perfect for all k, where k-perfect graphs are defined as in Lovász [31] (see Greene and Kleitman [21]). A class of box perfect graphs which includes comparability graphs is described in Cameron [5], along with a survey of other results on box perfect graphs. Grötschel, Lovász, and Schrijver [24], Lovász [31], and Lubiw (see Cameron [5]) have shown that the class of perfect graphs is in co-NP. It will be shown below that the class of box perfect graphs is also in co-NP.

It follows directly from the definition of box perfect graphs that a graph is box perfect if and only if

$$\begin{aligned}&\sum\{x_v\colon v\in K\}\leqslant 1 && \text{for each maximal clique } K \text{ of } G,\\ &x_v\geqslant 0 && \text{for each node } v \text{ of } G\end{aligned} \tag{4.2}$$

is a box TDI system. Moreover, Theorem 2.4, together with the theorem of Fulkerson [17] and Lovász [30] mentioned above, implies the following characterization (where the stable set polytope of a graph is the convex hull of the incidence vectors of the stable sets of the graph).

Proposition 4.5. *A graph G is box perfect if and only if G is perfect and the stable set polytope of G is a box TDI polyhedron.* □

This characterization will be used to prove the following result.

Theorem 4.6. *The class of box perfect graphs is in co-NP.*

Proof. Suppose that G is not a box perfect graph. Since the class of perfect graphs is in co-NP, if G is not perfect then there exists a polynomial-time proof of this, which also proves that G is not box perfect. Suppose that G is perfect. By the results of Grötschel, Lovász, and Schrijver [24, 25], one can optimize linear functions over the stable set polytope of G in polynomial time. By Theorem 4.2, using the Grötschel, Lovász, and Schrijver algorithm polynomially many times we can prove that the stable set polytope of G is not a box TDI polyhedron. (Notice that to verify that the Grötschel, Lovász, and Schrijver algorithm has optimized a linear function over the stable set polytope of G we do not need to verify that G is perfect, but only check that the primal and dual solutions produced by the algorithm are feasible solutions to the corresponding linear programs and that they give equal objective values, which can be done in polynomial time since the primal solutions are integral and the dual solutions are basic.) □

Acknowledgement

I thank Professor L. Lovász for his help with the applications of the ellipsoid method.

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