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CHVÁTAL CLOSURES FOR MIXED INTEGER PROGRAMMING PROBLEMS

W. COOK*
Institut für Ökonometrie und Operations Research, Universität Bonn, D-5300 Bonn, FR Germany

R. KANNAN**
Department of Computer Science, Carnegie-Mellon University, Pittsburgh, PA, USA

A. SCHRIJVER
Mathematical Centre, 1098 SJ Amsterdam, The Netherlands

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Chvátal introduced the idea of viewing cutting planes as a system for proving that every integral solution of a given set of linear inequalities satisfies another given linear inequality. This viewpoint has proven to be very useful in many studies of combinatorial and integer programming problems. The basic ingredient in these cutting-plane proofs is that for a polyhedron P and integral vector w, if $\max(wx \mid x \in P, wx \text{ integer}\} = t$, then $wx \leqslant t$ is valid for all integral vectors in P. We consider the variant of this step where the requirement that wx be integer may be replaced by the requirement that $\bar{w}x$ be integer for some other integral vector $\bar{w}$. The cutting-plane proofs thus obtained may be seen either as an abstraction of Gomory's mixed integer cutting-plane technique or as a proof version of a simple class of the disjunctive cutting planes studied by Balas and Jeroslow. Our main result is that for a given polyhedron P, the set of vectors that satisfy every cutting plane for P with respect to a specified subset of integer variables is again a polyhedron. This allows us to obtain a finite recursive procedure for generating the mixed integer hull of a polyhedron, analogous to the process of repeatedly taking Chvátal closures in the integer programming case. These results are illustrated with a number of examples from combinatorial optimization. Our work can be seen as a continuation of that of Nemhauser and Wolsey on mixed integer cutting planes.

1. Introduction

Cutting-plane techniques have been one of the most studied topics in the theory of integer programming. Early, fundamental work was carried out by Dantzig, Fulkerson and Johnson [12] and Gomory [14], resulting in Gomory's well known integer programming algorithm. Although a very important theoretical development, this method turned out to be considerably less important from a practical point of view, where enumerative techniques have generally ruled. In recent years, however, cutting planes have also come to the forefront of practical methods. One of the

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developments which sparked this resurgence was Chvátal's [8] treatment of Gomory's early work. Rather than viewing Gomory's technique as an algorithm, Chvátal looked at cutting planes as a method for proving that every integral solution to a given set of linear inequalities satisfies another given linear inequality. His approach is as follows: Consider a system of linear inequalities

$$a_i x \leqslant b_i \quad (i = 1, \ldots, k). \tag{1}$$

If we have nonnegative numbers $y_i, \ldots, y_k$ such that $y_1 a_1 + \cdots + y_k a_k$ is integral, then every integral solution of (1) is satisfied by the inequality

$$(y_1 a_1 + \cdots + y_k a_k) x \leqslant \gamma \tag{2}$$

for any number γ which is greater than or equal to $\lfloor y_1 b_1 + \cdots + y_k b_k \rfloor$ (the number $y_1 b_1 + \cdots + y_k b_k$ rounded down to the nearest integer). We say that the inequality (2) is *derived* from (1) using the numbers $y_1, \ldots, y_k$. A *cutting-plane proof* of an inequality $wx \leqslant t$ from (1) is a list of inequalities $a_{k+i} x \leqslant b_{k+i}$ ($i = 1, \ldots, M$), together with nonnegative numbers y_{ij} ($i = 1, \ldots, M, j = 1, \ldots, k+i-1$), such that for each i the inequality $a_{k+i} x \leqslant b_{k+i}$ is derived from the inequalities $a_j x \leqslant b_j$ ($j = 1, \ldots, k+i-1$) using the numbers y_{ij} ($j = 1, \ldots, k+i-1$) and where the last inequality in the list is $wx \leqslant t$. Clearly, an inequality which has a cutting-plane proof satisfies every integral solution of the given system. Conversely, Chvátal [8] and Schrijver [23; 24, Corollary 23.2b] showed:

Theorem 1. *Let $P = \{x \mid Ax \leqslant b\}$ be a nonempty polyhedron which is either rational or bounded.*

(i) *If $wx \leqslant t$ is satisfied by all integral vectors in P (w being integral) and P contains at least one such vector, then there is a cutting-plane proof of $wx \leqslant t$ from $Ax \leqslant b$.*

(ii) *If P contains no integral vectors, then there is a cutting-plane proof of* $0x \leqslant -1$ *from $Ax \leqslant b$.* □

This result may be viewed geometrically as giving a procedure which takes a polyhedron P and generates a linear description of P_I, the convex hull of the integral vectors in P, in the following sense. Call an inequality $wx \leqslant \lfloor \delta \rfloor$ a *Chvátal cutting plane* for P if w is integral and $wx \leqslant \delta$ is satisfied by all vectors in P (so if $P = \{x \mid Ax \leqslant b\}$ then $wx \leqslant \lfloor \delta \rfloor$ can be derived from $Ax \leqslant b$). Now denote by P' the *Chvátal closure* of P, that is, the set of vectors which satisfy every Chvátal cutting plane for P, and let $P^{(0)} = P$ and $P^{(i)} = P^{(i-1)\prime}$ for all $i \geqslant 1$. The result of Chvátal and Schrijver gives:

Theorem 2. *Let P be a rational polyhedron. Then*:

(i) *P' is again a polyhedron.*

(ii) *$P_I = P^{(k)}$ for some integer k.* □

Chvátal [8, 9, 10, 11] has shown that the viewpoints given in these two theorems lead to many nice results in combinatorics, and cutting-plane proof arguments can be found in papers such as Barahona, Grötschel, and Majoub [2], Grötschel and Padberg [15], Grötschel and Pulleyblank [16], and others, which have laid the foundation for subsequent computational work. The frequency of cutting-plane proof arguments in these papers lies in the fact that they provide a concrete model for approaching the task of finding useful valid inequalities for the problem at hand.

In the description of Chvátal cutting planes, we implicitly use the following simple principle.

Principle A. *For an integral vector w, if* $\max\{wx \mid Ax \leqslant b,\ wx \text{ integer}\} = t$, *then* $wx \leqslant t$ *is satisfied by all integral solutions of* $Ax \leqslant b$. □

Chvátal cutting planes are precisely those inequalities $wx \leqslant t$ which can be defined using this principle. In this paper we study the cutting planes which arise by relaxing this to the following, equally simple principle.

Principle B. *For an integral vector c, if* $\max\{wx \mid Ax \leqslant b,\ cx \text{ integer}\} = t$, *then* $wx \leqslant t$ *is satisfied by all integral solutions of* $Ax \leqslant b$. □

Here we do not require that c and w be identical. The cutting-plane proofs which can be obtained with this second principle can be seen either as an abstraction of Gomory's [13] mixed integer programming technique or as a proof version of a simple class of the disjunctive cutting planes studied by Balas [1], and Jeroslow [18] as we will make clear in the next section. We study the extent to which these cuts, when generalized to the context of mixed integer programming, preserve the nice features of Chvátal's cutting-plane proofs. Our main result is the analogue of Theorem 2(i) for these cutting planes, which gives, together with a rounding operation, analogues of Theorem 1 and Theorem 2(ii) for mixed integer programming problems. These theorems can be seen as a continuation of the work of Nemhauser and Wolsey [21] on cutting planes in the spirit of Chvátal cuts, for mixed integer programming. The results are presented and discussed in Section 2 and proven in Section 3. The applicability of these cutting plane proofs is illustrated in Section 4 with a number of examples from combinatorial optimization. Throughout the paper we make use of results in polyhedral theory, for which we refer the reader to the book of Schrijver [24].

2. Split cuts

An important feature of Chvátal cutting planes is that, given the nonnegative multipliers y_i, it is trivial to verify that a derived inequality is indeed satisfied by all integral solutions of the given system. The cutting planes we study have a similar property. First note the following.

Principle C. *For a given system $Ax \leqslant b$, integral vector c and integer k, if $wx \leqslant t$ is valid for both $\{x \mid Ax \leqslant b, cx \leqslant k\}$ and $\{x \mid Ax \leqslant b, cx \geqslant k+1\}$, then $wx \leqslant t$ is satisfied by all integral solutions of $Ax \leqslant b$.* □

Now observe that by letting $k = \lfloor cx^* \rfloor$ where x^* is any optimal solution to $\max\{wx \mid Ax \leqslant b\}$, we obtain Principle B. (If $\max\{wx \mid Ax \leqslant b\}$ does not exist, then either $\max\{wx \mid Ax \leqslant b, cx \text{ integer}\}$ does not exist or there is an integer k with $k < cx < k+1$ for all solutions of $Ax \leqslant b$. To see the implication in general, notice that if $\bar{x}$ is any optimal solution to $\max\{wx \mid Ax \leqslant b, cx \leqslant \lfloor cx^* \rfloor\}$ then $c\bar{x} = \lfloor cx^* \rfloor$. Since $\lfloor cx^* \rfloor$ is integer, we must have $\max\{wx \mid Ax \leqslant b, cx \leqslant \lfloor cx^* \rfloor\} \leqslant t$. Similarly, $\max\{wx \mid Ax \leqslant b, cx \geqslant \lfloor cx^* \rfloor + 1\} \leqslant t$.)

Thus, an inequality $wx \leqslant t$ can be verified by checking separately that it is valid for $\{x \mid Ax \leqslant b, cx \leqslant k\}$ and valid for $\{x \mid Ax \leqslant b, cx \geqslant k+1\}$ (see Figure 1), each of which can be done, via Farkas' lemma, by using the appropriate nonnegative multipliers. Due to the form of this verification, we refer to the cutting planes we propose to study as split cuts. So $wx \leqslant t$ is a *split cut* for a polyhedron P if for some integral vector c and integer k, it is a valid inequality for both $\{x \in P \mid cx \leqslant k\}$ and $\{x \in P \mid cx \geqslant k+1\}$. It follows immediately that these cutting planes are a simple class of disjunctive cuts, as mentioned in the introduction. In a mixed integer programming problem, only a subset of the variables are restricted to integral values. So the set of feasible solutions to such a problem has the form

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \;\middle|\; Ax + By \leqslant b,\, x \text{ integral} \right\}.$$

We extend the definition of a split cut to such sets in the following way: An inequality $wx + vy \leqslant t$ is a *split cut for $P \subseteq \mathbb{R}^{m+n}$ with respect to the integer variables x* if there

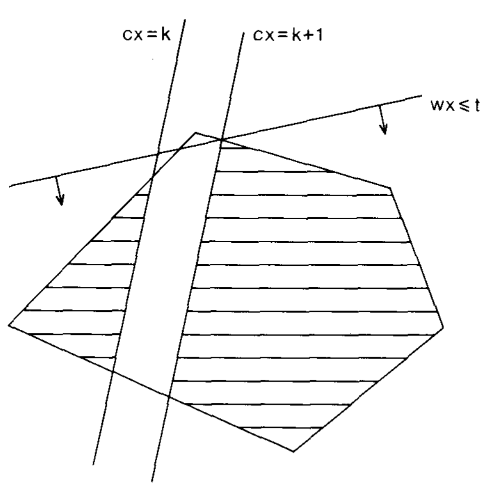

Fig. 1.

exists an integral vector $c \in \mathbb{R}^m$ and an integer k such that $wx + vy \leqslant t$ is valid for both

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, cx \leqslant k\right\} \quad \text{and} \quad \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, cx \geqslant k+1\right\}.$$

It is again a simple fact that such an inequality is satisfied by all x-integral vectors in P (those vectors

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in P$$

such that $\bar{x}$ is integral). Analogous to the definition of the Chvátal closure of a polyhedron, we define the *split closure* of P with respect to the integer variables x as the set of all vectors that satisfy every split cut for P with respect to x. One way to view this is as follows. For each $c \in \mathbb{Z}^m$ let

$$P^c = \text{convex hull}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, cx \text{ integer}\right\}. \tag{3}$$

Clearly, P^c is a polyhedron (see [24, the proof of Theorem 16.1, p. 231]) and the split closure of P is

$$\bigcap_{c \in \mathbb{Z}^n} P^c.$$

Our main result is the following.

Theorem 3. *The split closure of a rational polyhedron P, with respect to any subset of integer variables, is again a polyhedron.*

The proof of this theorem is given in the next section.

For a polyhedron

$$P = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,\middle|\, Ax + By \leqslant b\right\},$$

let $P_{\mathrm{I}(x)}$ denote the convex hull of the x-integral vectors in P. From Motzkin's decomposition theorem for polyhedra, it follows that if P is rational then $P_{\mathrm{I}(x)}$ is a polyhedron (see [24, Section 16.7]). Thus, given Theorem 3, one may suspect that repeatedly taking the split closure of P would give $P_{\mathrm{I}(x)}$ after a finite number of iterations, which of course follows from Theorem 2 when all variables are integer. This however is not the case. To see the difficulty, first consider the following direct extension of Chvátal closures, based on Principle A rather than Principle B: For a given polyhedron $P \subseteq \mathbb{R}^{m+n}$, let P'_x be the set of all vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P$$

which satisfy each inequality $wx + vy \leqslant \delta$, where w is integral and

$$\delta = \max\left\{ wx + vy \colon \begin{pmatrix} x \\ y \end{pmatrix} \in P,\ wx \text{ integer}\right\}.$$

Now if all variables are integral, then P'_x is the Chvátal closure of P, so, in that case, repeating the closure finitely many times gives P_{I}. In the general case however, it may happen that $P \neq P_{\mathrm{I}(x)}$ but $P = P'_x$, that is, the procedure may get 'stuck' before reaching the convex hull of the x-integral vectors. Consider the following:

Example 1. Let $P \subseteq \mathbb{R}^{2+1}$ be defined by

$$x_1 + y_1 \leq \tfrac{1}{2}, \qquad x_2 + y_1 \leq 1,$$

and suppose that only x_1 and x_2 are restricted to integer values. Clearly $P \neq P_{\mathrm{I}(x)}$. (For instance, $(\frac{1}{2}, 1, 0) \in P \backslash P_{\mathrm{I}(x)}$.) Now consider an inequality $w_1x_1 + w_2x_2 + v_1y_1 \leq \delta$ where w_1 and w_2 are integers and

$$\delta = \max\{w_1x_1 + w_2x_2 + v_1y_1 | (x_1, x_2, y_1) \in P,\ w_1x_1 + w_2x_2 \text{ integer}\}.$$

Since δ is finite,

$$\max\{w_1x_1 + w_2x_2 + v_1y_1 | (x_1, x_2, y_1) \in P\} \tag{4}$$

is also finite. So (4) is achieved by all vectors on the unique minimal face $F = \{(x_1, x_2, y_1) | (x_1 + y_1 = \frac{1}{2},\ x_2 + y_1 = 1\}$ of P. We must have $w_1 \geq 0$ and $w_2 \geq 0$ (since (4) is finite), so there exists a number q such that $\frac{1}{2}w_1 + w_2 - (w_1 + w_2)q = 0$. Now $(\frac{1}{2} - q, 1 - q, q)$ is a vector on F with $w_1x_1 + w_2x_2$ integer. Thus δ is equal to (4). Since this is true for any choice of w_1, w_2, and v_1, it follows that $P'_x = P$.

It is easy to see that the split closure cannot get 'stuck' in the sense of the above example. Indeed, if $P \neq P_{\mathrm{I}(x)}$ then there exists a minimal face F of P that contains no x-integral vectors. As F is a rational affine subspace of $\mathbb{R}^{m+n}$, the projection of F onto the x variables is a rational affine subspace of $\mathbb{R}^m$ that contains no integral vectors. So the 'integer Farkas lemma' [24, Corollary 4.1a] implies that there exists an integral vector $c \in \mathbb{R}^m$ and a rational (nonintegral) number γ such that

$$cx = \gamma \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in F.$$

Now

$$F \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,\middle|\, \lfloor \gamma \rfloor < cx < \lfloor \gamma \rfloor + 1 \right\}.$$

Thus, letting $wx + vy \leq q$ be a valid inequality for P with

$$F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, wx + vy = q \right\},$$

for a small enough $\varepsilon > 0$ the inequality $wx + vy \leq q - \varepsilon$ is a split cut for P. So, letting $\hat{P}$ denote the split closure of P, we have $F \cap \hat{P} = \emptyset$, which implies $\hat{P} \neq P$.

The problem that arises with split closures is that although $\hat{P} \neq P$, the difference between these two polyhedra may become arbitrarily small after the split closure operation has been repeated a number of times, as in the following:

Example 2. Let $P \subseteq \mathbb{R}^{2+1}$ be the convex hull of the four vectors

$$(0,0,0), \quad (2,0,0), \quad (0,2,0), \quad (\tfrac{1}{2}, \tfrac{1}{2}, \varepsilon)$$

for some rational $0 < \varepsilon < 1$, where the first two variables, x_1 and x_2, are required to be integers and the third variable, y_1, may be noninteger. Clearly, $P_{I(x)}$ is simply the convex hull of $(0,0,0)$, $(2,0,0)$ and $(0,2,0)$. Now for any integers c_1, c_2, and k, there exists a rational $\varepsilon_{(c_1,c_2,k)} > 0$ such that $(\frac{1}{2}, \frac{1}{2}, \varepsilon_{(c_1,c_2,k)})$ is contained in the convex hull of the two polytopes

$$\{(x_1, x_2, y_1) \in P \mid c_1x_1 + c_2x_2 \leq k\}$$

and

$$\{(x_1, x_2, y_1) \in P \mid c_1x_1 + c_2x_2 \geq k+1\}.$$

(As suggested by the referee, one way to see this is by noting that if $0 < \lambda_1 \leq 1$ is chosen such that either $\frac{1}{2}\lambda_1(c_1+c_2) \leq k$ or $\frac{1}{2}\lambda_1(c_1+c_2) \geq k+1$ and $\lambda_2 = \lambda_3 = \frac{1}{4} - \frac{1}{4}\lambda_1$, then the convex combination $\lambda_1(\frac{1}{2}, \frac{1}{2}, \varepsilon) + \lambda_2(2,0,0) + \lambda_3(2,0,0) + (1-\lambda_1-\lambda_2-\lambda_3) \times (0,0,0)$ gives a point of the desired form.)

Thus, since the split closure, $\hat{P}$, of P is a polyhedron, it follows that there exists an $\varepsilon_1 > 0$ such that $(\frac{1}{2}, \frac{1}{2}, \varepsilon_1) \in \hat{P} \neq P_{I(x)}$ and contains a polytope of the same form as P. So repeating the argument, for any k the polytope obtained from P by taking the split closure k times contains the vector $(\frac{1}{2}, \frac{1}{2}, \varepsilon_k)$ for some $\varepsilon_k > 0$. Therefore, we cannot obtain $P_{I(x)}$ after a finite number of split closures.

An immediate way to deal with this problem is to treat the continuous variables y in a discrete fashion by examining the numbers that appear in the inequalities $Ax + By \leq b$. Indeed, if P is rational then we may assume that A, B and b are integral. Thus, for any vector

$$\begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^{m+n}$$

such that

$$\max\{wx + vy \mid Ax + By \leq b,\ x \text{ integral}\} \tag{5}$$

exists, there is an optimal solution

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix}$$

such that $(\det \bar{B})y^*$ is integral for some submatrix $\bar{B}$ of B. (This follows from Cramer's rule and the fact that if

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

is an optimal solution to (5) then so is

$$\begin{pmatrix} \bar{x} \\ y^* \end{pmatrix}$$

for any optimal solution y^* of $\max\{vy\colon By \leqslant b + A\bar{x}\}$.) So if we replace each variable y_i by My_i where M is an upper bound on the product of the subdeterminants of B, then we may treat all variables as integers and consider Chvátal cuts on this transformed problem. Interpreting this directly on P gives that if

$$\begin{pmatrix} w \\ v \end{pmatrix}$$

is integral and $wx + vy \leqslant \delta$ is valid for P, then

$$wx + vy \leqslant \lfloor M\delta \rfloor / M \tag{6}$$

is satisfied by all x-integral vectors in P. The trouble with this is that the size of M (in binary notation) may be exponential in the size of $Ax + By \leqslant b$. Thus it may be impossible to verify that (6) is valid for all x-integral solutions in polynomial time, which is counter to the idea behind cutting-plane proofs.

As suggested by Éva Tardos (private communication), the difficulty with (6) can be overcome by employing a more sophisticated type of rounding. For an integral $r \times n$ matrix B, let Δ_B denote the number $n!\beta^n$ where β is the maximum of the absolute values of the entries of B. As Δ_B is trivially an upper bound on the largest subdeterminant of B, it follows that if (5) exists then the maximum is achieved by a vector

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix}$$

such that sy^* is integral for some integer $1 \leqslant s \leqslant \Delta_B$. Thus, if w and v are integral and $wx + vy \leqslant \delta$ is valid for P then

$$wx + vy \leqslant [\delta]_{\Delta_B} \tag{7}$$

is satisfied by all x-integral vectors in P, where $[\delta]_{\Delta_B}$ is the greatest rational number $p/q \leqslant \delta$ such that $1 \leqslant q \leqslant \Delta_B$. The point of this type of rounding is that $[\delta]_{\Delta_B}$ can be calculated easily (in polynomial time) using continued fractions (see [17, Chapter 3; 20, Section 1.1; 24, Section 6.1]). Let $\mathrm{ROUND}(P, \Delta_B)$ denote the set of vectors which satisfy every inequality of the type given in (7). Example 1 given above again shows that it may happen that $P \neq P_{\mathrm{I}(x)}$ but $\mathrm{ROUND}(P, \Delta_B) = P$. However, suppose we combine split closures and rounding by letting $\mathrm{SPLIT}(P, \Delta_B) = \mathrm{ROUND}(\hat{P}, \Delta_B)$ where $\hat{P}$ is the split closure of P with respect to the integer variables x. Then letting

$$\mathrm{SPLIT}^0(P, \Delta_B) = P$$

and

$$\mathrm{SPLIT}^i(P, \Delta_B) = \mathrm{SPLIT}(\mathrm{SPLIT}^{i-1}(P, \Delta_B), \Delta_B)$$

we have:

Theorem 4. *Let*

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,\middle|\, Ax + By \leq b \right\}$$

for some integral A, B and b. Then, with respect to the set x of integer variables:

(i) $\mathrm{SPLIT}(P, \Delta_B)$ *is again a polyhedron.*

(ii) $\mathrm{SPLIT}^k(P, \Delta_B) = P_{\mathrm{I}(x)}$ *for some integer k.*

The proof of this theorem is also given in the next section.

This result gives a finite cutting-plane proof system for mixed integer programming problems. Of course, when looking for such proofs one would hope that the rounding cuts would not be required, as is the case in the combinatorial examples presented in Section 4.

Remarks. (i) Although there is no finite bound on the number of split closures needed to obtain $P_{\mathrm{I}(x)}$ in general, it is easy to see that if the integer variables are bounded between 0 and 1 then m closures will suffice.

(ii) A different recursive procedure for proving the validity of mixed integer cutting planes was developed by Nemhauser and Wolsey [23]. The cuts used in their proofs are a special type of split cut, as shown to us by Chvátal (private communication).

(iii) For another approach to the problem of generalizing Chvátal's methods to mixed integer programming, we refer the reader to the papers of Blair and Jeroslow [5, 6], where the theory is treated in terms of 'Chvátal functions'.

3. Proofs of Theorems 3 and 4

Throughout this section, we let

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,\middle|\, A\begin{pmatrix} x \\ y \end{pmatrix} \leq b \right\},$$

where A is an integral $r \times (m+n)$ matrix and b is an integral vector, and consider the split cuts with respect to the integer variables x. (To shorten the notation, we have combined the matrices A and B and the vectors v and w of the previous sections into a single matrix and vector.)

Proof of Theorem 3. Define for each $c \in \mathbb{Z}^m$,

$$P^c = \text{convex hull}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, cx \text{ integer}\right\}.$$

Since P^c is a polyhedron and

$$\hat{P} = \bigcap_{c \in \mathbb{Z}^n} P^c.$$

the following claim immediately implies the theorem.

Claim. There exists a finite subset $\mathscr{C}_p$ of $\mathbb{Z}^m$ such that

$$\hat{P} = \bigcap_{c \in \mathscr{C}_p} P^c.$$

Proof of claim. The proof is by induction on the dimension of P, the case when this is zero being trivial. We may assume that $P^c \neq \emptyset$ for all $c \in \mathbb{Z}^m$ since otherwise the claim is trivial. This implies that

$$\text{char. cone}(P^c) = \text{char. cone}(P) \tag{8}$$

(char. cone(K) denotes the characteristic cone of K) for each $c \in \mathbb{Z}^m$ (see [24, Theorem 16.1]). This also implies that the affine hull of P must contain x-integral vectors, since otherwise, by the 'integer Farkas lemma' (see [24, Corollary 4.1a]), there would exist a hyperplane

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,\middle|\, cx = \delta\right\}$$

with $c \in \mathbb{Z}^m$ and δ nonintegral, which contains P and hence $P^c = \emptyset$. Using this, we may assume that P is of full dimension, by taking an appropriate affine transformation of $\mathbb{R}^{m+n}$ if necessary (see [24, p. 341]).

By induction, for each facet F of P there exists a finite subset $\mathscr{C}_F$ of $\mathbb{Z}^m$ so that

$$\hat{F} = \bigcap_{c \in \mathscr{C}_F} F^c$$

where F^c denotes convex hull

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in F \,\middle|\, cx \text{ integral}\right\}.$$

Let

$$\mathscr{C} = \bigcup_{F \text{ facet of } P} \mathscr{C}_F.$$

So $\mathscr{C}$ is finite. Let Q be the polyhedron

$$Q = \bigcap_{c \in \mathscr{C}} P^c.$$

We use later that for each facet F of P,

$$F \cap Q = F \cap \bigcap_{c \in \mathscr{C}} P^c = \bigcap_{c \in \mathscr{C}} (F \cap P^c) = \bigcap_{c \in \mathscr{C}} F^c \subseteq \bigcap_{c \in \mathscr{C}_F} F^c = \hat{F}. \tag{9}$$

If $Q = \emptyset$ we can take $\mathscr{C}_p = \mathscr{C}$. So we may assume $Q \neq \emptyset$. Let

$$\mathscr{G} = \left\{ g \,\middle|\, g \text{ is a minimal face of } Q; g \not\subseteq \bigcup_{F \text{ facet of } P} F \right\}$$

and

$$K = \left\{ w \in \mathbb{R}^{m+n} \,\middle|\, \|w\| = 1; \right.$$

$$\left. \max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\} \text{ is finite and attained at some } g \in \mathscr{G} \right\}.$$

So K is compact. Let $h: K \to \mathbb{R}$ be defined by

$$h(w) = \max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\} - \max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\}.$$

(This is well defined, since if $w \in K$ then

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\}$$

if finite, since by (8), char. cone(P) = char. cone(Q).) The function h is continuous, as it is the difference of two piece-wise linear functions. Moreover, $h(w) > 0$ for all w in K. Indeed, suppose to the contrary that

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\} = \max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\} = \mu$$

for some $w \in K$ and some μ. As $w \in K$,

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\}$$

is attained at some $g \in \mathscr{G}$. Then

$$g \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in Q \,\middle|\, w\begin{pmatrix} x \\ y \end{pmatrix} = \mu \right\} \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, w\begin{pmatrix} x \\ y \end{pmatrix} = \mu \right\}.$$

and hence g would be contained in some facet of P, a contradiction.

It follows that there exists an $\varepsilon > 0$ so that $h(w) \geq \varepsilon$ for all $w \in K$. Obviously, there exists a $\rho > 0$ (depending on ε and P only) so that for each minimal face f of P the set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,\middle|\, \operatorname{dist}\left(\begin{pmatrix} x \\ y \end{pmatrix}, f \right) \leq \varepsilon \right\}$$

contains a ball of radius ρ. Now for each $w \in K$, there exists a minimal face f of P so that

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\}$$

is attained at f, and therefore

$$\begin{aligned} &\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \middle| w\begin{pmatrix} x \\ y \end{pmatrix} \geqslant \max\left\{ w\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \middle| \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in Q \right\} \right\} \\ &\quad = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \middle| w\begin{pmatrix} x \\ y \end{pmatrix} \geqslant \max\left\{ w\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \middle| \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in P \right\} - h(w) \right\} \\ &\quad \supseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \middle| w\begin{pmatrix} x \\ y \end{pmatrix} \geqslant \max\left\{ w\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \middle| \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in P \right\} - \varepsilon \right\} \\ &\quad \supseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \middle| \operatorname{dist}\left(\begin{pmatrix} x \\ y \end{pmatrix}, f \right) \leqslant \varepsilon \right\} \end{aligned}$$

(as $\|w\| = 1$). Therefore:

$$\text{For each } w \in K, \text{ the set } \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \middle| w\begin{pmatrix} x \\ y \end{pmatrix} \geqslant \max\left\{ w\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \middle| \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in Q \right\} \right\}$$

contains a ball of radius ρ. (10)

We finally show that this implies that for each $c \in \mathbb{Z}^m$ with $\|c\| > 1/\rho$ one has

$$Q \subseteq P^c. \tag{11}$$

This implies that we may take

$$\mathscr{C}_P := \mathscr{C} \cup \{ c \in \mathbb{Z}^m \mid \|c\| \leqslant 1/\rho \}$$

since

$$\bigcap_{c \in \mathscr{C}_P} P^c = Q \cap \bigcap_{\substack{c \in \mathbb{Z}^m \\ \|c\| \leqslant 1/\rho}} P^c = \bigcap_{c \in \mathbb{Z}^m} P^c = \hat{P}.$$

In order to prove (11), observe that it suffices to show that

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\} \leqslant \max\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \in P^c \right\}$$

for each $w \in \mathbb{R}^{m+n}$ with $\|w\| = 1$ and

$$\max\left\{ w\begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\}$$

finite (since by (9), char. cone(Q) = char. cone(P^c)).

We consider two cases:

Case 1.

$$\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in Q\right\}$$

is attained at a minimal face g of Q with $g\notin\mathcal{G}$. So $g\subseteq F$ for some facet F of P. Then by (9):

$$\begin{aligned}\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in Q\right\}&=\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in F\cap Q\right\}\\&\leqslant\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in \hat{F}\right\}\\&\leqslant\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in \hat{P}\right\}\\&\leqslant\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in P^c\right\}.\end{aligned}$$

Case 2.

$$\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in Q\right\}$$

is attained at a minimal face g of Q with $g\in\mathcal{G}$. Then $w\in K$. Hence by (10),

$$B\subseteq\left\{\begin{pmatrix}x\\y\end{pmatrix}\in P\,\middle|\,w\begin{pmatrix}x\\y\end{pmatrix}\geqslant\max\left\{w\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\middle|\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\in Q\right\}\right\}\tag{12}$$

for some ball B of radius ρ. As $\|c\|>1/\rho$,

$$B\cap\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^{m+n}\,\middle|\,c\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{Z}\right\}\neq\emptyset$$

(since, for any t, the distance between the hyperplanes defined by

$$c\begin{pmatrix}x\\y\end{pmatrix}=t\quad\text{and}\quad c\begin{pmatrix}x\\y\end{pmatrix}=t+1$$

is $1/\|c\|$).

Hence $B\cap P^c\neq\emptyset$. Choose

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\in B\cap P^c.$$

So by (12),

$$w\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\geqslant\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in Q\right\},$$

implying

$$\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in Q\right\}\leqslant w\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\leqslant\max\left\{w\begin{pmatrix}x\\y\end{pmatrix}\middle|\begin{pmatrix}x\\y\end{pmatrix}\in P^c\right\}.\qquad\square$$

To prove Theorem 4, we need the following lemma, which will be used in an inductive argument.

Lemma 5. *If F is a face of P, then for any $\Delta > 0$ we have*

$$\mathrm{SPLIT}(F, \Delta) = F \cap \mathrm{SPLIT}(P, \Delta).$$

Proof. For each $c \in \mathbb{Z}^m$ we have $F^c = F \cap P^c$. Therefore $\hat{F} = F \cap \hat{P}$. So if $\hat{F} = \emptyset$ the result follows. Suppose this is not the case. Then $\hat{F}$ is a face of the polyhedron $\hat{P}$. So there is a linear system

$$M^0\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^0, \qquad M^1\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^1,$$

such that

$$\hat{P} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| M^0\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^0, M^0\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^1 \right\}$$

and

$$\hat{F} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| M^0\begin{pmatrix} x \\ y \end{pmatrix} = d^0, M^1\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^1 \right\}$$

where M^0, M^1, d^0, and d^1 are all integral. Let

$$w\begin{pmatrix} x \\ y \end{pmatrix} \leqslant \delta$$

be a valid inequality for $\hat{F}$ with $w \in \mathbb{Z}^{m+n}$. For a large enough positive integer T, adding T times each inequality in

$$M^0\begin{pmatrix} x \\ y \end{pmatrix} \leqslant d^0$$

to

$$w\begin{pmatrix} x \\ y \end{pmatrix} \leqslant \delta$$

we obtain an inequality

$$\bar{w}\begin{pmatrix} x \\ y \end{pmatrix} \leqslant \bar{\delta}$$

that is valid for $\hat{P}$. (This follows from Farkas' lemma.) Furthermore, since $\delta \equiv \bar{\delta}$ (mod 1), we have

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \hat{F} \middle| w\begin{pmatrix} x \\ y \end{pmatrix} \leqslant [\delta]_\Delta \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \hat{F} \middle| \bar{w}\begin{pmatrix} x \\ y \end{pmatrix} \leqslant [\bar{\delta}]_\Delta \right\}.$$

It follows that

$$\mathrm{ROUND}(\hat{F}) = \hat{F} \cap \mathrm{ROUND}(\hat{P}),$$

which proves the lemma. $\square$

Proof of Theorem 4. (i) By Theorem 3, it suffices to show that ROUND(P, Δ) is a polyhedron for any $\Delta > 0$. Now since A and b are integral, Farkas' lemma implies that ROUND(P, Δ) is defined by the set of all inequalities of the form

$$\omega \binom{x}{y} \leqslant [\delta]_\Delta$$

where $\omega \in \mathbb{Z}^{m+n}$ and

$$\omega = zA \quad \text{and} \quad \delta = zb$$

for some $z = (z_1, \ldots, z_r) \in \mathbb{R}^r$ with $0 \leqslant z_i < 1$ for each $i = 1, \ldots, r$ (if $z_i \geqslant 1$ for some i, then we could replace z_i by $z_i - 1$). As there are only finitely many such inequalities, the result follows.

(ii) The proof is again by induction on the dimension of P, the case when this is 0 being trivial. If the affine hull of P does not contain x-integral vectors, then, as in the proof of Theorem 3, we have $\hat{P} = \emptyset$. So we may assume this is not the case.

Suppose $P_{\mathrm{I}(x)} \neq \emptyset$. There exists a linear system

$$D \binom{x}{y} \leqslant f$$

such that D is integral and

$$P_{\mathrm{I}(x)} = \left\{ \binom{x}{y} \,\middle|\, D \binom{x}{y} \leqslant f \right\}$$

since $P_{\mathrm{I}(x)}$ is a polyhedron. Let

$$d \binom{x}{y} \leqslant \alpha$$

be an inequality in this system. Then it suffices to show that

$$\mathrm{SPLIT}^k(P, \Delta_B) \subseteq \left\{ \binom{x}{y} \,\middle|\, d \binom{x}{y} \leqslant \alpha \right\}$$

for some k, where $B = [A_{m+1}, \ldots, A_{m+n}]$, the last n columns of A. Letting

$$\delta = \max \left\{ d \binom{x}{y} \,\middle|\, \binom{x}{y} \in P \right\},$$

we have

$$\mathrm{SPLIT}(P, \Delta_B) \subseteq \left\{ \binom{x}{y} \,\middle|\, d \binom{x}{y} \leqslant [\delta]_{\Delta_B} \right\}.$$

If $[\delta]_{\Delta_B} \leqslant \alpha$ then we are done, so suppose this is not the case. Let

$$F = \mathrm{SPLIT}(P, \Delta_B) \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| d \begin{pmatrix} x \\ y \end{pmatrix} = [\delta]_{\Delta_B} \right\}.$$

If $F = \emptyset$, then

$$\mathrm{SPLIT}^2(P, \Delta_B) \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| d \begin{pmatrix} x \\ y \end{pmatrix} \leqslant [\delta]_{\Delta_B} - \left(\frac{1}{\Delta_B} \right)^2 \right\}$$

since the difference between any two distinct rationals $p_1/q_1, p_2/q_2$, with $1 \leqslant q_1, q_2 \leqslant \Delta_B$, is at least $(1/\Delta_B)^2$. If $F \neq \emptyset$, then it is a proper face of $\mathrm{SPLIT}(P, \Delta_B)$ and $F_{\mathrm{I}(x)} = \emptyset$. By induction we have

$$\mathrm{SPLIT}^l(F, \Delta_B) = \emptyset$$

for some integer l. Thus, by applying Lemma 5 l times, we have

$$F \cap \mathrm{SPLIT}^{l+1}(P, \Delta_B) = \emptyset.$$

So

$$\mathrm{SPLIT}^{l+2}(P, \Delta_B) \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| d \begin{pmatrix} x \\ y \end{pmatrix} \leqslant [\delta]_{\Delta_B} - \left(\frac{1}{\Delta_B} \right)^2 \right\}.$$

Thus, repeating this procedure at most

$$([\delta]_{\Delta_B} - \alpha)/(1/\Delta_B)^2$$

times we obtain the result.

Suppose $P_{\mathrm{I}(x)} = \emptyset$. As the affine hull of P contains x-integral vectors, we know P is not an affine subspace. Furthermore, the dimension of the characteristic cone of P is less than the dimension of P (since $P_{\mathrm{I}(x)} = \emptyset$ and the affine hull of P contains x-integral vectors.) So

$$P \subseteq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \alpha_1 \leqslant d \begin{pmatrix} x \\ y \end{pmatrix} \leqslant \alpha_2 \right\}$$

for some nonzero $d \in \mathbb{Z}^{m+n}$ and integers α_1, α_2, where for any number t we have

$$P \neq \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| d \begin{pmatrix} x \\ y \end{pmatrix} = t \right\}.$$

Thus, proceeding as above (letting

$$\delta = \max \left\{ d \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\},$$

etc.), we have $\mathrm{SPLIT}^k(P, \Delta_B) = \emptyset$ for some integer k. $\square$

4. Examples

4.1. Integer programs with circular ones

Split cuts occur in a natural way in the work of Bartholdi, Orlin and Ratliff [4] on cyclic scheduling problems, as pointed out to us by Jim Orlin (private communication). The problems they consider are of the form

$$\min\{wx \mid Ax \geqslant b, x \geqslant 0, x \text{ integer}\}$$

where w and b are nonnegative integral vectors and A is a 0–1 matrix with the circular 1's property, that is, in each row of A the 1's occur consecutively, where the first and last components are defined to be consecutive. Their work shows that if $P = \{x \mid Ax \leqslant b, x \geqslant 0, x \text{ integer}\}$ then we have:

The split closure of P is identical to P_I. (13)

Indeed, if $wx \geqslant t$ is valid for P then it may be obtained by letting $c \equiv 1 \equiv (1, 1, \ldots, 1)$. To see this, let x^* be an optimal solution to $\min\{wx \mid Ax \leqslant b, x \geqslant 0\}$ and let $k = \lfloor wx^* \rfloor$. Consider the two linear programs

$$\min\{wx \mid Ax \geqslant b, 1x \leqslant k, x \geqslant 0\} \tag{14}$$

and

$$\min\{wx \mid Ax \geqslant b, 1x \geqslant k+1, x \geqslant 0\}. \tag{15}$$

By the choice of k, if (14) is feasible then it has an optimal solution $\bar{x}$ with $1\bar{x} = k$ (by taking a convex combination of x^* and any optimal solution to (14)). Thus we may subtract $1x = k$ from some of the inequalities in $Ax \geqslant b$ without changing the value of (14). We may do this is such a way that we obtain a linear program

$$\min\{wx \mid Ax \geqslant \bar{b}, 1x = k, x \geqslant 0\} \tag{16}$$

with $\bar{A}$ a $\{0, 1, -1\}$ matrix where each row is either 0, 1 or 0, -1 and the nonzeros occur consecutively, where the first and last components are *not* considered to be consecutive. Such a matrix A (together with the row $1 = (1, 1, \ldots, 1)$) is well known to be totally unimodular. So (16), and hence (14), has an integral optimal solution. Applying the same argument, we have that (15) also has an integral optimal solution. It follows that t is at most the minimum of (14), if it is feasible, and (15). Thus $wx \geqslant t$ is a split cut for P.

4.2. Fixed charge problems

Sets of the form

$$Q = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+n} \;\middle|\; \sum_{i=1}^{n} y_i \leqslant d_0, 0 \leqslant y_i \leqslant m_i x_i, x_i \in \{0, 1\}, i = 1, \ldots, n \right\}$$

arise in a number of models in operations research. The integer variable x_i represent the decision to make y_i positive, and are used to incorporate fixed costs into the objective function. Padberg, van Roy and Wolsey [22] introduced a class of valid inequalities for Q, as a step towards developing an efficient cutting-plane, optimization algorithm. This class of inequalities is defined as follows: A set $S \subseteq \{1, \ldots, n\}$ is a cover if

$$\lambda \equiv \sum_{i \in S} m_i - d_0 > 0.$$

For a cover S and a set $L \subseteq \{1, \ldots, n\} \backslash S$ let

$$\bar{m} = \max_{i \in S} m_i,$$

$$\bar{m}_i = \max(\bar{m}, m_i) \quad \text{for all } i \in L.$$

If $\bar{m} \geqslant \lambda$, the (S, L) *flow cover inequality*

$$\sum_{i \in S \cup L} y_i + \sum_{i \in S} (m_i - \lambda)^+(1 - x_i) - \sum_{i \in L} (\bar{m}_i - \lambda) x_1 \leqslant d_0 \tag{17}$$

is valid for Q, where $(m_i - \lambda)^+ = \max\{0, m_i - \lambda\}$.

This can be proven with split cuts by letting

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+n} \,\middle|\, \sum_{i=1}^{n} y_i \leqslant d_0, 0 \leqslant y_i \leqslant m_i x_i, 0 \leqslant x_i \leqslant 1, i = 1, \ldots, n \right\}$$

and noting that:

The (S, L) inequality (17) is a split cut for P. (18)

Proof of (18). Let $\bar{S} = \{i \in S | m_i > \lambda\}$. We will chop P with the inequalities $\sum_{i \in \bar{S} \cup L} x_i \geqslant |\bar{S}|$ and $\sum_{i \in \bar{S} \cup L} x_i \leqslant |\bar{S}| - 1$. In the first case, write $\sum_{i \in \bar{S} \cup L} x_i \geqslant |\bar{S}|$ as

$$\sum_{i \in \bar{S}} (1 - x_i) - \sum_{i \in L} x_i \leqslant 0. \tag{19}$$

Multiplying (19) by $(\bar{m} - \lambda)$ and taking its sum with

$$\sum_{i=1}^{n} y_i \leqslant d_0,$$

$$-y_i \leqslant 0 \quad \text{for all } i \notin S \cup L,$$

we obtain

$$\sum_{i \in S \cup L} y_i + \sum_{i \in \bar{S}} (\bar{m} - \lambda)(1 - x_i) - \sum_{i \in L} (\bar{m} - \lambda) x_i \leqslant d_0 \tag{20}$$

as a valid inequality for

$$P \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \sum_{i \in \bar{S} \cup L} x_i \geqslant |\bar{S}| \right\}.$$

Now since $(m_j - \lambda)^+ = 0$ for all $i \in S \backslash \bar{S}$ and $(\bar{m} - \lambda) \leqslant (\bar{m}_i - \lambda)$ for all $i \in L$, (20) implies the inequality (17).

In the second case, write $\sum_{i\in\bar{S}\cup L} x_i \leqslant |\bar{S}|-1$ as

$$-\sum_{i\in\bar{S}}(1-x_i)+\sum_{i\in L}x_i\leqslant -1. \tag{21}$$

Multiplying (21) by λ and taking its sum with

$$(m_i-\bar{m}_i)x_i\leqslant 0 \quad \text{for all } i\in L,$$

$$y_i-m_ix_i\leqslant 0 \quad \text{for all } i\in S\cup L,$$

$$\sum_{i\in S} m_ix_i+\sum_{i\in\bar{S}} m_i(1-x_i)\leqslant \sum_{i\in S} m_i,$$

$$0=d_0+\lambda-\sum_{i\in S} m_i,$$

we obtain

$$\sum_{i\in S\cup L} y_i+\sum_{i\in\bar{S}}(m_i-\lambda)(1-x_i)-\sum_{i\in L}(\bar{m}_i-\lambda)x_i\leqslant d_0 \tag{22}$$

as a valid inequality for

$$P\cap\left\{\begin{pmatrix}x\\y\end{pmatrix}\,\middle|\,\sum_{i\in\bar{S}\cup L} x_i\leqslant|\bar{S}|-1\right\}.$$

Now since $(m_i-\lambda)^+=0$ for all $i\in S\backslash\bar{S}$, (22) is identical to the (S,L) inequality (17). $\square$

4.3. Plant location and lot-sizing problems

A number of results on valid inequalities for mixed integer programming formulations of plant location problems and economic lot-sizing problems have been obtained by Bárány, van Roy and Wolsey [3], Cho, Johnson, Padberg and Rao [7], Leung and Magnanti [19] and others. We do not discuss these inequalities in detail, but mention that (a) the validity of the 'residual capacity inequalities' for the capacitated plant location problem described in [19] can be established by showing they are split cuts for the linear programming relaxation (in fact, this is the way they are shown to be valid in [19]); (b) the validity of the '(S, L) inequalities' for the uncapacitated economic lot-sizing problem treated in [3] can be proven using at most m split cuts, where m is the number of integer variables (this is easy, the main point of [3] is that these inequalities completely describe the corresponding mixed integer hull); (c) the inequalities for the uncapacitated plant location problem given in (7) do not appear to have short split cut proofs, but this is not surprising since a polynomial length split cut proof for these inequalities would imply that NP = co-NP, as it would give a good characterization for the set cover problem.

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