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CUTTING-PLANE PROOFS IN POLYNOMIAL SPACE

William COOK
Institut für Ökonometrie und Operations Research, Universität Bonn, 5300 Bonn 1, FR Germany

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Following Chvátal, cutting planes may be viewed as a proof system for establishing that a given system of linear inequalities has no integral solution. We show that such proofs may be carried out in polynomial workspace.

Key words: Integer programming, cutting planes.

The integer programming problem is to decide if a given system of linear inequalities has an integral solution. Recent progress on this algorithmic question has involved techniques from the geometry of numbers, in the celebrated paper of Lenstra [20] and in results of Babai [1], Grötschel, Lovász and Schrijver [14] and Kannan [16]. One of the things that is apparent in these results is the importance of the fact that if a polyhedron contains no integral vectors then there must be some direction in which it is not very 'wide'. This idea has been developed more fully by Kannan and Lovász [17], who obtained a theorem which provides much more information on the appearance of such polyhedra. These 'width' results have consequences for the construction and analysis of proof systems for verifying that a polyhedron contains no integral vectors. Whereas the integer programming problem is directly related to the question of the equality of P and NP, the existence of a polynomial-length proof system for integer programming is equivalent to NP = co-NP.

One of the fundamental concepts in the theory of integer programming is that of cutting planes, going back to the work of Dantzig, Fulkerson and Johnson [11] and Gomory [12]. On the practical side, cutting-plane techniques are the basis of very successful algorithms for the solution of large-scale combinatorial and 0–1 programming problems in Crowder, Johnson and Padberg [9], Crowder and Padberg [10], Grötschel, Jünger and Reinelt [13], Padberg, van Roy and Wolsey [21] and elsewhere. On the theoretical side, Chvátal [3, 4, 5, 6] has shown that the notion of cutting planes leads to many nice results and proofs in combinatorics. We will adopt Chvátal's point of view and consider cutting planes as a proof system, in our case for verifying that polyhedra contain no integral vectors.

Perhaps the best known of all proof systems is the resolution method for proving the unsatisfiability of formulas in the propositional calculus. Haken [15] settled a

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long-standing open problem by showing that resolution is nonpolynomial. It is easy to see that proving the unsatisfiability of a formula is a special case of proving that a polyhedron contains no integral vectors, and, using Haken's result, it can be shown that cutting planes are a strictly more powerful proof system than the resolution system (see [7] for a treatment of this and the relationship of cutting planes and extended resolution).

To define Chvátal's [5] concept of a cutting-plane proof, consider a system of linear inequalities

$$a_i x \leqslant b_i \quad (i = 1, \dots, k). \tag{1}$$

If we have nonnegative numbers $y_1, \dots, y_k$ such that $y_1 a_1 + \cdots + y_k a_k$ is integral, then every integral solution of (1) satisfies the inequality

$$(y_1 a_1 + \cdots + y_k a_k) x \leqslant \gamma \tag{2}$$

for any number γ which is greater than or equal to $\lfloor y_1 b_1 + \cdots + y_k b_k \rfloor$ (the number $y_1 b_1 + \cdots + y_k b_k$ rounded down to the nearest integer). We say that the inequality (2) is *derived* from (1) using the numbers $y_1, \dots, y_k$. A *cutting-plane proof* of the fact that the linear system (1) has no integral solution is a list of inequalities $a_{k+i} x \leqslant b_{k+i}$ $(i = 1, \dots, M)$, together with nonnegative numbers y_{ij} $(i = 1, \dots, M$, $j = 1, \dots, k+i-1)$, such that for each i the inequality $a_{k+i} x \leqslant b_{k+i}$ is derived from the inequalities $a_j x \leqslant b_j$ $(j = 1, \dots, k+i-1)$ using the numbers y_{ij} $(j = 1, \dots, k+i-1)$ and where the last inequality in the sequence is $0x \leqslant -1$. Results of Chvátal [3] and Schrijver [24] imply that a system of rational linear inequalities has no integral solution if and only if this fact has a cutting-plane proof.

The *length* of a cutting-plane proof is the number, M, of derived inequalities. Cook, Coullard, and Turán [7] have shown that results on the 'width' of polyhedra imply that if a rational linear system has no integral solution then there exists a cutting-plane proof of this with length bounded above by a function depending only on the number of variables in the system. A consequence of this is that in fixed dimension, the total number of binary digits needed to write down a cutting-plane proof that a rational system $Ax \leqslant b$ has no integral solution can be bounded above by a polynomial function of the size, in binary notation, of $Ax \leqslant b$ (see [2, 7]). Unfortunately, the bound on the length of the cutting-plane proofs is necessarily exponential in the number of variables, so for varying dimension we have no guarantee that we can write down our cutting-plane proof in polynomial space. (Again, this is possible if and only if NP = co-NP.) Notice, however, that during the course of a proof it may happen that some of the derived inequalities are no longer needed and so could be removed from our workspace. Thus the amount of space we need in order to carry out a proof may be considerably less than the amount of space it would take to write down the entire list of derived inequalities. So perhaps we can still bound the amount of workspace we need by a polynomial function of the size of $Ax \leqslant b$.

A notion of the amount of space required by general proof systems was developed by Kozen [18, 19]. To specialise his definition to cutting planes we will view our proofs as certain acyclic directed graphs, as suggested by Chvátal [5]. Suppose that $a_{k+i}x \leq b_{k+i}$ $(i=1,\ldots,M)$, together with nonnegative y_{ij} $(i=1,\ldots,M,\ j=1,\ldots,k+i-1)$, is a cutting-plane proof of the fact that $a_i x \leq b_i$ $(i=1,\ldots,k)$ has no integral solution. An associated directed graph has nodes $1,2,\ldots,k+M$ and a directed edge from node i to node j if and only if the inequality $a_i x \leq b_i$ is used in the derivation of $a_j x \leq b_j$. (By 'used' we mean that a positive multiple of the inequality $a_i x \leq b_i$ is taken in the derivation of $a_j x \leq b_j$.) So to derive inequality $a_j x \leq b_j$, we only need to know the inequalities corresponding to the immediate predecessors of node j in our directed graph. Thus, once we have reached node j, the only previously derived inequalities we need to remember are those for which there is a directed edge going from it to a node greater than j. So the greatest number of inequalities which must be stored during the proof is the maximum number, over all nodes $k+i$ $(i=1,\ldots,M)$, of directed edges going from nodes $\{1,\ldots,k+i\}$ to nodes $\{k+i+1,\ldots,k+M\}$. As our bound on the space requirement of the proof we take this number multiplied by the maximum size of an inequality used in the proof. (We have not considered the numbers y_{ij} in calculating our bound, since, using linear programming results, these can always be chosen to be of size polynomial in the size of the inequalities used in the derivation and the size of the inequality to be derived; see, for example, [23].) With this definition, we will show that there exist cutting-plane proofs with length depending only on the dimension and which can be carried out in polynomial workspace, that is, in an amount of workspace bounded above by a polynomial function of the size of $Ax \leq b$. We refer the reader to the book of Schrijver [23] for results in the theory of polyhedra and integer programming which are used in the proof.

Theorem 1. *Let A be a rational $m \times n$ matrix and b a rational $m \times 1$ vector such that $Ax \leq b$ has no integral solution. Then there exists a cutting-plane proof of $0x \leq -1$ from $Ax \leq b$ of $O(n^{3^n})$ length which can be carried out in polynomial workspace.*

The proof of this result will involve an inductive argument, making use of the following lemma (see [22]) which allows one to 'rotate' a cutting plane for a face of a polyhedron so that it is also a cutting plane for the polyhedron itself.

Lemma 2. *Suppose $wx \leq \alpha$ is derivable from the linear system $(Ax \leq b,\ Cx = d)$, where C, d, w and α are integral, and that the system has a solution x with $wx > \alpha - 1$. Then there exists an inequality $w'x \leq \alpha'$ that is derivable from $(Ax \leq b,\ Cx \leq d)$ such that:*

(i) $\{x\colon Ax \leq b,\ Cx = d,\ w'x \leq \alpha'\} \subseteq \{x\colon Ax \leq b,\ Cx = d,\ wx \leq \alpha\}$;

(ii) $\{x\colon Ax \leq b,\ Cx = d,\ w'x = \alpha'\} = \{x\colon Ax \leq b,\ Cx = d,\ wx = \alpha\}$.

Furthermore, letting $\sigma(A, b, C, d)$ denote the greatest absolute value amongst the entries of A, b, C and d, the absolute value of each coefficient of $w'x \leq \alpha'$ can be bounded above by $n\sigma(A, b, C, d)$.

Proof. By Caratheodory's theorem and the definition of a derivation, there exist vectors u and v with $u \geqslant 0$, $uA+vC=w$, and $\lfloor ub+vd \rfloor = \alpha$, such that at most n components of u and v are nonzero. For each inequality $a_i x \leqslant b_i$ let

$$\bar{u}_i = \begin{cases} u_i - 1 & \text{if } u_i \text{ is a positive integer,} \\ \lfloor u_i \rfloor & \text{otherwise,} \end{cases}$$

and for each equation $c_i x = d_i$ let

$$\bar{v}_i = \lfloor v_i \rfloor .$$

Now let

$$w' = w - \bar{u}A - \bar{v}C = (u-\bar{u})A + (v-\bar{v})C,$$

$$\alpha' = \alpha - \bar{u}b - \bar{v}d = \lfloor (u-\bar{u})b + (v-\bar{v})d \rfloor .$$

We claim that $w'x \leqslant \alpha'$ is the desired inequality.

Firstly, since $u-\bar{u}$ and $v-\bar{v}$ are nonnegative and w' is integral, $w'x \leqslant \alpha'$ may be derived from $(Ax \leqslant b,\ Cx \leqslant d)$. Secondly, since each component of $u-\bar{u}$ and $v-\bar{v}$ is at most 1, the absolute value of each coefficient in $w'x \leqslant \alpha'$ is at most $n\sigma(A, b, C, d)$. Thirdly, (i) follows from the fact that $wx \leqslant \alpha$ can be obtained from $w'x \leqslant \alpha'$ by adding nonnegative multiples of inequalities from $Ax \leqslant b$ and multiples of equations from $Cx = d$. Finally, (ii) follows in the same way, using the fact that if $u_i > 0$, then $a_i x \leqslant b_i$ may be replaced by $a_i x = b_i$ in both sides of (ii) without altering the solution sets (since $u_i - \bar{u}_i$ is also positive). □

To obtain the $O(n^{3n})$ bound on the length of the cutting-plane proofs, we will use the following 'width' result of Kannan and Lovász [17].

Theorem 3. *For any rational polyhedron P of dimension n that contains no integral vectors, there exists an integral vector $w \neq 0$ such that*

$$\max\{wx\colon x \in P\} - \min\{wx\colon x \in P\} < \psi n^2$$

where ψ is a positive constant, independent of P and n. □

Proof of Theorem 1. As we may scale the inequalities if necessary, we may assume A and b are integral. We may also assume n is at least 2, since the result is trivial otherwise. The theorem will be proven by showing that the following result holds for each $k \in \{0, 1, \ldots, n\}$:

(A) Let C be a $k \times n$ integral matrix of rank k, let d be a $k \times 1$ integral vector and let $\sigma(A, b, C, d)$ denote the greatest absolute value amongst the entries of A, b, C, d. Then there exists an inequality $c_{k+1}x \leqslant d_{k+1}$ with $\{x\colon c_{k+1}x \leqslant d_{k+1}\} \cap \{x\colon Ax \leqslant b,\ Cx = d\} = \emptyset$ and a cutting-plane proof of $c_{k+1}x \leqslant d_{k+1}$ from $(Ax \leqslant b,\ Cx \leqslant d)$ of length at most $\Psi^{n-k} n^{2.5(n-k)}$ (where Ψ is the positive constant given in Theorem 3) needing only $n-k+1$ inequalities, besides $(Ax \leqslant b,\ Cx \leqslant d)$, to be stored at any one time and where each inequality in the proof has all coefficients of absolute value at most $\Psi^{n-k} n^{3(n-k)+1} \sigma(A, b, C, d)$.

The theorem follows from the case $k=0$, since $\{x: c_1x \leq d_1\} \cap \{x: Ax \leq b\} = \emptyset$ implies, by Farkas' lemma, that $0x \leq -1$ may be derived from $(Ax \leq b,\ c_1 x \leq d_1)$.

The proof is by induction on k, beginning with the case $k=n$. So suppose C is an $n \times n$ matrix. If $\{x: Ax \leq b, Cx = d\} = \emptyset$, then there is nothing to prove. So we may assume that $\{x: Ax \leq b, Cx = d\}$ consists of a single vector, say v. (Since $Cx = d$ has a unique solution). Now since $Ax \leq b$ has no integral solution, v must be nonintegral. Thus there exists trivially an inequality $wx \leq \alpha$ which can be derived from $(Ax \leq b,\ Cx \leq d,\ -Cx \leq -d)$ with $\{x: wx \leq \alpha\} \cap \{x: Ax \leq b, Cx = d\} = \emptyset$. By Lemma 2, we can 'rotate' $wx \leq \alpha$ to obtain an inequality $c_{n+1}x \leq d_{n+1}$, that can be derived from $(Ax \leq b,\ Cx \leq d)$, such that

$$\{x: c_{n+1}x \leq d_{n+1}\} \cap \{x: Ax \leq b, Cx = d\} = \emptyset$$

and the absolute value of each coefficient in $c_{n+1}x \leq d_{n+1}$ is at most $n\sigma(A, b, C, d)$. So (A) is true when $k=n$.

Now assume, by induction, that (A) is true for all $k \geq r$. We will show that (A) holds when $k = r-1$, which will complete the proof. So suppose C is an $(r-1) \times n$ matrix. Letting $A^0x \leq b^0$ be those inequalities in $Ax \leq b$ that hold as equality for each vector in $\{x: Ax \leq b, Cx = d\}$ we have that $M = \{x: A^0x = b^0, Cx = d\}$ is the affine hull of $\{x: Ax \leq b, Cx = d\}$.

Claim 1. We may assume that M contains integral vectors.

Proof of Claim 1. If M contains no integral vectors, then there exist vectors y^0 and y such that $y^0A^0 + yC$ is integral and $y^0b^0 + yd$ is nonintegral (see, for example, [23]). Letting $w = y^0A^0 + yC$ and $\alpha = \lfloor y^0b^0 + yd \rfloor$ we have $\{x: wx \leq \alpha, Ax \leq b, Cx = d\} = \emptyset$. Also, by Farkas' lemma, $wx \leq \alpha$ may be derived from $(Ax \leq b,\ Cx = d)$. So, rotating $wx \leq \alpha$ via Lemma 2(i), we obtain an inequality $c_r x \leq d_r$ which satisfies the conditions in (A). So we may assume that M contains integral vectors. This completes the proof of Claim 1.

Claim 2. There exists an integral vector $w \neq 0$ such that $wx' \neq wx''$ for some x', x'' in M and

$$\lfloor \max\{wx: Ax \leq b, Cx = d\} \rfloor - \lceil \min\{wx: Ax \leq b, Cx = d\} \rceil < \Psi(n-r+1)^2. \quad (3)$$

Proof of Claim 2. We cannot apply Theorem 3 directly, since we do not want $M \subseteq \{x: wx = k\}$ for some integer k. So we first transform M so that we may work with polyhedra of full dimension.

Let s be the dimension of M. Clearly $s \leq n-r+1$, as there are $r-1$ equations in $Cx = d$. Since M contains integral vectors, there exists an affine transformation T which maps $\mathbb{Z}^n$ onto $\mathbb{Z}^n$ and M onto $\{x \in \mathbb{R}^n: x_{s+1} = 0, \ldots, x_n = 0\}$ (see [23, p. 341]).

Let $P = \{\bar{x} \in \mathbb{R}^S: (\bar{x}, 0) \in T(\{x: Ax \leq b,\ Cx = d\})\}$. Since T maps $\mathbb{Z}^n$ onto $\mathbb{Z}^n$, we have $P \cap \mathbb{Z}^s = \emptyset$. Thus, by Theorem 3, there exists a vector $\bar{w} \in \mathbb{Z}^s$ such that $\bar{w} \neq \emptyset$ and

$$0 < \max\{\bar{w}x: x \in P\} - \min\{\bar{w}x: x \in P\} < \Psi s^2 \quad (4)$$

(where the first inequality comes from the fact that P is of full dimension).

We may assume that the components of $\bar{w}$ are relatively prime and hence that for any integer k the equation $\bar{w}\bar{x} = k$ has integral solutions. Let $t_1 - \lfloor \max\{\bar{w}\bar{x}: \bar{x} \in P\} \rfloor$

and $t_2 = \lceil \min\{\bar{w}\bar{x}: \bar{x} \in P\} \rceil$. Since $\{\bar{x} \in \mathbb{R}^s: \bar{w}\bar{x} = t_1\}$ contains integral vectors, so does the hyperplane $H = T^{-1}(\{(\bar{x}, 0) \in \mathbb{R}^n: \bar{w}\bar{x} = t_1\})$. So there exists a vector $w \in \mathbb{Z}^n$ with relatively prime components such that $H = \{x: wx = \alpha\}$ for some integer α. Furthermore, for any integer k the hyperplane $T^{-1}(\{(\bar{x}, 0) \in \mathbb{R}^n: \bar{w}\bar{x} = k\})$ contains integral vectors and so is of the form $\{x: wx = k'\}$ for some k'. Thus, the fact that $t_1 - t_2 < \Psi s^2$ implies that (3) holds. Now, by the first inequality in (4), we know that there does not exist an integer k such that $M \subseteq \{x: wx = k\}$. So $wx' \neq wx''$ for some x', x'' in M. This completes the proof of Claim 2.

Let $w \in \mathbb{Z}^n$ satisfy the conditions in Claim 2 and let $\alpha = \lfloor \max\{wx: Ax \leq b, Cx = d\} \rfloor$. Since $\{x: Ax \leq b, Cx = d\} \subseteq \{x: wx < \alpha + 1\}$, Farkas' lemma implies that $wx \leq \alpha$ can be derived from $(Ax \leq b,\ Cx \leq d,\ -Cx \leq -d)$. By rotating, via Lemma 2(i), we obtain an inequality $c'_r x \leq d'_r$ which can be derived from $(Ax \leq b,\ Cx \leq d)$, such that $\{x: Ax \leq b,\ Cx = d,\ c'_r x \leq d'_r\} \subseteq \{x: Ax \leq b,\ Cx = d,\ wx \leq \alpha\}$ and the greatest amongst the absolute values of the components of c'_r and d'_r is at most $n\sigma(A, b, C, d)$. It also follows from the rotation procedure that for any integer k,

$$\{x: Ax \leq b,\ Cx = d,\ c'_r x \leq d'_r - k\} \subseteq \{x: Ax \leq b,\ Cx = d,\ wx \leq \alpha - k\}.$$

So, by (3), $\{x: Ax \leq b,\ Cx = d\} \subseteq \{x: c'_r x > d'_r - \Psi(n - r + 1)^2\}$. Let $c''_r x \leq d''_r$ be obtained by summing $c'_r x \leq d'_r$ and the inequalities $Cx \leq d$, that is, $c''_r = c'_r + \mathbf{1}C$, $d''_r = d'_r + \mathbf{1}d$, where $\mathbf{1}$ is the vector of all 1's. We have that

$$\{x: Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r,\ c''_r x = d''_r\} = \{x: Ax \leq b,\ Cx = d,\ c'_r x = d'_r\}. \quad (5)$$

Notice that $\{x: Ax \leq b,\ Cx = d\} \subseteq \{x: c''_r x > d''_r - \Psi(n - r + 1)^2 + 1\}$ and that the numbers appearing in $c''_r x \leq d''_r$ have absolute value at most $2n\sigma(A, b, C, d)$.

Claim 3. There exists a cutting-plane proof of $c''_r x \leq d''_r - 1$ from $(Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r)$ of length at most $\Psi^{n-r} n^{2.5(n-r)} + 1$ which requires at most $n - r + 1$ inequalities besides $(Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r)$ to be stored at any one time and with each inequality in the proof having all coefficients of absolute value at most $\Psi^{n-r} n^{3(n-r)+2} \sigma(A, b, C, d)$.

Proof of Claim 3. By Claim 2, the dimension of $\{x: Ax \leq b,\ Cx = d,\ wx = \alpha\}$ is less than that of $\{x: Ax \leq b,\ Cx = d\}$ (since $wx' \neq wx''$ for some x', x'' in M). So $\{x: Ax \leq b,\ Cx = d,\ c'_r x = d'_r\}$ has dimension less than that of $\{x: Ax \leq b,\ Cx = d\}$ (by Lemma 2(ii)). Thus c'_r is not a linear combination of the rows of C. This implies that there exists an inequality $c_{r+1} x \leq d_{r+1}$ with $\{x: Ax \leq b,\ Cx = d,\ c'_r x = d'_r,\ c_{r+1} x \leq d_{r+1}\} = \emptyset$ and a cutting-plane proof of $c_{r+1} x \leq d_{r+1}$ from $(Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r)$ of length at most $\Psi^{n-r} n^{2.5(n-r)}$ which requires at most $n - r + 1$ inequalities besides $(Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r)$ to be stored at any one time and with each inequality in the proof having all coefficients of absolute value at most $\Psi^{n-r} n^{3(n-r)+1} n\sigma(A, b, C, d)$ (by the induction hypothesis). Now, by (5), we have

$$\{x: Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r,\ c''_r x = d''_r,\ c_{r+1} x \leq d_{r+1}\} = \emptyset$$

and hence $c''_r x \leq d''_r - 1$ may be derived from $(Ax \leq b,\ Cx \leq d,\ c'_r x \leq d'_r,\ c_{r+1} x \leq d_{r+1})$. This completes the proof of Claim 3.

If $\{x\colon Ax \leq b,\ Cx = d,\ c''_r x \leq d''_r - 1\} = \emptyset$ we are finished. Otherwise, arguing as in Claim 3, we can find a cutting-plane proof of $c'''_r x \leq d'''_r - 2$ from $(Ax \leq b,\ Cx \leq d,\ c''_r x \leq d''_r - 1)$, where $c'''_r x \leq d'''_r$ is obtained by summing $c''_r x \leq d''_r$ and the inequalities $Cx \leq d$. (Notice that each coefficient in $c'''_r x \leq d'''_r$ is of absolute value at most $3n\sigma(A, b, C, d)$.) Repeating this at most $\Psi(n-r+1)^2$ times, we obtain a cutting-plane proof of an inequality $c_r x \leq d_r$ from $(Ax \leq b,\ Cx \leq d)$ with $\{x\colon Ax \leq b,\ Cx = d,\ c_r x \leq d_r\} = \emptyset$.

The absolute values of the coefficients of $c_r x \leq d_r$ are at most $(\Psi(n-r+1)^2 + 1)\sigma(A, b, C, d)$. So the greatest absolute value amongst the coefficients of the inequalities in the cutting-plane proof is at most

$$\begin{aligned} &\Psi^{n-r} n^{3(n-r)+1}((\Psi(n-r+1)^2+1)\sigma(A, b, C, d)) \\ &\quad \leq \Psi^{n-r+1} n^{3(n-r+1)+1} \sigma(A, b, C, d). \end{aligned}$$

The length of the cutting-plane proof is at most

$$\Psi(n-r+1)^2(\Psi^{n-r} n^{2.5(n-r)} + 1) + 1 \leq \Psi^{n-r+1} n^{2.5(n-r+1)}.$$

Finally, the proof requires at most $n-r+2$ inequalities, besides $(Ax \leq b,\ Cx \leq d)$, to be stored at any one time. So (A) holds when $k = r-1$, which completes the proof of the theorem. □

Remarks. (a) For bounded polyhedra, this theorem without the restriction on the lengths of the proofs may also be derived from Chvátal's [3] technique, since, as observed by Coullard [8], the cutting-plane proofs given in [3] require only polynomial workspace. The restriction on the length does not follow in this way since the number of derived inequalities in these proofs depends on the least integer N such that $\{x\colon Ax \leq b\} \subseteq \{x\colon |x_i| \leq N,\ i = 1, \ldots, n\}$ and so may be arbitrarily high, even in the 2-dimensional case.

(b) Chvátal [5] defines cutting-plane proofs in general as a method for showing that every integral solution of $Ax \leq b$ satisfies another specified inequality $wx \leq \beta$, by requiring that the last inequality in the proof be $wx \leq \beta$, rather than $0x \leq -1$. Such a cutting-plane proof always exists if either $\{x\colon Ax \leq b\}$ is bounded, as shown by Chvátal [3], or if A and b are rational and $Ax \leq b$ has at least one integral solution, as shown by Schrijver [24]. The lengths of these proofs, even when the inequalities have only two variables, may necessarily be arbitrarily long (see the example of J.A. Bondy given in [3]). But, as the proof of our theorem only requires that $(Ax \leq b,\ wx = t)$ have no integral solution in order to obtain a cutting-plane proof of $wx \leq t-1$ from $(Ax \leq b,\ wx \leq t)$, if A and b are rational then in either Chvátal's case or Schrijver's case there exist proofs which can be carried out in polynomial workspace.

(c) It should be noted that Theorem 1 is of an existential nature—it states only that there exists such a cutting-plane proof and not how to go about finding it. For related algorithmic work we refer the reader to the paper of Kannan [16].

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