

THE DISCIPLINE NUMBER OF A GRAPH

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1. Introduction

The *domination number* $\gamma(G)$ of a graph G is the size of a smallest set D of vertices such that every vertex outside D has at least one neighbour in D ; Fink, Jacobson, Kinch, and Roberts [4] defined the *bondage number* $b(G)$ of a graph G as the least number of edges whose deletion makes $\gamma(G)$ increase. As we are about to point out, computing $b(G)$ amounts to solving an integer linear program.

Define a *whip* in a graph G as any spanning subgraph F of G such that each component of F is a star and F has precisely $\gamma(G)$ components; let $E(G)$ denote the set of edges of G and let $W(G)$ denote the set of all whips in G . Obviously, $b(G)$ is the optimal value of the problem

$$\begin{aligned} & \text{minimize } \sum \{x_e : e \in E(G)\} \\ & \text{subject to } \sum \{x_e : e \in E(F)\} \geq 1 \quad \text{for all } F \text{ in } W(G), \\ & \quad x_e \geq 0 \quad \quad \quad \text{for all } e \text{ in } E(G). \\ & \quad x_e = \text{integer} \quad \quad \text{for all } e \text{ in } E(G). \end{aligned} \tag{1}$$

By the *fractional bondage number* $b^*(G)$ we shall mean the optimal value of the 'linear programming relaxation' of (1),

$$\begin{aligned} & \text{minimize } \sum \{x_e : e \in E(G)\} \\ & \text{subject to } \sum \{x_e : e \in E(F)\} \geq 1 \quad \text{for all } F \text{ in } W(G), \\ & \quad x_e \geq 0 \quad \quad \quad \text{for all } e \text{ in } E(G). \end{aligned} \tag{2}$$

By the duality theorem of linear programming, $b^*(G)$ equals the optimal value of

the dual of (2),

$$\begin{aligned} & \text{maximize } \sum \{y_F: F \in W(G)\} \\ & \text{subject to } \sum \{y_F: e \in E(F)\} \leq 1 \quad \text{for all } e \text{ in } E(G), \\ & \quad y_F \geq 0 \quad \quad \quad \text{for all } F \text{ in } W(G). \end{aligned} \quad (3)$$

Since (3) can be seen as the linear programming relaxation of

$$\begin{aligned} & \text{maximize } \sum \{y_F: F \in W(G)\} \\ & \text{subject to } \sum \{y_F: e \in E(F)\} \leq 1 \quad \text{for all } e \text{ in } E(G), \\ & \quad y_F \geq 0 \quad \quad \quad \text{for all } F \text{ in } W(G), \\ & \quad y_F = \text{integer} \quad \quad \quad \text{for all } F \text{ in } W(G), \end{aligned} \quad (4)$$

problems (1) and (4) are in a sense dual. Therefore we refer to the optimal value of (4) as the *discipline number* $\text{dis}(G)$ of G .

We have

$$1 \leq \text{dis}(G) \leq b^*(G) \leq b(G) \quad (5)$$

for all graphs G . Apart from establishing upper bounds on $b(G)$, Fink et al. computed the bondage number of cycles, paths, and complete multipartite graphs and studied the bondage number of trees (several of these results can also be found in Bauer, Harary, Nieminen, and Suffel [1]). The purpose of this paper is to provide ties with analogous results for the fractional bondage number and for the discipline number.

2. The fractional bondage number

The principle restraining device of this section goes as follows.

Theorem 1. *If G has n vertices and m edges then $b^*(G) \leq m/(n - \gamma(G))$.*

Proof. Observe that the constraints of (2) are satisfied by $x_e = 1/(n - \gamma(G))$ for all e . \square

As usual, let $\Delta(G)$ denote the largest degree of a vertex in G . Fink et al. conjectured that $b(G) \leq \Delta(G) + 1$.

Theorem 2. $b^*(G) \leq \Delta(G)$.

Proof. Consider any maximal set S of pairwise nonadjacent vertices: trivially, every vertex outside S has at least one neighbour in S and the number of edges in G is at most the sum of the degrees of all the vertices outside S . Hence the desired conclusion follows from Theorem 1: we have $\gamma(G) \leq |S|$ and $m \leq \Delta(G)(n - |S|)$. \square

Let C_n denote the cycle with n vertices. Fink et al. proved that $b(C_n) = 3$ if $n \equiv 1 \pmod{3}$, and $b(C_n) = 2$ otherwise. Now we shall prove a theorem that includes a formula for $b^*(C_n)$ as a special case. Recall that a graph G is called *edge-transitive* if for every choice of its edges e_1, e_2 some automorphism of G sends e_1 onto e_2 .

Theorem 3. *If G is edge-transitive with n vertices and m edges then $b^*(G) = m/(n - \gamma(G))$.*

Proof. Since G is edge-transitive, (2) has an optimal solution with all x_e equal to each other. Hence $b^*(G)$ is the optimal value of the problem

$$\text{minimize } mx \quad \text{subject to } (n - \gamma(G))x \geq 1, x \geq 0. \quad \square$$

Since C_n is edge-transitive and $\gamma(C_n) = \lceil n/3 \rceil$, Theorem 3 yields $b^*(C_n) = n/\lfloor 2n/3 \rfloor$.

Let P_n denote the path with n vertices. Fink et al. proved that $b(P_n) = 2$ if $n \equiv 1 \pmod{3}$, and $b(P_n) = 1$ otherwise.

Theorem 4. *$b^*(P_n) = \frac{3}{2}$ if $n \equiv 1 \pmod{3}$, and $b^*(P_n) = 1$ otherwise.*

Proof. We may assume that $n = 3k + 1$, for otherwise the desired conclusion follows from (5). Since $\gamma(P_{3k+1}) = k + 1$, Theorem 1 guarantees that $b^*(P_{3k+1}) \leq \frac{3}{2}$; to prove the reversed inequality, we only need exhibit a feasible solution of (3) in which precisely three variables have value $\frac{1}{2}$. To put it differently, we only need to find three whips in P_{3k+1} so that each edge belongs to precisely two of the three whips. For this purpose, label the edges of P_{3k+1} as e_1, e_2, \dots, e_m in such a way that e_i and e_{i+1} share an endpoint whenever $1 \leq i \leq m - 1$. Now the j th whip arises by deleting all the edges e_i with $i \equiv j \pmod{3}$. \square

In dealing with complete multipartite graphs, we shall distinguish between those having a positive number k of classes of size one and those in which all classes have size at least two. For the first kind, Fink et al. proved that the bondage number equals $\lceil k/2 \rceil$.

Theorem 5. *Let G have n vertices and let precisely k vertices of G have degree $n - 1$. If $k = 1$ then $b^*(G) = 1$; if $k \geq 2$ then $b^*(G) = k/2$.*

Proof. We may assume that $k \geq 2$, for otherwise the desired conclusion follows from (5). Setting $x_e = 1/(k-1)$ if both endpoints of e have degree $n-1$, and $x_e = 0$ otherwise, we obtain a feasible solution of (2); hence $b^*(G) \leq k/2$. On the other hand, there are precisely k whips; setting $y_F = \frac{1}{2}$ for all of them, we obtain a feasible solution of (3); hence $b^*(G) \geq k/2$. \square

For complete multipartite graphs G with class sizes n_1, n_2, \dots, n_t such that $2 \leq n_1 \leq n_2 \leq \dots \leq n_t$, Fink et al. proved that $b(G) = n - n_t$ unless $n_1 = n_2 = \dots = n_t = 2$, in which case $b(G) = n - 1$.

Theorem 6. Let G be a complete multipartite graph with n vertices and m edges. If all classes of G have size at least two then $b^*(G) = m/(n-2)$.

Proof. Theorem 1 guarantees that $b^*(G) \leq m/(n-2)$; to prove the reversed inequality, we shall exhibit an appropriate feasible solution of (3). For this purpose, let S_1, S_2, \dots, S_t denote the classes of G ; write $n_k = |S_k|$. By a *center* of a star, we shall mean a vertex in the star adjacent to all the other vertices in the star (unless the star has precisely two vertices, its center is uniquely determined); by a *pointed whip*, we shall mean a whip with a center distinguished in each of the two components; the pointed whip is of *type* (i, j) if its two centers belong to S_i and S_j . Clearly, there are precisely

$$n_i n_j 2^{n-(n_i+n_j)}$$

pointed whips of type (i, j) ; each edge with one endpoint in S_i and the other endpoint in S_j belongs to precisely

$$(n_i + n_j - 2) 2^{n-(n_i+n_j)}$$

pointed whips of type (i, j) , to precisely

$$n_k 2^{n-(n_i+n_k)-1}$$

pointed whips of type (i, k) with $k \neq i, j$, and to precisely

$$n_k 2^{n-(n_j+n_k)-1}$$

pointed whips of type (k, j) with $k \neq i, j$. Hence the desired feasible solution of (3) can be obtained by setting first

$$z_H = \frac{1}{n-2} 2^{n_i+n_j-n}$$

for every pointed whip H of type (i, j) , and then

$$y_F = \sum z_H$$

with the summation running through all pointed whips H such that $E(H) = E(F)$. \square

Fink et al. proved that $b(T) \leq 2$ for every tree T .

Theorem 7. $b^*(T) \leq (n-1)/\lceil n/2 \rceil$ for every tree T with n vertices.

Proof. As S.T. Hedetniemi pointed out to us, Theorem 13.1.3 in Ore's book [5] implies that $\gamma(G) \leq \lceil n/2 \rceil$ for every graph without isolated vertices; the rest follows from Theorem 1. \square

To show that the bound of Theorem 7 cannot be improved (at least not for even values of n), consider the tree with vertices u_i, v_i ($1 \leq i \leq k$) and edges $u_i u_{i+1}$ ($1 \leq i \leq k-1$), $u_i v_i$ ($1 \leq i \leq k$). We shall refer to any such tree as a *Justine* [7]. (One of the referees pointed out that the same trees have been called *combs* by Fink et al. [3]. However, *combs* is also the name of graphs used by Padberg and Rinaldi [6] in solving a traveling salesman problem. To avoid confusion, we prefer the descriptive and unambiguous term *Justine*.)

Theorem 8. $b^*(T) = 2(n-1)/n$ for the *Justine* T with n vertices.

Proof. By virtue of Theorem 7, we only need prove that $b^*(T) \geq 2(n-1)/n$; to do this we only need exhibit a feasible solution of (3) in which precisely $n-1$ variables have value $2/n$. To put it differently, we only need find whips $F_1, F_2, \dots, F_{2k-1}$ in a *Justine* with $2k$ vertices so that each edge belongs to precisely k of these whips. We propose to do so by induction on k . The case of $k=1$ is trivial; now assume that appropriate whips $F_1, F_2, \dots, F_{2k-3}$ have been found in the *Justine* with $2k-2$ vertices. Without loss of generality, assume that F_1, F_2, \dots, F_{k-2} do not include the edge $u_{k-2} u_{k-1}$. Next, observe that each of these $k-2$ whips must include the edge $u_{k-1} v_{k-1}$. Extend each F_i with $1 \leq i \leq k-2$ by adding the edge $u_{k-1} u_k$ and extend each F_i with $k-1 \leq i \leq 2k-3$ by adding the edge $u_k v_k$. Finally, let F_{2k-2} consist of all $u_i v_i$ with i odd, all $u_i u_{i+1}$ with i even and less than k , and $u_{k-1} u_k$. Let F_{2k-1} consist of all $u_i v_i$ with i even, all $u_i u_{i+1}$ with i odd and less than k , and $u_{k-1} u_k$. \square

3. The discipline number

Theorem 3 combined with (5) implies that $\text{dis}(C_n) = 1$ whenever $n \geq 5$; Theorem 7 combined with (5) implies that $\text{dis}(T) = 1$ for every tree T ; in addition, it is easy to see that $\text{dis}(G) = 1$ whenever $\gamma(G) = 1$. However, we are about to show that $\text{dis}(G)$ can be arbitrarily large even when $\gamma(G) = 2$.

Theorem 9. Let G be a complete multipartite graph with no classes of size one, a classes of size two, and b classes of size at least three. If $a + \lfloor b/2 \rfloor \geq 3$, then

$$\text{dis}(G) = a + \lfloor b/2 \rfloor.$$

If $(a, b) = (0, 4), (0, 5)$ or $(1, 3)$, then $\text{dis}(G) = 3$. If $(a, b) = (0, 3), (1, 1), (1, 2), (2, 0)$ or $(2, 1)$, then $\text{dis}(G) = 2$. If $(a, b) = (0, 2)$, then $\text{dis}(G) = 1$.

Proof. Enumerate the classes of G as S_1, S_2, \dots, S_{a+b} so that $|S_i| = 2$ whenever $1 \leq i \leq a$ and $|S_i| \geq 3$ whenever $a+1 \leq i \leq a+b$.

Claim 1. $\text{dis}(G) \geq a + \lfloor b/2 \rfloor$.

Proof of Claim 1. Write $S_i = \{u_i, v_i\}$ for $i = 1, 2, \dots, a$ and choose vertices u_{a+j}, v_{a+j} with $1 \leq j \leq \lfloor b/2 \rfloor$ so that $u_{a+j} \in S_{a+2j-1}, v_{a+j} \in S_{a+2j}$. For every choice of i and j such that $1 \leq i < j \leq a + \lfloor b/2 \rfloor$, set

$$u_i u_j \in F_i, \quad v_i v_j \in F_i, \quad u_i v_j \in F_j, \quad u_j v_i \in F_j.$$

For all the remaining vertices w , set $wu_i \in F_i$ if w and v_i belong to the same S_k , and $wv_i \in F_i$ otherwise. \square

Claim 2. If $a + b \geq 4$, then $\text{dis}(G) \geq 3$.

Proof of Claim 2. Choose vertices u_1, u_2, u_3 so that $u_i \in S_i$ and choose a vertex x in S_4 . Set

$$u_1 u_2 \in F_1, \quad u_2 u_3 \in F_2, \quad u_3 u_1 \in F_3, \quad u_1 x \in F_2, \quad u_2 x \in F_3, \quad u_3 x \in F_1.$$

For all the remaining vertices w , set $wx \in F_i$ if $w \in S_i$ and $wu_i \in F_i$ otherwise. \square

Claim 3. If $a \geq 1$, then $\text{dis}(G) \geq 2$.

Proof of Claim 3. Write $S_1 = \{x_1, x_2\}$. For every vertex w outside S_1 , set $wx_1 \in F_1$ and $wx_2 \in F_2$. \square

These three claims guarantee that the values stated in Theorem 9 provide correct lower bounds on $\text{dis}(G)$; now we shall establish the upper bounds. For this purpose, consider arbitrary pairwise edge-disjoint whips F_1, F_2, \dots, F_k in G . For each $i = 1, 2, \dots, k$, choose vertices u_i, v_i that are centers of the two components of F_i . Write

$$Q = \{u_1, v_1, u_2, v_2, \dots, u_k, v_k\}.$$

Claim 4. If $|Q| = 2k$, then $k \leq a + \lfloor b/2 \rfloor$.

Proof of Claim 4. Consider the graph H whose set of vertices is Q , two vertices being adjacent in H if and only if they are adjacent in some F_i . Since no u_i is adjacent to v_i in H , all the remaining pairs of vertices must be adjacent in H : we have $\binom{2k}{2} - k = k(2k - 2)$ and each F_i contributes $2k - 2$ edges to H .

Now call an S_j *special* if it includes at least two vertices from Q . As we have just observed, each special S_j includes some u_i and v_i and it includes no other vertices from Q ; since each vertex outside Q is adjacent to at least one of u_i and v_i , we must have $|S_j| = 2$. It follows that $|Q| \leq 2a + b$. \square

Claim 5. If $k \geq 3$, then $|Q| \geq 4$.

Proof of Claim 5. Assume the contrary: $k \geq 3$ but $|Q| \leq 3$. Since G has at least four vertices, some vertex w lies outside Q ; since F_1, F_2, \dots, F_k are edge-disjoint, w is adjacent to at least k distinct vertices in Q . Hence $|Q| = k = 3$. Now no S_j can include a vertex from Q and a vertex w outside Q (w has to be adjacent to at least three distinct vertices in Q); since $|S_j| \geq 2$ for all j , it follows that $Q = S_j$ for some j . Finally, this S_j includes some vertex w distinct from u_1 and v_1 , a contradiction: w must be adjacent to at least one of u_1 and v_1 . \square

Claim 6. If $k \geq 4$, then $k \leq a + \lfloor b/2 \rfloor$.

Proof of Claim 6. By virtue of Claim 4, we only need show that $|Q| = 2k$. For this purpose, assume the contrary: without loss of generality $u_1 = u_2$. Write

$$Q_0 = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}$$

and consider the graph H_0 whose set of vertices is Q_0 , two vertices being adjacent in H_0 if and only if they are adjacent in some F_i with $1 \leq i \leq 4$. Since each F_i with $1 \leq i \leq 4$ contributes $|Q_0| - 2$ edges to H_0 , we have

$$4(|Q_0| - 2) \leq \binom{|Q_0|}{2};$$

observing that $|Q_0| \leq 7$ (since $u_1 = u_2$) and $|Q_0| \geq 4$ (by Claim 5), we conclude that $|Q_0| = 7$. Now H_0 has twenty edges, which is a contradiction: $\binom{7}{2} = 21$ and no u_i with $2 \leq i \leq 4$ is adjacent to v_i in H_0 . \square

Claim 7. If $k = 3$, then $k \leq a + \lfloor b/2 \rfloor$ or $a + b \geq 4$.

Proof of Claim 7. Claim 4 allows us to assume that $|Q| \leq 5$; Claim 5 guarantees that $|Q| \geq 4$. Defining H as in the proof of Claim 4, observe that H has $3(|Q| - 2)$ edges. It follows that H (and hence also G) contains four pairwise adjacent vertices. \square

Claim 8. If $a = 0$ and $b = 2$, then $k \neq 2$.

Proof of Claim 8. Assume the contrary: $k = 2$ but $a = 0$ and $b = 2$. Claim 4 implies that $|Q| \leq 3$ and so, without loss of generality, $u_1 = u_2 \in S_1$. Since S_1 includes a vertex distinct from both u_1, v_1 but adjacent to at least one of them, we must have $v_1 \in S_2$; a symmetric argument shows that $v_2 \in S_2$. But then S_2 includes a vertex outside Q and adjacent to only one vertex in Q , a contradiction. \square

This ties down the proof of Theorem 9. \square

The reader interested in additional results in a similar vein is directed to [2, Chapter 5].

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