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Integral infeasibility and testing total dual integrality

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A number of well known results in combinatorial optimization, such as Hoffman's circulation theorem and the matching theorems of Hall and Tutte, can be interpreted as stating that either a certain linear system has a solution or there exists a simple combinatorial reason why it is infeasible. We give a characterization of total dual integrality in terms of such infeasibility results. This leads to a method for testing total dual integrality which is tractable for small linear systems. In particular, a computer implementation of the method settled a conjecture of Barahona and Mahjoub concerning feedback sets in directed graphs.

polytedra * integer programming * graph theory

1. Introduction

A rational linear system $Ax \leqslant b$ is called *totally dual integral* when the minimum in the linear programming duality equation

$$\max\{wx:\ Ax \leqslant b\} = \min\{yb:\ yA = w,\ y \geqslant 0\} \tag{1}$$

can be achieved by an integral solution for all integral vectors w for which the optima exist. Hoffman [11] and Edmonds and Giles [6] showed that if $AX \leqslant b$ is totally dual integral and b is integral, then the maximum in (1) can also be achieved by an integral solution for all vectors w for which the optima exist. Total dual integrality is a natural framework for the study of min-max relations in combinatorial optimization.

It is often the case that a class of totally dual integral systems is associated with a well known theorem stating that if a given system has no solution, then this can be explained by a simple combinatorial argument. Examples of such results are Hoffman's circulation theorem [11], the matching theorems of Hall and Tutte [14], and Frank's submodular-flow theorem [8]. In the next section, we give a characterization of total dual integrality in terms of such 'integral infeasibility' results. Then in Section 3 we show how this characterization can be used to improve the test for total dual integrality given in Cook, Lovász, and Schrijver [5], and in Section 4 we report on the solution of a problem of Barahona and Mahjoub concerning feedback sets in directed graphs.

2. Integral infeasibility

For a system $Ax \leqslant b$ of m linear inequalities and a set $T \subseteq \{1, \ldots, m\}$, we let

$$A_T x = b_T, \quad A_{\bar{T}} x \leqslant b_{\bar{T}} \tag{2}$$

denote the system obtained by setting each inequality in T to equality while keeping each inequality in $\bar{T}=\{1,\ldots,m\}\setminus T$ as an inequality. Suppose (2) has no solution for some specified set T. Although it may be far from obvious that this system really does have no solution, with the help of Farkas' lemma this fact can be readily verified. Indeed, by exhibiting a vector $(y_T, y_{\bar{T}})$ such that

$$\begin{aligned} &y_T b_T + y_{\bar{T}} b_{\bar{T}} < 0,\\ &y_T A_T + y_{\bar{T}} A_{\bar{T}} = 0,\\ &y_{\bar{T}} \geqslant 0, \end{aligned} \tag{3}$$

we prove the infeasibility of the system. Due to the equality constraints, the vector y_T might necessarily have some negative components. Notice, however, that we may scale $(y_T, y_{\bar{T}})$ so that it satisfies

$$\begin{aligned} &y_T b_T + y_{\bar{T}} b_{\bar{T}} < 0,\\ &y_T A_T + y_{\bar{T}} A_{\bar{T}} = 0,\\ &y_T \geqslant -1, \quad y_{\bar{T}} \geqslant 0. \end{aligned} \tag{4}$$

Now (3) necessarily has an integral solution (since it has a rational solution), but (4), although solvable, may not be solvable in integers. We say that the infeasibility of (2) can be *proven integrally* if (4) does in fact have an integral solution. (Note that (4) slightly generalizes the classic form of integral infeasibility since y_T can be negative.)

A set of vectors $\{h_1,\ldots,h_k\}$ is called a *Hilbert basis* if each integral vector in the cone $\{\lambda_1 h_1 + \cdots + \lambda_k h_k: \lambda_i \geqslant 0, i=1,\ldots,k\}$ can be written as a nonnegative integral combination of $h_1,\ldots,h_k$ (see Giles and Pulleyblank [9]).

Theorem 1. *Let A be an integral matrix and b a rational vector such that the linear system $Ax \leqslant b$ has at least one solution. Then $Ax \leqslant b$ is totally dual integral if and only if*

(i) *the rows of A form a Hilbert basis and*

(ii) *for each subset T of inequalities from $Ax \leqslant b$, if (2) is infeasible, then this can be proven integrally.*

Proof. Let $Ax \leqslant b$ be a totally dual integral system which has at least one solution. The validity of (i) follows directly from the definition of total dual integrality. To check (ii), suppose (2) has so solution for some specified set T. We must show that the dual system (4) has an integral solution. To this end, let $(y_T^*, y_{\bar{T}}^*)$ be an integral optimal solution to the linear programming problem

$$\min\{y_T b_T + y_{\bar{T}} b_{\bar{T}}: y_T A_T + y_{\bar{T}} A_{\bar{T}} = 1 \cdot A_T,\ y_T \geqslant 0,\ y_{\bar{T}} \geqslant 0\}, \tag{5}$$

where $1 \cdot A_T$ denotes the vector obtained by summing the rows in A_T. We claim that $(y_T^* - 1, y_{\bar{T}}^*)$ is a solution to (4). Indeed, by the choice of right-hand-side, we only need to prove that

$$(y_T^* - 1) b_T + y_{\bar{T}}^* b_{\bar{T}} < 0,$$

and this can be seen as follows: Letting $(\bar{y}_T, \bar{y}_{\bar{T}})$ be a solution to (4) (which exists, via Farkas' lemma) we have that $(\bar{y}_T + 1, \bar{y}_{\bar{T}})$ is a feasible solution to (5). Therefore,

$$(y_T^* - 1) b_T + y_{\bar{T}}^* b_{\bar{T}} \leqslant \bar{y}_T b_T + \bar{y}_{\bar{T}} b_{\bar{T}} < 0.$$

For the other direction, suppose $Ax \leqslant b$ has a solution and (i) and (ii) hold. Let w be an integral vector such that the optima in (1) exist. We need to show that

$$\min\{yb: yA = w,\ y \geqslant 0\} \tag{6}$$

has an integral optimal solution. So let y^* be an integral solution to (6) having y^*b as small as possible. (By condition (i), we know such an integral solution exists.) Now let T consist of those inequalities $a_i x \leqslant b_i$ such that y_i^* is positive. If the system $A_T x = b_T$, $A_{\bar{T}} \leqslant b_{\bar{T}}$ is feasible, then by complementary slackness y^* is an optimal solution. Otherwise, by (ii), there exists an integral solution $\bar{y}$ to (4). But since $y_T^* > 0$, $y^* + \bar{y}$ is a solution to (6) and $(y^* + \bar{y})b < y^*b$, a contradiction. □

This characterization is used by Ervolina and McCormick [7] to give a new proof of the total dual integrality of submodular-flow systems. When condition (ii) is satisfied, they say that $Ax \leqslant b$ has an *Integral Infeasibility Theorem*.

3. Testing for total dual integrality

A polynomial-time test for total dual integrality in fixed dimension is given in Cook, Lovász and Schrijver [5] (see also Chandrasekaran and Shirali [4]). The core of the algorithm is a method for testing if a given set of integer vectors is a Hilbert basis. The general problem is easily reduced to this by the observation that a system $Ax \leqslant b$ is totally dual integral if and only if for each minimal face F of $\{x: Ax \leqslant b\}$, the set of active rows of A for F (those for which $a_i x = b_i$ for all $x \in F$), form a Hilbert basis. (This geometric interpretation of total dual integrality was first used in Giles and Pulleyblank [9].) The routine which tests for a Hilbert basis makes repeated use of Lenstra's integer programming algorithm [13] (the number of times is an exponential function of the dimension). Therefore, to obtain a practical method for testing small linear systems it is important to avoid Hilbert basis tests whenever possible. The following consequence of the proof of Theorem 1 helps in certain cases:

Theorem 2. *Let A be an $m \times n$ integral matrix and b a rational vector such that $Ax \leqslant b$ has at least one solution. Then $Ax \leqslant b$ is totally dual integral if and only if*

(i) *the rows of A form a Hilbert basis, and*

(ii) *for each subset T of at most n inequalities from $Ax \leqslant b$, the linear programming problem*

$$\min\{yb:\ yA = 1 \cdot A_T,\ y \geqslant 0\}$$

has an integral optimal solution. □

The improvement comes about in two ways. First, integer programming problems of the form

$$\min\{yb:\ yA = w,\ y \geqslant 0\}$$

for a totally dual integral system $Ax \leqslant b$ (with A integral) and integral vector w can be solved in polynomial time (even for varying dimension) by a sequence of linear programming problems and linear diophantine equation problems (Chandrasekaran [3], see also Schrijver [16]). Second, in some cases condition (i) can be checked without resorting to the method of Cook, Lovász and Schrijver.

Example 1. If $\{x: Ax \leqslant b\}$ is bounded, then (i) can be verified by checking that, for each $j = 1, \ldots, m$, both the j-th unit vector and its negative can be written as a nonnegative integer combination of the row of A. This can again be done in polynomial time (even for varying dimension) by the algorithm of Chandrasekaran.

Example 2. If the system is of the form $Ax \geqslant b$, $x \geqslant 0$, with A nonnegative, then (i) is trivially satisfied. Furthermore, in this case the algorithm of Chandrasekaran reduces to just a sequence of linear programming problems (see Schrijver [16]).

4. Feedback sets

Barahona and Mahjoub (see [1], as well as Jünger [12]) posed the problem of determining whether or not the linear system

$$\begin{aligned} &\textstyle\sum\{x_e:\ e \in C\} \geqslant 1 \\ &\quad \text{for each directed circuit } C \text{ of } D, \\ &x_e \geqslant 0 \quad \text{for each arc } e \text{ of } D, \end{aligned} \tag{7}$$

is totally dual integral for D_5, the complete symmetric directed graph on 5 nodes (that is, the directed graph having nodes $1, \ldots, 5$ and arcs ij and ji for all $i \neq j$). As described below, a computer implementation of the check for condition (ii) in Theorem 2 (for systems of the form given in Example 2) showed that the system is indeed totally dual integral.

A 0–1 solution of (7) corresponds to a subset of arcs which meets every directed circuit in the directed graph D. Such a subset of arcs is called a *feedback set*. An elegant theorem of Lucchesi and Younger [15] states that in a planar directed graph the minimum number of arcs in a feedback set is equal to the maximum number of pairwise arc-disjoint directed circuits. The motivation for proving that (7) is totally dual integral for D_5 is that it provides the last step in an extension of the Lucchesi–Younger theorem to the class of directed graphs having no $K_{3,3}$ minor (see Barahona and Mahjoub [1]).

Note that D_5 has 84 directed circuits, which leads to far too many inequalities to be able to check condition (ii) directly. We need the following lemma of Barahona and Mahjoub [1] to cut down the size of D_5:

Lemma 3. *Suppose that D is a directed graph with arcs ij and ji, that D_1 is D with ji deleted, and that D_2 is D with ij deleted. Then if (7) is totally dual integral for both D_1 and D_2, it is totally dual integral for D.*

Proof. Let

$$w = \{w_e:\ e \text{ an arc of } D\}$$

be an integral objective vector for (7). We can assume that $w_{ij} \geqslant w_{ji} > 0$. Now solve the problem on D_1 with $w^1_{ij} = w_{ij} - w_{ji}$. If ij is *not* in the optimal feedback set in D_1, then add ji to get a feedback set for D; the primal objective increases by w_{ji}, but we can feasibly increase the dual objective by setting the dual variable on the two-arc circuit using ij and ji to w_{ji}. If ij is the optimal feedback set in D_1, then the same feedback set is feasible for D (the only uncovered circuits must use ji; but ij being in a minimal feedback set in D_1 implies that there is a directed circuit C covered only by ij; now C together with the uncovered ji circuit gives an uncovered directed walk in D_1, a contradiction). Once again, the primal objective goes up by w_{ji}, but we again compensate by setting the dual variable on the two-arc circuit to w_{ij}. □

Thus (7) is totally dual integral for D_5 if and only if it is totally dual integral for each orientation of K_5 (that is, for each directed graph obtained from the complete undirected graph on 5 nodes by orienting its arcs). This reduces the number of directed circuits from 84 to a maximum of 12 for any orientation of K_5.

To reduce the number of orientations of K_5 that need to be checked we make use of the following lemma of Barahona and Mahjoub:

Lemma 4. *Let D be an orientation of K_5. Then if either some node of D meets all directed circuits or some arc is in no directed circuit,* (7) *is totally dual integral for D.*

Proof. The first case follows from the max flow-min cut theorem, while the second case follows from the Lucchesi–Younger theorem. □

Checking for isomorphisms, we determined that there are only 3 distinct orientations of K_5 that cannot be eliminated by Lemma 4. These 3 directed graphs have 12, 10, and 9 directed circuits respectively, and are given in Table 1 where a 0 indicates that the arc is directed from i to j and a 1 indicates j to i.

We checked the total dual integrality of the systems on a Sun 3-60 workstation, using R.E. Bixby's LOPT 3.0 linear programming package [2] to solve the linear programming problems that arose. For each of the three orientations, and for each subset T of inequalities in (7), the linear program had an integral optimal solution, thus proving Barahona and Mahjoub's conjecture.

The times for the three runs, in hours: minutes: seconds, were

$D(12)$ 14:33:21,

$D(10)$ 2:52:07,

$D(9)$ 1:06:03.

In the computations we took advantage of the fact that if we encounter a linear programming problem where one or more of the variables has a coefficient of 0 in the objective function, then the dual problem has an integral optimal solution. (This follows from Lemma 4.) This type of savings is typical for systems involving problems that are minimal in the sense that any smaller problem (of the given type) is known to be totally dual integral.

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Table 1

	12	13	14	15	23	24	25	34	35	45
$D(12)$	0	0	1	1	0	0	1	0	0	0
$D(10)$	0	0	0	1	0	1	0	0	0	0
$D(9)$	0	0	0	1	0	1	0	0	1	0

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