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A NOTE ON MATCHINGS AND SEPARABILITY

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It is observed that minimal totally dual integral systems for a class of independence systems are closely related to a type of separability. This observation is used to give a short proof, using Edmonds' matching polytope theorem, of a theorem of Cunningham and Marsh which gives a description of the minimal totally dual integral defining system for the matching polytope and also to show the relationship between that result and a theorem of F.R. Giles.

1. Introduction

Combinatorial optimisation problems are often of the form "maximise wx over all vectors x in S" where S is a set of integer vectors in $\mathbb{Q}^n$ and $w \in \mathbb{Q}^n$ is an integer 'weight' vector. To solve such a problem in polynomial time it is necessary to have a good characterisation for this maximum value. One method for obtaining such a good characterisation is to find a system of linear inequalities which defines the convex hull of S (that is, the convex hull of S is identical to the set of solutions to the linear system) – the duality theorem of linear programming then gives a min–max relation, and hence a good characterisation, for the problem (see Pulleyblank [16]). In fact, Grötschel, Lovász, and Schrijver [11] have shown that in many cases finding such a system of linear inequalities also gives a polynomial time algorithm for the optimisation problem.

One way to strengthen a min–max relation obtained in the above way is to require that the variables in the corresponding dual linear program take on integer values in an optimal solution, that is, to require that the defining system for the convex hull of S be a totally dual integral system (Edmonds and Giles [8] defined a rational linear system $Ax \leq b$ to be a *totally dual integral* system if the dual linear program $\min\{yb: yA = w, y \geq 0\}$ has an integral optimal solution for each integer vector w for which the optimum exists). Further motivation for searching for such a totally dual integral system is provided by the fact that often integer solutions to the dual linear program correspond to combinatorial objects such as 'coverings' or 'cuts', in which case the min–max relation gives a nice combinatorial theorem (see Schrijver [20,21]).

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Once such a totally dual integral system is found, the min–max relation can be further strengthened by removing some of the inequalities to obtain a minimal totally dual integral defining system for the convex hull of S (here 'minimal' means that if any inequality in the linear system is removed, the resulting system is either no longer totally dual integral or else no longer defines the convex hull of S). Schrijver [8] has shown that for each polyhedron P of full dimension there exists a unique minimal totally dual integral system $Ax \le b$ with A integral such that $Ax \le b$ defines P. Define the *Schrijver system* for P to be this unique system $Ax \le b$. Thus, if the convex hull of S is of full dimension, a 'best possible' min–max relation for S can be obtained by finding the Schrijver system for S.

In this note, it is observed that Schrijver systems for a class of general independence systems are closely related to a type of separability. This observation is used to give a short proof, using Edmonds' matching polytope theorem, of a result of Cunningham and Marsh [4] which gives a characterisation of the Schrijver system of the matchings of a graph and also to show the relationship between that result and a theorem of F.R. Giles. This observation is also used extensively in Cook and Pulleyblank [2].

2. Schrijver systems and separability

Let E be a finite set and let I be a finite set of nonnegative integer vectors $a = (a_e : e \in E)$. The pair (E, I) is a *general independence system* if $0 \in I$ and for each $a \in I$ and nonnegative integral $b \le a$ it is the case that $b \in I$ (so a general independence system with the property that each $a \in I$ is 0, 1-valued is an independence system). The *rank*, $r(A)$, of a set $A \subseteq E$ is the maximum value of $x(A)$ over all vectors $x \in I$, where $x(A) = \sum \{x_e : e \in A\}$. A set $A \subseteq E$ is *closed* if $r(A \cup \{e\}) > r(A)$ for each $e \in E - A$. A *separation* of a set $A \subseteq E$ is a pair of nonempty subsets A_1, A_2 of A such that $A_1 \cup A_2 = A$ and $r(A_1) + r(A_2) = r(A)$. If there exists a separation of $A \subseteq E$, then A is *separable* (otherwise A is *nonseparable*). Let $C(I)$ denote the convex hull of I. A characterisation of the Schrijver system for $C(I)$ for a class of general independence systems is given in the following lemma.

Lemma 2.1. *Let (E, I) be a general independence system such that $r(\{e\}) \ge 1$ for each $e \in E$. Suppose that the linear system*

$$\begin{aligned} x(A) &\le r(A) \quad \forall A \subseteq E,\ A \ne \emptyset, \\ x_e &\ge 0 \qquad \forall e \in E \end{aligned} \tag{2.1}$$

is a totally dual integral defining system for $C(I)$. An inequality $x(A) \le r(A)$ is in the Schrijver system for $C(I)$ if and only if $A \ne \emptyset$ is a closed non-separable set.

Proof. Since $r(\{e\}) \ge 1$ for each $e \in E$, the polyhedron $C(I)$ is of full dimension. If

$x(A) \le r(A)$ is in the Schrijver system for $C(I)$, then clearly A is nonseparable and closed. Conversely, suppose that $D \neq \emptyset$ is a nonseparable closed set. By assumption, for each integral w the linear program

$$\begin{aligned} &\min \quad \sum \{r(A) Y_A : A \subseteq E,\ A \neq \emptyset\}, \\ &\text{subject to} \quad \sum \{Y_A : A \subseteq E,\ e \in A\} \ge w_e \quad \forall e \in E, \\ &\qquad\qquad\qquad\qquad Y_A \ge 0 \quad \forall A \subseteq E,\ A \neq \emptyset \end{aligned} \tag{2.2}$$

has an integral optimal solution. To prove that $x(D) \le r(D)$ is in the Schrijver system for $C(I)$ it suffices to show that for some integer vector w the linear program (2.2) has no integer optimal solution with $Y_D = 0$. Let $w_e = 1$ for each $e \in D$ and let $w_e = 0$ for each $e \in E - D$. An optimal solution to (2.2) is $\bar{Y}_D = 1$ and $\bar{Y}_B = 0$ for all other $B \subseteq E$, $B \neq \emptyset$, with objective value $r(D)$. Any integral optimal solution to (2.2) with $Y_D = 0$ corresponds to a collection of nonempty sets $A_1, \ldots, A_j \subseteq E$ with $A_i \neq D$, $i = 1, \ldots, j$, $D \subseteq A_1 \cup \cdots \cup A_j$ and $r(A_1) + \cdots + r(A_j) = r(D)$. Since D is closed, j must be at least 2. However, since D is nonseparable, j must be equal to 1. So there does not exist such a solution. □

Before applying this lemma to matchings in graphs, several direct applications of it will be mentioned.

Edmonds [7] proved that if $M = (E, I)$ is a matroid with rank function r, then linear system (2.1) is a totally dual integral defining system for $P(M)$, the convex hull of the independent sets of M. Thus, Lemma 2.1 implies that the Schrijver system for $P(M)$ is (2.1) with an inequality $x(A) \le r(A)$ only for those $A \subseteq E$, $A \neq \emptyset$ that are closed and nonseparable in the matroidal sense. (Note that Cunningham [3] has given an algorithm which may be used to test in polynomial time whether or not A has this property.) This result of Edmonds can be found in Pulleyblank [16]. A similar result of Giles [10] for matroid intersection polyhedra follows in the same manner from Lemma 2.1 using the matroid intersection theorem of Edmonds [6]. Lemma 2.1 also provides a characterisation of the Schrijver systems for the convex hulls of 'perfect independence systems' – see Euler [9].

3. Matchings

Let G be a graph and $M(G)$ the convex hull of (the incidence vectors of) the matchings of G. Edmonds [5] proved that $M(G)$ is defined by the following linear system

$$\begin{aligned} x_e &\ge 0 \quad \forall e \in EG, \\ x(\delta(v)) &\le 1 \quad \forall v \in VG, \\ x(\gamma(T)) &\le \lfloor |T|/2 \rfloor \quad \forall T \subseteq VG,\ |T| \ge 3,\ |T| \text{ odd} \end{aligned} \tag{3.1}$$

where EG is the edge set of G, VG is the node set of G, for each $v \in VG$ the set of edges which meet v is denoted by $\delta(v)$, for each $S \subseteq VG$ the set of edges which have both ends in S is denoted by $\gamma(S)$, and for any number d, $\lfloor d \rfloor$ is the greatest integer less than or equal to d. Moreover, it follows directly from Edmonds' matching algorithm that adding the inequalities

$$x(\gamma(T)) \leq \lfloor |T|/2 \rfloor \quad \forall T \subseteq VG,\ |T| \geq 3,\ |T| \text{ even} \tag{3.2}$$

to (3.1) gives a totally dual integral system (see Hoffman & Oppenheim [12] and Schrijver [19,20] for different proofs of strengthened forms of this result).

A matching M of G is *perfect* if each node in VG is met by an edge in M. If G is connected, then G is *hypomatchable* if for each $v \in VG$ the graph obtained by deleting v from G has a perfect matching. For each $v \in VG$ let $N(v)$ denote the set of nodes in $VG - \{v\}$ which are adjacent to v. Let V' be the set of nodes $v \in VG$ such that either $|N(v)| \geq 3$ *or* $|N(v)| = 2$ and $\gamma(N(v)) = \emptyset$ *or* $|N(v)| = 1$ and v is in a two node connected component of G.

Since $\{e\}$ is a matching for each $e \in EG$, the polytope $M(G)$ is of full dimension. Pulleyblank and Edmonds [17] proved that the unique minimal defining system (unique up to positive scalar multiples of the inequalities) for $M(G)$ is

$$\begin{aligned} x_e &\leq 0 \quad \forall e \in EG, \\ x(\delta(v)) &\leq 1 \quad \forall v \in V', \\ x(\gamma(T)) &\leq \lfloor |T|/2 \rfloor \quad \forall T \subseteq VG,\ |T| \geq 3, \\ &\qquad G[T] \text{ hypomatchable with no cutnode.} \end{aligned} \tag{3.3}$$

where, for each $T \subseteq VG$, $G[T]$ denotes the subgraph of G with node set T and edge set $\gamma(T)$. Cunningham and Marsh [4] proved that this linear system is totally dual integral, which implies the following result.

Theorem 3.1. *The Schrijver system for* $M(G)$ *is* (3.3).

Using Lemma 2.1, a short proof of this theorem, which does not use the result of Pulleyblank and Edmonds, can be given.

Proof of Theorem 3.1. Let (E, I) be the independence system with $E = EG$ and I the set of matchings of G. Since Edmonds' matching algorithm gives that (3.1) and (3.2) is a totally dual intergral defining system for $M(G)$, Lemma 2.1 implies that the Schrijver system for $M(G)$ consists of $x_e \geq 0$ for each $e \in EG$ and an inequality $x(F) \leq r(F)$ for each $F \subseteq EG$, $F \neq \emptyset$, which is nonseparable and closed for (E, I), where $r(F)$ is the cardinality of a maximum cardinality matching of G contained in F.

Clearly, each set of edges of the form $\delta(v)$ for some $v \in VG$ is nonseparable. A set $\delta(v)$ for some $v \in VG$ is closed if and only if $v \in V'$. So an inequality $x(\delta(v)) \leq 1$ is in the Schrijver system for $M(G)$ if and only if $v \in V'$.

Since (3.1) and (3.2) is a totally dual integral defining system for $M(G)$, each closed nonseparable set of edges not of the form $\delta(v)$ for some $v \in VG$ is of the form

$\gamma(T)$ for some $T \subseteq VG$ with $r(\gamma(T)) = \lfloor |T|/2 \rfloor$. Suppose that $\gamma(T)$, for some $T \subseteq VG$, $T \neq \emptyset$, is such a closed nonseparable set. The cardinality of T must be odd, since otherwise $(\delta(v) \cap \gamma(T), \gamma(T) - \delta(v))$ is a separation of $\gamma(T)$ for any $v \in T$. If there exists a node $v \in T$ such that $G[T - \{v\}]$ does not have a perfect matching, then, again, $(\delta(v) \cap \gamma(T), \gamma(T) - \delta(v))$ is a separation of $\gamma(T)$. So $G[T]$ must be hypomatchable. Furthermore, $G[T]$ does not have a cutnode v, since otherwise $(\gamma(T_1 \cup \{v\}), \gamma(T - T_1))$ is a separation of $\gamma(T)$ where $T_1 \subseteq T$ is a set such that $G[T_1]$ is a connected component of $G[T - \{v\}]$ (note that $|T_1|$ must be even, since $G[T - \{v\}]$ has a perfect matching). So (3.3) is a totally dual integral defining system for $M(G)$.

To complete the proof it must be shown that if $T \subseteq VG$, $|T| \geq 3$, is such that $G[T]$ is hypomatchable with no cutnode, then $\gamma(T)$ is a closed nonseparable set. Suppose that $T \subseteq VG$ is such a subset. Let $e \in EG - \gamma(T)$ be an edge with ends v_1 and v_2 (if $EG - \gamma(T) \neq \emptyset$). If neither v_1 nor v_2 is in T, then clearly $r(\gamma(T) \cup \{e\}) = r(\gamma(T)) + 1$. It cannot be the case that both v_1 and v_2 are in T, so suppose that $v_1 \in T$ and $v_2 \notin T$. If M is a perfect matching of $G[T - \{v_1\}]$, then $M \cup \{e\}$ is a matching of rank $r(\gamma(T)) + 1$ in $\gamma(T) \cup \{e\}$. So $\gamma(T)$ is closed. Suppose that $\gamma(T)$ is separable. Let $F_1, \ldots, F_k$ be subsets of $\gamma(T)$ such that F_i is nonseparable for each $i \in \{1, \ldots, k\}$, $r(\gamma(T)) = r(F_1) + \cdots + r(F_k)$, and $\gamma(T) = F_1 \cup \cdots \cup F_k$. Since $r(\gamma(T) - \delta(v)) = r(\gamma(T))$ for each $v \in T$, it can be assumed that for each $i \in \{1, \ldots, k\}$ there exists a set $T_i \subseteq T$ such that $F_i = \gamma(T_i)$ and $r(F_i) = \lfloor |T_i|/2 \rfloor$.

Claim. $|T_1| + \cdots + |T_k| \geq |T| + k$.

Once the claim is shown, the proof will be complete, since it implies that $\lfloor |T_1|/2 \rfloor + \cdots + \lfloor |T_k|/2 \rfloor > \lfloor |T|/2 \rfloor$, a contradiction. To see the claim, let H be a graph with nodes $t_1, \ldots, t_k$ and with an edge (t_i, t_j) for all $i \neq j$ such that $T_i \cap T_j \neq \emptyset$. Observe that $|T_1| + \cdots + |T_k| \geq |T| + |EH|$. Since $G[T]$ is connected and each edge in $\gamma(T)$ is in $\gamma(T_i)$ for some $i \in \{1, \ldots, k\}$, the graph H is connected. So $|EH| \geq k - 1$. Suppose that $|EH| = k - 1$. Let t_i be a node of degree 1 in H and let t_j be the node adjacent to t_i (by assumption, k is at least 2). Since $G[T]$ has no cutnode, $|T_i \cap T_j| \geq 2$ and $|T_1| + \cdots + |T_k| \geq |T| + |EH| + 1$. □

Part of Theorem 3.1 can be stated in a different way. Let G be a graph and k the cardinality of a maximum cardinality matching of G. Let E_1 and E_2 be nonempty subsets of EG with $E_1 \cup E_2 = EG$. Let k_i be the cardinality of a largest matching of G contained in E_i, $i = 1, 2$. If $k_1 + k_2 = k$, then (E_1, E_2) is a *matching separation* of G. Using Lemma 2.1, the above theorem implies the following result due to F.R. Giles.

Theorem 3.2. *A graph G is matching nonseparable if and only if G is isomorphic to $K_{1,n}$ for some n or G is hypomatchable with no cutnode.*

In fact, by virtue of Lemma 2.1, this result is the major portion of the content of Theorem 3.1 (if one assumes Edmonds' matching result that (3.1) and (3.2) is a totally dual integral defining system for $M(G)$).

Observe that in both the matching case and the matroid intersection case the Schrijver systems for the polyhedra in question are identical to the minimal defining systems for the polyhedra scaled so that the left hand sides of the inequalities are 0, 1-valued. An example of a class of full dimensional polyhedra where these two systems differ arise from b-matchings in graphs.

Let G be a graph and $b=(b_v\colon v\in VG)$ a positive integer vector. A *b-matching* of G is a nonnegative integer vector $x=(x_e\colon e\in EG)$ such that

$$x(\delta(v)) \le b_v \quad \text{for each } v\in VG.$$

Let $P(G,b)$ denote the convex hull of the b-matchings of G. Pulleyblank [13] characterised the minimal defining system for $P(G,b)$. This minimal system, when scaled so that the left hand sides are 0, 1-valued, is not in general totally dual integral. Cook [1] and Pulleyblank [15] independently characterised the Schrijver system for $P(G,b)$ (see also Pulleyblank [14] and Cook and Pulleyblank [2]). This characterisation, together with Lemma 2.1, gives a generalisation of Theorem 3.2. If G is connected, then G is *b-critical* if for each $v\in VG$ there exists a b-matching $\bar{x}$ of G such that

$$\bar{x}(\delta(v)) = b_v - 1 \quad \text{and} \quad \bar{x}(\delta(u)) = b_u \quad \text{for each } u\in VG-\{v\}.$$

If G is connected, then G is *b-bicritical* if for each $v\in VG$ there exists a b-matching $\bar{x}$ of G such that

$$\bar{x}(\delta(v)) = b_v - 2 \quad \text{and} \quad \bar{x}(\delta(u)) = b_u \quad \text{for each } u\in VG-\{v\}.$$

The result of Cook [1] and Pulleyblank [15] implies the following theorem, where b-matching separability is defined analogously to matching separability.

Theorem 3.3. *A graph G is b-matching nonseparable if and only if either G is isomorphic to $K_{1,n}$ for some n and either $n\le 1$ or $b(N(v))\ge b_v+1$ where v is the node of degree n* **or** *G is b-critical with no cutnode v having $b_v=1$* **or** *G is b-bicritical.*

If $b_v=1$ for each $v\in VG$, this result implies Theorem 3.2.

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