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ON THE MATRIX-CUT RANK OF POLYHEDRA

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Lovász and Schrijver (1991) described a semidefinite operator for generating strong valid inequalities for the 0–1 vectors in a prescribed polyhedron. Among their results, they showed that n iterations of the operator are sufficient to generate the convex hull of 0–1 vectors contained in a polyhedron in n-space. We give a simple example, having Chvátal rank 1, that meets this worst case bound of n. We describe another example requiring n iterations even when combining the semidefinite and Gomory-Chvátal operators. This second example is used to show that the standard linear programming relaxation of a k-city traveling salesman problem requires at least $\lfloor k/8 \rfloor$ iterations of the combined operator; this bound is best possible, up to a constant factor, as $k+1$ iterations suffice.

Many structures in combinatorial optimization can be modeled as a set of 0–1 vectors in a prescribed polyhedron in R^n, the n-dimensional Euclidean space. The utility of such a formulation depends to a large degree on our ability to derive, from the polyhedron, linear inequalities that are valid for the 0–1 vectors in the polyhedron. In some cases, these inequalities directly answer important combinatorial questions; in other cases, they permit linear programming methods to effectively analyze the given structure.

A general approach for obtaining valid inequalities was proposed by Lovász and Schrijver. The main version of their method uses an operator that lifts a polyhedron P to a higher dimensional space, applies a semidefinite relaxation, and projects it back to a convex set that better approximates the convex hull of the 0–1 vectors in P. An important property of the operator is that it is possible to optimize linear functions over the resulting convex set in polynomial time (provided we can optimize over the original polytope in polynomial time). Furthermore, for any polyhedron in $[0,1]^n$, at most n iterations of the operator are sufficient to obtain the convex hull of its 0–1 vectors.

The power of this semidefinite operator is illustrated by the result of Lovász and Schrijver (1991) that the stable set polytope of a perfect graph can be obtained in a single iteration from a certain polytope having a defining system of polynomial size (in the number of vertices of the graph). This implies the polynomial solvability of the weighted stable set problem for perfect graphs.

For general polyhedra, Goemans (1997) raised the question of determining the worst case behavior of the operator in terms of the number of iterations required to obtain the convex hull of 0–1 vectors. Stephen and Tunçel (1999) showed that a well-known relaxation of the matching polytope of a complete graph requires roughly $\sqrt{n/2}$ iterations where n is the dimension of the problem. Recently Goemans and Tunçel (Goemans 1998) presented an example where $n/2$ iterations of the operator are necessary. In this paper, we present two examples where the upper bound of n is attained (one of the examples has also been discovered by Goemans and Tunçel 2000). The first of these examples has Chvátal rank 1, while the second has Chvátal rank n. Moreover, if we combine the semidefinite operator with the Gomory-Chvátal cutting-plane procedure, the second example still requires n iterations. We use this result to show that the standard relaxation of the traveling salesman problem requires at least $\lfloor k/8 \rfloor$ iterations of the combined operator, where k is the number of cities. We also show that $k+1$ iterations of the combined operator suffice.

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The paper is organized as follows. In §1 we describe the semidefinite operator, as well as two others defined by Lovász and Schrijver (1991). Some of the basic properties of this family of operators are collected in §2, and the worst-case examples are discussed in §3. In §4 we apply the results to the traveling salesman problem. We will assume that the reader is familiar with the theory of linear inequalities and polyhedra; an excellent general reference is the book of Schrijver (1986).

1. The matrix-cut operators. Let Q_n be the 0–1 cube in R^n, that is $Q_n = [0, 1]^n$. If the dimension is obvious from the context, we denote the 0–1 cube by Q. A system of linear inequalities $a_i^T x \le b_i$ $(i = 1, \ldots, m)$ in R^n is denoted by $Ax \le b$ (here $A \in R^{m\times n}$ and $b \in R^m$). Given a set $S \subseteq R^n$, S_I denotes the convex hull of integral vectors in S (also called the *integer hull*); in particular, $S \subseteq Q \Rightarrow S_I = \text{conv}(S \cap \{0, 1\}^n)$ where $\text{conv}(X)$ is the convex hull of vectors in the set X.

For $x \in R^n$, let $\bar{x} = \binom{1}{x} \in R^{n+1}$. The additional coordinate will be referred to as the 0th coordinate; thus $\bar{x}_0 = 1$. Given a convex set $S \subseteq R^n$, we define an associated convex cone $\overline{S}$ by

$$\overline{S} = \text{cone}\left(\left\{\begin{pmatrix}1\\x\end{pmatrix} \in R^{n+1} : x \in S\right\}\right), \tag{1}$$

where $\text{cone}(X)$ is the set of nonnegative linear combinations of vectors in X. If $P \subseteq Q$ is defined by $P = \{x \in R^n : Ax \le b\}$, it follows that

$$\overline{P} = \left\{\begin{pmatrix}x_0\\x\end{pmatrix} \in R^{n+1} : bx_0 - Ax \ge \mathbf{0}\right\}.$$

For the empty set $\varnothing$, we adopt the convention that $\overline{\varnothing} = \{\mathbf{0}\}$ (here $\mathbf{0}\,(\mathbf{1})$ refers to the vector of all zeros (all ones) in the appropriate dimension). Of special interest will be the cone $\overline{Q} = \{x \in R^{n+1} : x_0 - x_i \ge 0, x_i \ge 0, 1 \le i \le n\}$.

If K is a convex cone, its polar cone is K^*, where $K^* = \{y : y^T x \ge 0 \,\forall x \in K\}$. Let the ith unit vector be e_i and let f_i stand for $e_0 - e_i$. Then $\overline{Q}^*$ is spanned by the vectors e_i and f_i, for i between 1 and n.

Given a point $y \in \overline{Q}$ with $y_0 > 0$, we define the image of y in R^n by $\tilde{y}$ where $\lambda y = \binom{1}{\tilde{y}}$ for some $\lambda > 0$. For a sub-cone K of $\overline{Q}$, we let $\widetilde{K}$ denote the set $\{\tilde{y} : y \in K\}$. From this it follows that for a convex set $S \subseteq Q$, $\widetilde{\overline{S}} = S$ (note that $\overline{\varnothing} = \{\mathbf{0}\}$ and $\{\widetilde{\mathbf{0}}\} = \varnothing$).

We now introduce the matrix-cut operators of Lovász and Schrijver. Let $P \subseteq Q$ be a polytope defined by $\{x \in R^n : a_i^T x \le b_i, i = 1, \ldots, m\}$. We can rewrite $a_i^T x \le b_i$ as

$$u_i^T \bar{x} \ge 0, \quad \text{where } u_i = \begin{pmatrix}b_i\\-a_i\end{pmatrix}.$$

Since $x_j \ge 0$ and $1 - x_j \ge 0$ are also valid for P for $1 \le j \le n$, it follows that the quadratic inequalities $(u_i^T \bar{x})x_j \ge 0$ and $(u_i^T \bar{x})(1 - x_j) \ge 0$ are valid for P. Writing x_j as $e_j^T \bar{x}$ and $1 - x_j$ as $f_j^T \bar{x}$, we have

$$P = \{x : (u_i^T \bar{x})(e_j^T \bar{x}) \ge 0, \quad (u_i^T \bar{x})(f_j^T \bar{x}) \ge 0 \;\forall i, j\} \tag{2}$$

(the original inequalities $u_i^T \bar{x} \ge 0$ can be recovered by adding $(u_i^T \bar{x})x_j \ge 0$ and $(u_i^T \bar{x})(1 - x_j) \ge 0$). Rewriting $(u^T \bar{x})(v^T \bar{x})$ as $u^T(\bar{x}\bar{x}^T)v$, and using the fact that $\overline{P}^*$ is spanned by the vectors u_i, we obtain from (2) that

$$P = \left\{x : u^T(\bar{x}\bar{x}^T)v \ge 0 \quad \text{for } u \in \overline{P}^*, v \in \overline{Q}^*\right\}. \tag{3}$$

All 0–1 vectors in P satisfy $x_i^2 = x_i$. Therefore, if x is a 0–1 vector in P, then setting $Y = \bar{x}\bar{x}^T$ and $K = \overline{P}$ we have that

(4) $$Y \text{ is symmetric},$$

(5) $$Ye_0 = Y^T e_0 = \operatorname{diag}(Y), \quad \text{that is } Y_{i0} = Y_{0i} = Y_{ii} \text{ if } 1 \le i \le n,$$

(6) $$u^T Y v \ge 0 \quad \text{for } u \in K^*, v \in \overline{Q}^*,$$

(7) $$Y \text{ is positive semidefinite.}$$

(Recall that an $n \times n$ matrix A is positive semidefinite iff $x^T A x \ge 0$ for all $x \in R^n$; equivalently $A = U^T U$ for some matrix U.) Condition (6) is equivalent to

(8) $$Ye_i \in K \quad \text{and} \quad Y(e_0 - e_i) \in K \quad \text{if } 1 \le i \le n.$$

Also, if $Y = (y_{ij})$ is a matrix satisfying (8), then (since $K \subseteq \overline{Q}$) we have

(9) $$\begin{gathered} y_{ij} \ge 0, \qquad y_{0j} \ge y_{ij}, \qquad y_{i0} \ge y_{ij}, \\ y_{ij} \ge y_{i0} + y_{0j} - y_{00} \quad \text{whenever } i \ge 0, j \ge 0. \end{gathered}$$

Let $K \subseteq \overline{Q}$ be a closed convex cone, and consider the three derived cones:

(10) $$M(K) = \{Y \in R^{(n+1)\times(n+1)} : Y \text{ satisfies conditions (4)–(6)}\},$$

(11) $$M_0(K) = \{Y \in R^{(n+1)\times(n+1)} : Y \text{ satisfies conditions (5)–(6)}\},$$

(12) $$M_+(K) = \{Y \in R^{(n+1)\times(n+1)} : Y \text{ satisfies conditions (4)–(7)}\}.$$

Define $N(K) \subseteq R^{n+1}$ to be $\{Ye_0 : Y \in M(K)\}$. $N_0(K)$ and $N_+(K)$ are defined analogously.

Given a convex set $S \subseteq Q$, define $N(S)$ by $N(S) = \widetilde{N(\overline{S})}$. Thus $N(S)$ consists of all the vectors $x \in R^n$ such that $\bar{x} = Ye_0$ where $Y \in M(\overline{S})$. Whether $N(T)$ is a cone in $\overline{Q}$ or a convex set in Q will be clear from the context. Both $M(P)$ and $M(\overline{P})$ refer to the same cone. If P is a polytope in Q, then both $M(P)$ and $M_0(P)$ are polyhedral cones (in a higher dimensional space) and hence both $N(P)$ and $N_0(P)$ are polytopes. In general, $N_+(P)$ is nonpolyhedral (it is a convex set).

We will refer to N, N_0 and N_+ collectively as the *matrix-cut operators* ($N(P)$ is also defined in Sherali and Adams 1990, but used in a different setting). $N_0(P)$ is actually defined by Lovász and Schrijver via a geometric characterization (see Lemma 2.3).

Defining $N^0(P) = P$ and $N^{t+1}(P) = N(N^t(P))$ if t is a nonnegative integer, it follows from (8) that $P \supseteq N(P) \supseteq N^2(P) \supseteq \cdots \supseteq P_I$. Lovász and Schrijver (1991) proved the following important result.

THEOREM 1.1. *Let $P \subseteq Q_n$ be a polytope. Then $N^n(P) = P_I$.* □

Moreover, Lovász and Shrijver showed that for any fixed value of t, it is possible to optimize linear functions over $N^t(P)$ in polynomial time (see their 1991 paper for a precise statement). Identical results hold for the N_0 and N_+ operators; we can also replace polytopes by closed convex sets in Q_n.

We follow Lipták (1999) and define the *noncommutative rank* of a polytope P to be the least integer $t \ge 0$, such that $N_0^t(P) = P_I$. The *commutative rank* (also in Lipták 1999) and *semidefinite rank* are defined analogously for the N and N_+ operators, respectively.

Chvátal (1973) (and implicitly Gomory 1958) defined another method to obtain approximations of the integer hull of a polytope P. If $c^T x \le d$ is valid for P and $c \in Z^n$, then $c^T x \le \lfloor d \rfloor$ is a *Gomory-Chvátal cutting plane* for P. Define P' to be the set of points satisfying all Gomory-Chvátal cutting planes for P, and let $P^{(0)} = P$ and $P^{(t+1)} = (P^{(t)})'$ for nonnegative integers t (we will think of P' as defining an operator $' : P \to P'$, which we call the *Gomory-Chvátal operator*). Obviously $P \supseteq P^{(t)} \supseteq P_I$. Chvátal (1973) showed that if P is a polytope, there exists some $t \geq 0$ such that $P^{(t)} = P_I$ (see also Schrijver 1980); the smallest number t for which this holds is the *Chvátal rank* of P. Bockmayr and Eisenbrand (1997) proved that $P \subseteq Q_n \Rightarrow P^{(t)} = P_I$ for some $t \leq 6n^3 \log n$ (this upper bound has been improved to $3n^2 \log n$ by Eisenbrand and Schulz 1999). In contrast to the matrix-cut operators, the separation problem for P' is NP-complete in general (Eisenbrand 1998).

2. Basic properties. We collect some properties of the matrix-cut operators applied to polytopes. All of these properties also hold for closed convex sets contained in Q.

A function $f : R^n \to R^n$ corresponds to a *flipping* operation if it "flips" some coordinates. That is, if $J \subseteq \{1, \ldots, n\}$ and f flips the coordinates in J, then

$$y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } i \notin J, \\ 1 - x_i & \text{if } i \in J. \end{cases} \tag{13}$$

The function f corresponds to an *embedding* operation if $f : R^n \to R^{n+k}$ and

$$y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } n < i \leq n + k_1, \\ 1 & \text{if } n + k_1 < i \leq n + k, \end{cases} \tag{14}$$

where $0 \leq k_1 \leq k$. Note that we can always renumber the coordinates so that the additional coordinates with values 0 or 1 are interspersed with the original ones and not grouped at the end. Given a face F of Q, f_F will denote the embedding function defined by

$$f_F \quad \text{embeds } Q_{\dim(F)} \text{ in } F. \tag{15}$$

Consider a k-tuple of coordinates $\{j_1, \ldots, j_k\}$, which are not necessarily distinct, such that $j_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, k$. If $f : R^n \to R^{n+k}$ and

$$y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ x_{j_{i-n}} & \text{if } n < i \leq n + k, \end{cases} \tag{16}$$

then f corresponds to a *duplication* operation.

Given a set $S \subseteq R^n$, we define the set $f(S)$ by $f(S) = \{f(x) : x \in S\}$. It is straightforward to prove the following lemma (see the discussion on flipping and embedding in Lovász and Schrijver 1991 and the discussion on embedding in Stephen and Tunçel 1999).

LEMMA 2.1. *Let $f : R^n \to R^m$ correspond to a flipping operation, an embedding operation, or a duplication operation and let $P \subseteq Q$ be a polytope. Then $N_+(f(P)) = f(N_+(P))$. This equation is also valid for the N_0 and N operators.* □

PROOF. We will prove the result on duplication for the N_+ operator. Assume f duplicates only x_n, that is $f : R^n \to R^{n+1}$ and $y = f(x) \Rightarrow y_{n+1} = x_n$ and $y_i = x_i$ for $1 \leq i \leq n$. For a matrix $Y = (y_{ij})$, let

$$Y' = \begin{pmatrix} Y & Ye_n \\ e_n^T Y & y_{nn} \end{pmatrix}$$

(we repeat the last row and column in Y and also the last diagonal element). Obviously $M(f(P)) = \{Y' : Y \in M(P)\}$. If Y is positive semidefinite, then $Y = U^T U$, for some matrix U. Let

$$U' = \begin{pmatrix} U & Ue_n \\ \mathbf{0}^T & 0 \end{pmatrix};$$

then $Y' = U'^T U'$ and Y' is positive semidefinite. As Y is a principal minor of Y', Y is positive semidefinite if Y' is. We can conclude that $Y \in M_+(P) \Leftrightarrow Y' \in M_+(f(P))$ and $N_+(f(P)) = \{\binom{x}{x_n} : x \in N_+(P)\} = f(N_+(P))$. □

A useful property of the Gomory-Chvátal operator is that $P' \cap F = F'$ where F is face of a (rational) polyhedron P. A similar property holds for the matrix-cut operators.

LEMMA 2.2. *If F is a face of a polytope $P \subseteq Q$, then $N_+(F) = N_+(P) \cap F$. This equation is also valid for the N and N_0 operators.*

PROOF. Assume F is a face of P. Then there is a supporting hyperplane $H = \{x : c^T x \leq d\}$ of P such that $F = P \cap H$ (we will rewrite $c^T x \leq d$ as $u^T \bar{x} \geq 0$). By definition $N_+(F) = N_+(P \cap H) \subseteq N_+(P) \cap H = N_+(P) \cap F$. Let $x \in N_+(P) \cap H$. Then $\bar{x} = Ye_0$ for some $Y \in M_+(P)$. As H is a supporting hyperplane of P, we have $u^T Ye_i \geq 0$ and $u^T Y(e_0 - e_i) \geq 0$ for $i = 1, \ldots, n$. As $x \in H$, $0 = u^T \bar{x} = u^T Ye_0 = u^T Ye_i + u^T Y(e_0 - e_i)$. Hence, $Ye_i \in \overline{H}$ and $Y(e_0 - e_i) \in \overline{H}$ for $i = 1, \ldots, n$. This implies that $x \in N_+(P \cap H) = N_+(F)$, and the lemma follows. It is clear that the proof applies to the N and N_0 operators. □

For $1 \leq i \leq n$, let F_i^0 and F_i^1 be facets of Q defined by $F_i^0 = \{x \in Q : x_i = 0\}$ and $F_i^1 = \{x \in Q : x_i = 1\}$. Lovász and Schrijver (1991) gave the following characterization of N_0.

LEMMA 2.3. $N_0(P) = \cap_i \mathrm{conv}((P \cap F_i^0) \cup (P \cap F_i^1))$. □

We can conclude from Lemma 2.3 that if P does not intersect some facet of Q (say F_i^0), then $N_0(P)$ is contained in the opposite facet (F_i^1). This fact, together with Lemma 2.2, has a useful corollary (note that if F is a face of Q, then $N(P \cap F) = N(P) \cap F$).

COROLLARY 2.4. *If $P \cap F_i^0 = \varnothing$, then $N_0(P) = N_0(P) \cap F_i^1 = N_0(P \cap F_i^1)$.* □

If P does not intersect some pair of opposing facets of Q, then $N_0(P) = \varnothing$. As $N_+(P) \subseteq N(P) \subseteq N_0(P)$, the same (and Corollary 2.4) is true for the N and N_+ operators.

If a polytope has empty integer hull and Chvátal rank n, then (Eisenbrand and Schulz 1999, Proposition 1) a defining (linear) system for P must have at least 2^n inequalities. We adapt the proof of this result and obtain the following fact.

PROPOSITION 2.5. *Let $P \subseteq Q_n$ be a polytope with $P_I = \varnothing$ and noncommutative (commutative, semidefinite) rank n. Then any system of linear inequalities defining P must contain at least 2^n inequalities different from the bounds $\mathbf{0} \leq x \leq \mathbf{1}$.*

PROOF. It suffices to prove the result for noncommutative rank. We observe that if the noncommutative rank of P is n, then both $P \cap F_i^0$ and $P \cap F_i^1$ have noncommutative rank $n-1$. For if $P \cap F_i^0$ (and similarly $P \cap F_i^1$) has noncommutative rank $\leq n-2$, then $N_0^{n-2}(P) \cap F_i^0 = N_0^{n-2}(P \cap F_i^0) = \varnothing$ and hence $N_0^{n-1}(P) = N_0^{n-1}(P \cap F_i^1) = \varnothing$. We can argue as above for faces of $P \cap F_i^0$ and $P \cap F_i^1$ and obtain by induction that for any 1-dimensional face F of Q, $P \cap F$ has noncommutative rank 1 and hence $P \cap F \neq \varnothing$. As $P_I = \varnothing$, for every vertex of Q there must be some inequality in any linear system defining P that separates that vertex from P. If some inequality separates two 0–1 vectors from P, then it separates some 1-dimensional face of Q from P. But this is a contradiction and hence the proposition follows. □

Clearly the bound of 2^n in Proposition 2.5 cannot be raised; any polytope $P \subseteq Q_n$ with $P_I = \varnothing$ is contained in a polytope T with $T_I = \varnothing$ that has a defining system of 2^n inequalities (besides the bounds on the variables). In addition, if P has rank n, then so does T. In §3 we present a family of examples meeting the 2^n bound given above.

A nonempty convex set S is said to be of *antiblocking type* (or has the antiblocking property) if $S \subseteq R_+^n$ and $x \in S$, $0 \leq y \leq x \Rightarrow y \in S$. A convex set S is of *blocking type* if $S \subseteq R_+^n$ and $x \in S$, $y \geq x \Rightarrow y \in S$. See Schrijver (1986) for a discussion of antiblocking and blocking polyhedra. Obviously a polytope contained in Q cannot be of blocking type. However we modify the above definition and say that a nonempty convex set $S \subseteq Q$ is of blocking type if $y \in Q$ and $y \geq x \in S \Rightarrow y \in S$.

LEMMA 2.6. *Let $P \subseteq Q$ be a nonempty antiblocking (blocking) polytope. Then $N_+(P)$ is a convex set with the antiblocking (blocking) property. $N_0(P)$ and $N(P)$ are anitblocking (blocking) polytopes.*

PROOF. Let $x \in N_+(P)$. Then $\bar{x} = Ye_0$ with $Y \in M_+(P)$. Consider $K \subseteq \{1, \dots, n\}$ and define Y' by

$$Y'_{ij} = \begin{cases} 0 & \text{if } i \in K \text{ or } j \in K, \\ Y_{ij} & \text{otherwise.} \end{cases}$$

Let the ith column of Y be y_i and let $z_i = y_0 - y_i$ (we define y'_i and z'_i analogously). Then $y'_i \leq y_i \Rightarrow y'_i \in \overline{P}$ (as P is an antiblocking polytope). Similarly $z'_i \in \overline{P}$ ($z'_i \leq z_i$ with the 0th coordinate being the same). The matrix Y' is positive semidefinite as the nonzero elements in Y' form a principal minor of Y and Y is positive semidefinite. Hence $Y' \in M_+(P)$. Defining $x^K = \tilde{y}'_0$, it follows that $x^K \in N_+(P)$ (x^K is the same as x with the components in K being set to zero). Since $0 \leq y \leq x \Rightarrow y \in \text{conv}(\{x^K : K \subseteq \{1, \dots, n\}\}) \subseteq N_+(P)$, the result for antiblocking polytopes follows. A nonempty blocking polytope can be transformed into an antiblocking one, by flipping all coordinates; one can then apply the above result and Lemma 2.1. □

Let P be a polytope in Q and let $c^T x \leq d$ be an inequality, with $c \geq 0, d \geq 0$, which is valid for $P \cap F_i^1$ whenever $c_i > 0$. It is shown in Lovász and Schrijver (1991, Lemma 1.5), that the above assumptions imply $c^T x \leq d$ is valid for $N_+(P)$. We use this result to obtain an upper bound on the semidefinite rank of an antiblocking polytope (we generalize Corollary 2.19 in Lovász and Schrijver (1991) which provides a similar bound for stable set polytopes). This can also be derived from a result of Goemans (1998, Theorem 2).

LEMMA 2.7. *Let $P \subseteq Q$ be a nonempty antiblocking polytope with* $\max\{\mathbf{1}^T x : x \in P_I\} = k$. *Then the semidefinite rank of P is at most $k+1$.*

PROOF. We prove the theorem by induction on $\max\{\mathbf{1}^T x : x \in P_I\}$ (which we denote by k). Let $k = 0$. Then $P_I = \{\mathbf{0}\}$. For $1 \leq i \leq n$ we have $P \cap F_i^1 = \varnothing$; since $P \cap F_i^1 \neq \varnothing$ would imply (via the antiblocking property) that $e_i \in P_I$. This implies (by Corollary 2.4) that $N_+(P) \subseteq \cap_i F_i^0 = \{\mathbf{0}\} = P_I$. Now consider some $k > 0$ and assume that the theorem is true whenever $\max\{\mathbf{1}^T x : x \in P_I\} < k$. Let P satisfy the conditions of the theorem (with this value of k). As P is an antiblocking polytope, so is P_I, and $P_I = \{x \in Q : Ax \leq b\}$ for some matrix $A \geq 0$ and vector $b \geq 0$. If $f \equiv f_{F_i^1}$ for some i, then $P \cap F_i^1 = f(P_i)$ where P_i is a lower-dimensional antiblocking polytope satisfying $\max\{\mathbf{1}^T x : x \in (P_i)_I\} \leq k-1$. Hence $N_+^k(P) \cap F_i^1 = N_+^k(P \cap F_i^1) = P_I \cap F_i^1$. Let $c^T x \leq d$ be an inequality in the system $Ax \leq b$; $c^T x \leq d$ is valid for P_I and also for $N_+^k(P) \cap F_i^1$. We can conclude (from the lemma of Lovász and Schrijver referred to above) that $c^T x \leq d$ is valid for $N_+^{k+1}(P)$; thus $N_+^{k+1}(P) \subseteq P_I$. □

3. Rank of polytopes. Consider the polytope defined by

$$P_n = \left\{x \in Q_n : x_1 + \cdots + x_n \geq \tfrac{1}{2}\right\}. \tag{17}$$

It is obvious that $(P_n)_I = \{x \in Q_n : x_1 + \cdots + x_n \geq 1\}$ and the Chvátal rank of P_n is 1.

THEOREM 3.1. *Let P_n be defined as in* (17). *Then the semidefinite rank of P_n is n. Further, $N_0^k(P_n) = N^k(P_n) = N_+^k(P_n)$ for all integers $k \geq 0$.*

PROOF. We first show that

$$\frac{1}{2n-k}\mathbf{1} \in N^k(P_n) \quad \text{if } k \leq n, \tag{18}$$

by induction on n. Certainly (18) is true for $n = 1$ and $k = 0, 1$. Let $n \geq 2$ and $k \leq n$ be given, and assume (18) holds for P_{n-1}. We may assume $k > 0$. Consider the matrix $Y = (y_{ij}) \in R^{(n+1)\times(n+1)}$ defined by

$$(19) \qquad y_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ \frac{1}{2n-k} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let the ith column of Y be y_i. Then $\tilde{y}_i = e_i \in (P_n)_I$ for $i \geq 1 \Rightarrow \tilde{y}_i \in N^{k-1}(P_n)$. Let $z_i = y_0 - y_i$. Then

$$z_i = \frac{2n-k-1}{2n-k} e_0 + \sum_{j \neq i, 0} \frac{1}{2n-k} e_j \Rightarrow \tilde{z}_i = \sum_{j \neq i, 0} \frac{1}{2n-k-1} e_j.$$

Now $\tilde{z}_i \in F_i^0$. Let $f \equiv f_{F_i^0}$; then $P_n \cap F_i^0 = f(P_{n-1})$. By the induction hypothesis,

$$\frac{1}{2n-k-1}\mathbf{1} = \frac{1}{2(n-1)-(k-1)}\mathbf{1} \in N^{k-1}(P_{n-1})$$
$$\Rightarrow \tilde{z}_i \in f(N^{k-1}(P_{n-1})) = N^{k-1}(P_n \cap F_i^0) \subseteq N^{k-1}(P_n).$$

Hence $Y \in M(N^{k-1}(P_n))$ and (18) follows (take the vector Ye_0). Since $n/(2n-k) < 1$ for $k < n$, it follows that $(1/(2n-k))\mathbf{1} \notin (P_n)_I$ for $k < n$ and the commutative rank of P_n is exactly n.

We now show, by induction on k, that

$$(20) \qquad N_0^k(P_n) = N_+^k(P_n) \quad \text{for } k \leq n$$

(we will refer to P_n as P as we do not need to consider P_n for varying n any more). The case $k = 0$ is trivial. Assume (20) is true for $k - 1$ and let $T = N_0^{k-1}(P)$ $(T_I = P_I)$. Consider some $x \in N_0(T)$. If $\sum_{i=1}^n x_i \geq 1$ then $x \in P_I \Rightarrow x \in N_+(T)$. So assume $\sum_{i=1}^n x_i < 1$. Now $\bar{x} = Ye_0$ for some $Y = (y_{ij}) \in M_0(T)$. For $1 \leq i \leq n$ both y_i and $y_0 - y_i \in \overline{T}$. Define $Y' = (y'_{ij})$ by

$$(21) \qquad y'_{ij} = \begin{cases} y_{ij} & \text{if } i = j \text{ or } i = 0 \text{ or } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{y}'_i = e_i \in P_I \Rightarrow y'_i \in \overline{T}$. Now $y'_0 - y'_i$ equals $y_0 - y_i$ in the 0th coordinate and $y'_0 - y'_i \geq y_0 - y_i$ (as $y'_i \leq y_i$ and $y'_0 = y_0$). Further, $\bar{x} = y'_0$ and $\sum_{i=1}^n x_i < 1$ together imply that $y'_0 - y'_i \in \overline{Q}$. Since P is a nonempty blocking polytope, it follows from Lemma 2.6 that T is also a nonempty blocking polytope. Therefore $y'_0 - y'_i \in \overline{T}$. Y' is obviously symmetric. Now consider any $u \in R^{n+1}$. We have from (21) that

$$u^T Y' u = \sum_{i=0}^n u_i^2 y_{ii} + 2\sum_{i=1}^n u_0 u_i y_{i0}.$$

Using $y_{00} = 1$ and $y_{ii} = y_{i0} = x_i$,

$$u^T Y' u = u_0^2 + \sum_{i=1}^n x_i(u_i^2 + 2u_i u_0)$$
$$\geq \sum_{i=1}^n x_i(u_0^2 + u_i^2 + 2u_i u_0) = \sum_{i=1}^n x_i(u_0 + u_i)^2 \geq 0.$$

The first inequality follows from that fact that $\sum_{i=1}^n x_i < 1$ and the second from $x_i \geq 0$. Hence Y' is positive semidefinite and $Y' \in M_+(T)$. But $\bar{x} = y_0 = y'_0 \Rightarrow x \in N_+(T)$. This implies that $N_0(T) \subseteq N_+(T)$. Hence $N_0^k(P_n) = N^k(P_n) = N_+^k(P_n)$ and the theorem follows. □

The polytopes of the previous example have high semidefinite rank, but low Chvátal rank. There exist examples where the reverse is true. Some polytopes however have high semidefinite rank as well as high Chvátal rank, as we now discuss.

Consider the polytope P_n, with empty integer hull, defined by

$$P_n = \left\{ x \in Q_n : \sum_{i \in J} x_i + \sum_{i \notin J} (1 - x_i) \geq \frac{1}{2}, \quad \text{for all } J \subseteq \{1, \ldots, n\} \right\}. \tag{22}$$

If F is a face of Q with $\dim(F) = q$, and f_F is defined as in (15), then

$$P_n \cap F = f_F(P_q). \tag{23}$$

THEOREM 3.2. *Let P_n be defined as in* (22). *Then the semidefinite rank of P_n is n.*

PROOF. We will prove by induction on n, that

$$\frac{1}{2}\mathbf{1} \in N_+^{n-1}(P_n). \tag{24}$$

The case $n = 1$ is trivial; assume $\frac{1}{2}\mathbf{1} \in N_+^{n-2}(P_{n-1})$. Consider the matrix $Y = (y_{ij})$ defined by

$$y_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ \frac{1}{2} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{25}$$

Let the ith row of Y be y_i and let $z_i = y_0 - y_i$. Then, if $i \geq 1$, $\bar{y}_i \in P_n \cap F_i^1$; the ith coordinate of $\bar{y}_i$ has value 1 while the rest have value $\frac{1}{2}$. Let $f \equiv f_{F_i^1}$. By the induction hypothesis and (23), $\bar{y}_i = f\left(\frac{1}{2}\mathbf{1}\right) \in f(N_+^{n-2}(P_{n-1})) = N_+^{n-2}(P_n \cap F_i^1) \subseteq N_+^{n-2}(P_n)$. Similarly one shows that $\bar{z}_i \in N_+^{n-2}(P_n \cap F_i^0) \subseteq N_+^{n-2}(P_n)$. To show that Y is positive semidefinite, consider $u \in R^{n+1}$. Then

$$\begin{aligned} u^T Y u &= u_0^2 + \frac{1}{2}\sum_{i=1}^n u_i^2 + \sum_{i=1}^n u_i u_0 + \frac{1}{2}\sum_{i=1}^n \sum_{j>1} u_i u_j \\ &= \left(u_0 + \frac{1}{2}\sum_{i=1}^n u_i\right)^2 + \frac{1}{4}\sum_{i=1}^n u_i^2 \geq 0. \end{aligned}$$

Hence $Y \in M_+(N_+^{n-2}(P_n))$ and (24) follows. This implies that the semidefinite rank of P_n is n (since $(P_n)_I = \varnothing$). $\square$

This result has also been obtained by Goemans and Tunçel (2000). The Chvátal rank of P_n is shown in Chvátal et al. (1989) to be at least n; that the rank is exactly n follows from the fact that $(P_n)_I = \varnothing$ (such polytopes have Chvátal rank at most n; see Bockmayr and Eisenbrand 1997). Hence we have a family of polytopes that have high Chvátal rank as well as high semidefinite rank. Let us combine both the operators to obtain a stronger operator N_* defined by

$$N_*(P) = N_+(P) \cap P'. \tag{26}$$

The rank of a polytope with respect to N_* will be defined as in the case of the other operators. We will show that even with this strengthened operator, P_n has rank n.

We define S_j to be the set of all vectors which have j components equal to $\frac{1}{2}$ and the remaining components equal to 0 or 1.

Chvátal et al. (1989, Lemma 7.2), show that the rank of P_n is at least n by proving that $S_j \subseteq P_n^{(j-1)}$ for all $j = 1, \dots, n$. Their proof technique establishes the auxiliary result that for a polytope P,

$$S_j \subseteq P \Rightarrow S_{j+1} \subseteq P' \quad \text{for } j = 1, \dots, n. \tag{27}$$

To obtain a similar result for the N_+ operator observe that the proof of (24) yields

$$S_{n-1} \subseteq P \Rightarrow S_n = \left\{\tfrac{1}{2}\mathbf{1}\right\} \subseteq N_+(P) \tag{28}$$

for any $P \subseteq Q$ (since the vectors $\tilde{y}_i$ and $\tilde{z}_i$ defined in the proof belong to S_{n-1}). We use (28) to prove the following lemma.

LEMMA 3.3. *Let $P \subseteq Q$ be a polytope and let $S_j \subseteq P$, where $1 \le j < n$. Then $S_{j+1} \subseteq N_+(P)$.*

PROOF. Assume $S_j \subseteq P$ for some $j \ge 1$. Let $x \in S_{j+1}$ and consider the face F of Q defined by

$$F = \{y \in Q : y_i = 1 \text{ if } x_i = 1, y_i = 0 \text{ if } x_i = 0\}.$$

Then $\dim(F) = j+1$. Let S'_j denote the collection of vectors in R^{j+1} with j components equal to $\frac{1}{2}$ and the remaining component equal to 0 or 1. The polytope $P \cap F$ can be written as $f_F(P_1)$ for some polytope $P_1 \subseteq Q_{j+1}$ where f_F is defined as in (15). (P_1 is obtained by dropping the fixed components of $P \cap F$). Then $S_j \cap F = f_F(S'_j)$ and $S'_j \subseteq P_1$. From (28) we obtain that $x = f_F\left(\frac{1}{2}\mathbf{1}\right) \in f_F(N_+(P_1)) \Rightarrow x \in N_+(P)$. Hence $S_{j+1} \subseteq N_+(P)$. □

Since S_1 belongs to P_n, we can combine Lemma 3.3 with (27) and conclude that $S_j \subseteq N_*^{j-1}(P_n)$.

COROLLARY 3.4. *Let P_n be defined as in (22). Then $\frac{1}{2}\mathbf{1} \in N_*^{n-1}(P_n)$ and the rank of P_n is n with respect to the N_* operator.* □

The following easy result will be useful in applying Corollary 3.4 to the traveling salesman problem.

LEMMA 3.5. *Let $f : R^n \to R^m$ be a function defined as a composition of the embedding, flipping and duplication operations. Let $S \subseteq Q_n$ and $T \subseteq Q_m$ be polytopes such that $f(S) \subseteq T$. Then for any positive integer t, $f(N_*^t(S)) \subseteq N_*^t(T)$.*

PROOF. Lemma 2.1 implies that $f(N_+^t(S)) = N_+^t(f(S)) \subseteq N_+^t(T)$. It is obvious that f can be represented by $f(x) = Ax + b$ for some integral A and b. It is known that (see Chvátal et al. 1989, Lemma 2.2) for such f, $f(S) \subseteq T$ implies $f(S^{(t)}) \subseteq T^{(t)}$. Hence $f(S^{(t)} \cap N_+^t(S)) \subseteq f(S^{(t)}) \cap f(N_+^t(S)) \subseteq T^{(t)} \cap N_+^t(T)$ and the result follows. □

4. The traveling salesman problem. Let $G = (V, E)$ denote a complete graph with vertex-set V and edge-set E. If $x \in R^E$ and $D \subseteq E$, we define $x(D)$ to be the sum $\sum_{e \in D} x_e$. For a subset S of V, let $\delta(S) = \{(v, w) \in E : v \in S, w \in V \backslash S\}$ and let $\gamma(S) = \{(v, w) \in E : v, w \in S\}$. Consider the polytope $H(G)$ (or H) defined as the set of all $x \in R^E$ satisfying

$$\begin{aligned} x(\delta(\{v\})) &= 2 \quad \text{for all } v \in V, \\ x(\delta(W)) &\ge 2 \quad \text{for all } W \subseteq V \text{ with } \varnothing \ne W \ne V, \\ 0 \le x_e &\le 1 \quad \text{for all } e \in E. \end{aligned} \tag{29}$$

The integral vectors in H are the incidence vectors of Hamiltonian circuits in G; the problem of maximizing a linear function over this set of integral vectors is the *traveling salesman problem* (TSP). Dantzig et al. (1954) introduced H as a relaxation of the TSP and developed the *cutting-plane method* for optimizing over H_I. The most successful algorithms for solving large TSP instances all adopt the Dantzig et al. approach (see Jünger et al. 1995 for a survey of this work).

Chvátal (1973b) conjectured that the Chvátal rank of $H(G)$ tends to infinity with the number of vertices n; Chvátal et al. (1989) proved this by establishing that the Chvátal rank of $H(G)$ is at least $\lfloor n/8 \rfloor$. We will adapt the proof in the above paper to show that the N_* rank of $H(G)$ is also at least $\lfloor n/8 \rfloor$. This bound cannot be improved by more than a constant factor; we establish an upper bound of $n+1$ on the N_* rank of $H(G)$ (as pointed out by a referee, this can be improved slightly to $n-2$ using a result in Goemans 1998). The dimension of $H(G)$ is $\frac{1}{2}n(n-3)$ (see Grötschel and Padberg 1985), so these results establish that the N_* rank of $H(G)$ is within a constant factor of the square root of its dimension. This is similar to the Stephen and Tunçel (1999) result for the semidefinite rank of the standard relaxation of the matching polytope; note however that the Chvátal rank (and hence the N_* rank) of the matching relaxation is 1.

We begin by identifying two subsets of edges used in Chvátal et al. (1989). Let $k = \lfloor n/8 \rfloor$ and $r = n - 8k$. Label the n vertices in V as $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i$ for $i = 1, \ldots, k$, and w_j for $j = 1, \ldots, r$; for convenience we set $w_0 = e_k$ and $w_{r+1} = a_1$. Let $E_{1/2}$ denote the edge-set

$$(a_i, b_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (e_i, f_i),$$
$$(f_i, g_i), (g_i, h_i), (h_i, a_i), \qquad i = 1, \ldots, k,$$

and let E_1 denote the edge-set

$$\begin{aligned}&(h_i, d_i), (b_i, f_i), && i = 1, \ldots, k,\\ &(c_i, g_{i+1}), (e_i, a_{i+1}), && i = 1, \ldots, k-1,\\ &(c_k, g_1),\\ &(w_j, w_{j+1}), && j = 0, \ldots, r.\end{aligned}$$

The two sets are illustrated in Figure 1.

It is easy to verify that no Hamiltonian circuit contained entirely in $E_{1/2} \cup E_1$ can use every edge in E_1. In other words, each 0–1 vector in H satisfies the inequality

$$x(E_{1/2}) + 2x(E_1) \le (n-1) + |E_1|. \tag{30}$$

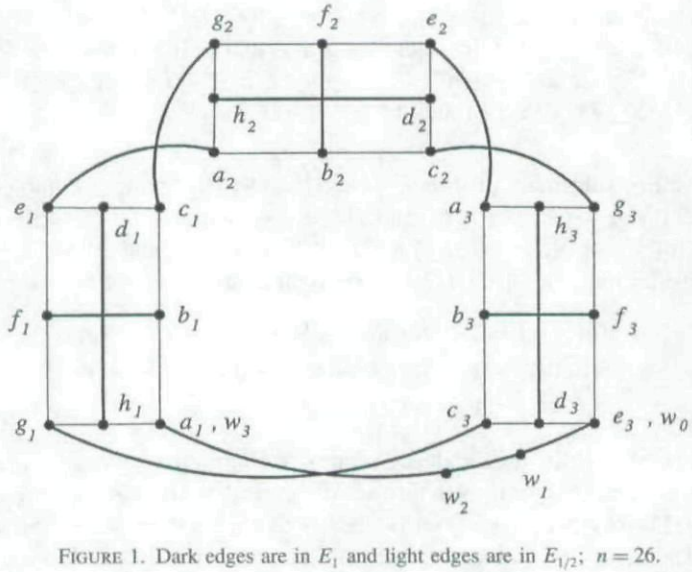

FIGURE 1. Dark edges are in E_1 and light edges are in $E_{1/2}$; $n = 26$.

Notice that this inequality is violated by the vector x^* obtained by setting $x^*_e = 1$ if $e \in E_1$, $x^*_e = \frac{1}{2}$ if $e \in E_{1/2}$, and $x^*_e = 0$ otherwise; therefore, x^* is not a member of H_I. We will obtain a lower bound on the N_* rank of H by showing that $N_*^{k-1}(H)$ contains x^*.

THEOREM 4.1. *Let G be a complete graph with n vertices. The rank of $H(G)$ (as defined in (29)) with respect to the N_* operator is at least $\lfloor n/8 \rfloor$ and at most $n+1$.*

PROOF. Consider the polytope $T(G)$ (or T) defined as the set of all $x \in R^E$ satisfying

$$\begin{aligned} x(\delta(\{v\})) &\le 2 && \text{for all } v \in V, \\ x(\gamma(W)) &\le |W| - 1 && \text{for all } W \subseteq V \text{ with } \varnothing \ne W \ne V, \\ 0 \le x_e &\le 1 && \text{for all } e \in E. \end{aligned}$$

Let $F = \{x \in Q : x(\delta(\{v\})) = 2 \text{ for all } v \in V\}$. It can be shown that $H(G)$ is a face of $T(G)$ (see Grötschel and Padberg 1985 for a discussion); in particular we have $H = T \cap F$ and $H_I = T_I \cap F$. Any 0–1 vector in T can have at most n ones; thus $\max\{\mathbf{1}^T x : x \in T_I\} = n$. Since T is an antiblocking polytope, it follows from Lemma 2.7 that the semidefinite rank of T is at most $n+1$. By Lemma 2.2 we have $N_+^{n+1}(H) = N_+^{n+1}(T \cap F) = N_+^{n+1}(T) \cap F = T_I \cap F = H_I$. Hence the N_* (or N_+) rank of H is at most $n+1$.

To obtain the lower bound, we show the existence of a function $f : Q_{\lfloor n/8 \rfloor} \to R^E$ satisfying the following properties (let $k = \lfloor n/8 \rfloor$):

(i) f is a composition of the embedding, flipping, and duplication operations;
(ii) $f(P_k) \subseteq H$, where P_k is defined as in (22);
(iii) $x^* = f(\frac{1}{2}\mathbf{1}) \notin H_I$.

Given such an f, Corollary 3.4 and Lemma 3.5 together imply that $x^* = f(\frac{1}{2}\mathbf{1}) \in f(N_*^{k-1}(P_k)) \subseteq N_*^{k-1}(H)$. Hence (iii) implies that the N_* rank of H is at least k.

We will construct f as in Chvátal et al. (1989). If $y \in Q_k$, let $f(y)$ be the vector $x \in R^E$ defined by

$$x_e = \begin{cases} 1 & \text{if } e \in E_1, \\ y_i & \text{if } e \in \{(a_i, b_i), (c_i, d_i), (e_i, f_i), (g_i, h_i)\}, \\ 1 - y_i & \text{if } e \in \{(b_i, c_i), (d_i, e_i), (f_i, g_i), (h_i, a_i)\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that (i) and (iii) hold for f defined in this way. A short proof that (ii) holds can be found in Lemma 8.2 of Chvátal et al. (1989); for completeness we repeat the argument below.

Consider an arbitrary vector $y \in P_k$ and let x' denote $f(y)$. We must show that x' satisfies each inequality in (29). Clearly $0 \le x' \le 1$ and $x'(\delta(\{v\})) = 2$ for all $v \in V$. It remains to show that x' satisfies $x(\delta(W)) \ge 2$ for all proper subsets $W \subseteq V$.

For $J \subseteq \{1, \ldots, k\}$, let $y_J \in Q_k$ denote the incidence vector of J and observe that $f(y_J)$ is the incidence vector of two circuits in G, one spanning the set

$$W_J = \left(\bigcup_{i \in J} \{g_i, h_i, d_i, c_i\} \right) \cup \left(\bigcup_{i \notin J} \{g_i, f_i, b_i, c_i\} \right),$$

and the other spanning $V \setminus W_J$. Therefore, $f(y_J)$ satisfies $x(\delta(W)) \ge 2$ for each proper subset $W \subseteq V$ other than W_J. Let W be any proper subset of V.

Case 1. $W \ne W_J$ for all J. Since y is a convex combination of vectors y_J (as y is in Q_k), we can conclude that x' satisfies $x(\delta(W)) \ge 2$.

Case 2. $W = W_J$ for some J. We have

$$x'(\delta(W_J)) = 4 \sum_{i \in J} (1 - y_i) + 4 \sum_{i \notin J} y_i.$$

Therefore, since y satisfies the inequalities (22), we know that $x'(\delta(W_J)) \ge 2$. □

A similar result can be proven for the standard relaxation of the asymmetric traveling salesman problem; the proof is again an easy application of Corollary 3.4 and the proof method used in Chvátal et al. (1989).

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