OPERATIONS THAT PRESERVE TOTAL DUAL INTEGRALITY

William COOK

Department of Combinatorics and Optimisation, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Received November 1982
Revised February 1983

There are many useful operations, such as adding slack variables, taking scalar multiples of inequalities, and applying Fourier–Motzkin elimination, that can be performed on a linear system such that if the system defines an integer polyhedron then so does the derived system. The topic dealt with here is whether or not these operations also preserve total dual integrality of linear systems.

Total dual integrality, integer polyhedra, Fourier-Motzkin

1. Introduction

A system of rational linear inequalities and rational linear equalities, $A_1x \leqslant b_1$, $A_2x = b_2$, is called *totally dual integral* if the (dual) linear program $\min\{y_1b_1 + y_2b_2: y_1A_1 + y_2A_2 = w, y_1 \geqslant 0\}$ has an integral optimal solution for each integral w for which the optimum exists. Hoffman [4] and Edmonds and Giles [2] have shown that if $A_1x \leqslant b_1$, $A_2x = b_2$ is totally dual integral and b_1 and b_2 are integral then the (primal) linear program $\max\{wx: A_1x \leqslant b_1, A_2x = b_2\}$ also has an integral optimal solution for each integral w for which the optimum exists (i.e. the polyhedron, $P = \{x: A_1x \leqslant b_1, A_2x = b_2\}$, defined by $A_1x \leqslant b_1$, $A_2x = b_2$ is an *integer polyhedron*—the restriction to integral w here is, of course, not necessary).

Many combinatorial min-max theorems, such as the max flow–min cut theorem, can be phrased as “a certain linear program and its dual each have integral optimal solutions” (the min-max equality following from the linear programming duality theorem). Often, a good way to prove such theorems is to work directly with the linear programs in question. The above mentioned theorem of Hoffman and Edmonds–Giles is useful in proving such results since to establish the result it only needs to be shown that the dual linear program always has an integral optimal solution—see Schrijver [7] for an example of using total dual integrality to prove a combinatorial min-max theorem. Combinatorial min-max theorems that arise by applying the theory of total dual integrality to problems are an improvement over the min-max theorems that arise by a straightforward application of the duality theorem, since the set of possible values for the dual variables is greatly reduced. Another reason that such theorems are useful is that integral solutions to the dual linear program often correspond to combinatorial objects that are of interest, such as cuts in the max flow–min cut theorem.

There are many operations that can be performed on a linear system such that if the system defines an integer polyhedron then so does the derived system. Some of these operations, such as adding slack variables or taking scalar multiples of inequalities, are useful because they allow certain assumptions to be made on the form of a linear system without losing any generality. Others, such as setting inequalities to equalities or applying Fourier–Motzkin elimination, are useful because often a system of interest arises by applying the operation to a system which is known to define an integer polyhedron. The topic dealt with here is whether or not these operations also preserve total dual integrality of linear systems. A survey of such operations is given, including a result on Fourier–Motzkin elimination.

The notion of a Hilbert basis is used in several of the proofs given in the next section. A set of vectors

0167-6377/83/$03.00

$\{a_1, \ldots, a_k\}$ is a Hilbert *basis* if every integer point in the convex cone they generate can be written as $\lambda_1 a_1 + \cdots + \lambda_k a_k$ for some nonnegative integer λ_i, $i = 1, \ldots, k$. Giles and Pulleyblank [3] used the result of Hilbert that every rational convex cone (i.e. a set of type $\{x: Ax \geqslant 0\}$, where A is rational) is generated by a finite integral Hilbert basis to show that every integer polyhedron can be defined by a totally dual integral system with integer right-hand sides.

Schrijver [6] has shown that every pointed rational convex cone is generated by a unique minimal integral Hilbert basis. He used this result to show that for every full dimensional rational polyhedron, P, there exists a unique minimal totally dual integral system $Ax \leqslant b$ such that A is integral and $P = \{x: Ax \leqslant b\}$ (call $Ax \leqslant b$ the *Schrijver system* for P).

The following lemma was used in both [3] and [6], it follows from complementary slackness.

Lemma 1.1. *A rational system $A_1 x \leqslant b_1$, $A_2 x = b_2$ is totally dual integral iff for every minimal nonempty face of $\{x: A_1 x \leqslant b_1, A_2 x = b_2\}$ the set of active rows of A_1, A_2 is a Hilbert basis (an* active row *is one for which the corresponding constraint holds as an equality for every point in the face).*

2. Operations

In this section α will denote a vector and β will denote a scalar.

2.1. Replacing an equality by two inequalities

Proposition 2.1. *The system $\{A_1 x \leqslant b_1,\ A_2 x = b_2,\ \alpha x = \beta\}$ is totally dual integral iff the system $\{A_1 x \leqslant b_1,\ A_2 x = b_2,\ \alpha x \leqslant \beta,\ -\alpha x \leqslant -\beta\}$ is totally dual integral.*

Proof. The result follows by noting that the change in the dual linear program is the variable z corresponding to $\alpha x = \beta$ being replaced by $z' - z''$ where $z' \geqslant 0$, $z'' \geqslant 0$. □

By this proposition, in the remainder of this section it can be assumed without loss of generality that the systems are of the form $Ax \leqslant b$.

2.2. Adding slack variables

Proposition 2.2. *Let α be integral. The system $\{Ax \leqslant b,\ \alpha x \leqslant \beta\}$ is totally dual integral iff the system $\{Ax \leqslant b,\ \alpha x + s = \beta,\ s \geqslant 0\}$ is totally dual integral.*

Proof. The following short proof is due to W. Pulleyblank. Let D1 denote the linear program $\min\{yb + z\beta: yA + z\alpha = w, y \geqslant 0, z \geqslant 0\}$ and let D2 denote the linear program $\min\{yb + z\beta: yA + z\alpha = w, z \geqslant w_s, y \geqslant 0\}$. Suppose that the system $\{Ax \leqslant b, \alpha x + s = \beta, s \geqslant 0\}$ is totally dual integral and let w be an integral vector such that D1 has an optimal solution. An integral optimal solution to D1 can be found by setting $w_s = 0$ in D2. Now suppose that $\{Ax \leqslant b, \alpha x \leqslant \beta\}$ is totally dual integral and let (w, w_s) be an integral vector such that D2 has an optimal solution. Let (y^*, z^*) be an integral optimal solution to D1 with right-hand side $w' = w - w_s\alpha$. An integral optimal solution to D2 is $(y^*, z^* + w_s)$. □

The result does not hold for arbitrary α, e.g. $\{x_1 \leqslant 0, \frac{2}{3}x_1 \leqslant 0\}$ is totally dual integral but $\{x_1 \leqslant 0, \frac{2}{3}x_1 + x_2 = 0, x_2 \geqslant 0\}$ is not.

2.3. Setting inequalities to equalities

The following proposition is given in Schrijver [8], for completeness a proof is included here.

Proposition 2.3. *If* $\{Ax \leqslant b, \alpha x \leqslant \beta\}$ *is totally dual integral, then* $\{Ax \leqslant b, \alpha x = \beta\}$ *is also totally dual integral.*

Proof. By Proposition 2.1, $\{Ax \leqslant b, \alpha x = \beta\}$ is totally dual integral iff $\{Ax \leqslant b, \alpha x \leqslant \beta, -\alpha x \leqslant -\beta\}$ is totally dual integral. Also, if $a_1,\dots,a_k$, α, $-\alpha$ are the active rows of a minimal face of $\{x\colon Ax \leqslant b, \alpha x \leqslant \beta, -\alpha x \leqslant -\beta\}$ then $a_1,\dots,a_k$, α are the active rows of a minimal face of $\{x\colon Ax \leqslant b, \alpha x \leqslant \beta\}$. So, by Lemma 1.1, it suffices to prove that if $a_1,\dots,a_k$ is a Hilbert basis then $a_1,\dots,a_k$, $-a_1$ is also a Hilbert basis. Suppose that r is integral and $r = \Sigma\{\lambda_i a_i\colon i = 1,\dots,k\} - \gamma a_1$, where $\lambda_i \geqslant 0$, $i = 1,\dots,k$ and $\gamma \geqslant 0$. Choose γ' such that $r + \gamma' a_1$ is in the cone generated by $a_1,\dots,a_k$, $\gamma' a_1$ is integral and γ' is integral. Since $a_1,\dots,a_k$ is a Hilbert basis, $r + \gamma' a_1 = \Sigma\{\lambda'_i a_i\colon i = 1,\dots,k\}$, where λ'_i is a nonnegative integer for $i = 1,\dots,k$. Now $\Sigma\{\lambda'_i a_i\colon i = 1,\dots,k\} - \gamma' a_1$ expresses r as a nonnegative integer combination of $a_1,\dots,a_k$, $-a_1$. □

This result corresponds to the well-known result that the face of an integer polyhedron is also an integer polyhedron. It can be used to find a totally dual integral defining system for any face of a polyhedron for which such a system is known, e.g. totally dual integral defining systems for the perfect matching polyhedron and the convex hull of the common bases of two matroids can be derived from the totally dual integral systems for the matching polyhedron and the matriod intersection polyhedron respectively—see Pulleyblank [5].

2.4. Splitting unrestricted variables

Variables which are not restricted to nonnegative values are usually disposed of in linear programming theory by replacing them by the difference of two nonnegative variables. This operation, however, cannot be used on totally dual integral systems, e.g. the system $\{x_1 + 5x_2 \leqslant 1, x_1 + 6x_2 \leqslant 1\}$ is totally dual integral but the system obtained by replacing x_1 by $x'_1 - x''_1$ and x_2 by $x'_2 - x''_2$, where $x'_1 \geqslant 0$, $x''_1 \geqslant 0$, $x'_2 \geqslant 0$, $x''_2 \geqslant 0$, is not totally dual integral. This example also shows that the property of defining an integer polyhedron is not preserved under such a splitting operation ($x'_1 = 0$, $x''_1 = 0$, $x'_2 = \frac{1}{6}$, $x''_2 = 0$ is a nonintegral vertex of the derived polyhedron).

2.5. Scalar multiplication of inequalities

It is not possible to multiply an inequality by an arbitrary positive scalar and maintain total dual integrality, e.g. $x_1 \leqslant 0$ is totally dual integral but $2x_1 \leqslant 0$ is not. In fact, Giles and Pulleyblank [3] have shown that for any rational system $Ax \leqslant b$ there exists a positive scalar d such that $dAx \leqslant db$ is totally dual integral. However, an inequality in a totally dual integral system can be multiplied by a scalar of the form $1/k$, where k is a positive integer, since solutions to the new dual linear program can be found by setting a single component $\bar{y}_i$ to $k\bar{y}_i$ in a solution to the original dual linear program. For linear systems with integer left-hand sides that define full dimensional polyhedra the following proposition shows that this is the only type of nontrivial scalar multiplication that is possible.

Proposition 2.4. *Let A and α be integral and let d be a positive rational scalar. Suppose that $Ax \leqslant b$, $\alpha x \leqslant \beta$ is totally dual integral and that $P = \{x\colon Ax \leqslant b, \alpha x \leqslant \beta\}$ is of full dimension. The system $Ax \leqslant b$, $d\alpha x \leqslant d\beta$ is also totally dual integral iff either $\alpha x \leqslant \beta$ is not the Schrijver system for P or d is of the form $1/k$ for some positive integer k.*

Proof. The sufficiency of either condition is easily seen. To show necessity, suppose $\{Ax \leqslant b, d\alpha x \leqslant d\beta\}$ is totally dual integral and $\alpha x \leqslant \beta$ is in the Schrijver system for P. There exists a convex cone, C, generated by the active rows of some minimal face, F, of P, that contains α in its unique minimal Hilbert basis (C is pointed since P is of full dimension). Let λ be the multiplier of $d\alpha$ in an expression of α as a nonnegative integer combination of the active rows of F in $\{A, d\alpha\}$. The multiplier λ must be positive since α cannot be expressed as a nonnegative integer combination of other integral vectors in C. Since $-\alpha \notin C$ (C is pointed),

λd must be less than or equal to 1. Since $\lambda d\alpha$ must be integral, $\lambda d\alpha = \alpha$ (α is the first nonzero integral point on the ray $\{y_1\alpha: y_1 \geqslant 0\}$). So $d = 1/\lambda$. □

2.6. Unimodular transformations

Let U be an integral matrix such that $\det(U) = \pm 1$. A simple and well-known result is that $Ax \leqslant b$ is totally dual integral iff $AU^{-1}x \leqslant b$ is totally dual integral. This corresponds to the result that P is an integer polyhedron iff $P' = \{Ux: x \in P\}$ is an integer polyhedron.

2.7. Fourier–Motzkin elimination

Let K be a set in $\mathbb{R}^n$ and $I \subseteq \{1,\ldots,n\}$. If $x \in \mathbb{R}^n$, let x_I denote the vector $(x_i: i \in I)$. The *projection* of K onto the I coordinates is the set $\{x_I: x \in K\}$. A well-known and useful result is that any projection of an integer polyhedron is again an integer polyhedron—for an example of its application see Balas and Pulleyblank [1]. To obtain a corresponding result for total dual integrality, linear systems must be considered.

If a defining system for a polyhedron is given then a defining system for any projection of the polyhedron can be found by repeated application of Fourier–Motzkin elimination—see [9]. Consider the linear system

$$\begin{aligned} &a_i x - a_i x_0 > b_i, && i \in I = \{1,\ldots,i_0\},\\ &c_j x + \gamma_j x_0 \leqslant d_j, && j \in J = \{1,\ldots,j_0\},\\ &f_k x \leqslant g_k, && k \in K_0 = \{1,\ldots,k_0\} \end{aligned} \tag{1}$$

where a_i, c_j, f_k are vectors, α_i, γ_j are positive scalars, and b_i, d_j, g_k are scalars, for $i \in I, j \in J, k \in K_0$. The following system is obtained by applying Fourier–Motzkin elimination to eliminate the variable x_0:

$$\begin{aligned} &(\gamma_j a_i + \alpha_i c_j)x \leqslant \gamma_j b_i + \alpha_i d_j, && i \in I, j \in J,\\ &f_k x \leqslant g_k, && k \in K_0 \end{aligned} \tag{2}$$

(the scaling used here is chosen so that the resulting system has integral data if the original system does).

It is not true that (2) must be totally dual integral if (1) is, even if all data is integral and the coefficients of any row of (2) have greatest common divisor 1. For example, $\{x_1 - x_3 \leqslant 0,\ x_2 - x_3 \leqslant 0,\ -x_2 + 2x_3 \leqslant 0\}$ is totally dual integral since the coefficient matrix is unimodular, but $\{2x_1 - x_2 \leqslant 0,\ x_2 \leqslant 0\}$ is not totally dual integral. However, the result is true in a special case.

Theorem 2.5. *Let $Ax \leqslant b$ be a totally dual integral system. If each coefficient of the variable x_0 is either* 0, 1, *or* -1 *then the system obtained by eliminating x_0 by Fourier–Motzkin elimination is also totally dual integral.*

Proof. What must be shown is that if (1) is totally dual integral and α_i, γ_j are equal to 1 for $i \in I, j \in J$, then (2) is also totally dual integral. Let D1 denote the linear program

$$\begin{aligned} \min\quad & \Sigma\{b_i y_i': i \in I\} + \Sigma\{d_j y_j'': j \in J\} + \Sigma\{g_k z_k: k \in K_0\},\\ \text{s.t.}\quad & bs\{a_i y_i': i \in I\} + \Sigma\{c_j y_j'': j \in J\} + \Sigma\{f_k z_k: k \in K_0\} = w,\\ & -\Sigma\{y_i': i \in I\} + \Sigma\{y_j'': j \in J\} = w_0,\\ & y_i' \geqslant 0, y_j'' \geqslant 0, z_k \geqslant 0, \quad i \in I, j \in J, k \in K_0 \end{aligned} \tag{3}$$

and let D2 denote the linear program

$$\begin{aligned} \min\quad & \Sigma\{(b_i + d_j)y_{ij}: i \in I, j \in J\} + \Sigma\{g_k z_k: k \in K_0\},\\ \text{s.t.}\quad & \Sigma\{(a_i + c_j)y_{ij}: i \in I, j \in J\} + \Sigma\{f_k z_k: k \in K_0\} = w,\\ & y_{ij} \geqslant 0, z_k \geqslant 0, \quad i \in I, j \in J, k \in K_0. \end{aligned} \tag{4}$$

Suppose that w is integral and D2 has an optimal solution $(\bar{y}, \bar{z})$. An optimal solution to D1 with $w_0 = 0$ is $\bar{z}$ together with

$$\bar{y}'_i = \Sigma\{\bar{y}_{ij}: j \in J\}, \quad i \in I,$$
$$\bar{y}''_j = \Sigma\{\bar{y}_{ij}: i \in I\}, \quad j \in J \tag{5}$$

since any solution (y', y'', z) to D1 with $w_0 = 0$ corresponds to a solution to D2 with the same objective value, by finding a nonnegative y such that

$$\Sigma\{y_{ij}: j \in J\} = y'_i, \quad i \in I,$$
$$\Sigma\{y_{ij}: i \in I\} = y''_j, \quad j \in J \tag{6}$$

($w_0 = 0$ implies that $\Sigma\{y'_i: i \in I\} = \Sigma\{y''_j: j \in J\}$).

An integral optimal solution to D2 can be found by finding an integral optimal solution, (y', y'', z) to D1 with $w_0 = 0$ and then finding a nonnegative integral y that satisfies (6). □

Acknowledgment

I thank U.S.R. Murty and William Pulleyblank for help in this work. In particular, W. Pulleyblank suggested the problem studied in Section 2.7.

References

[1] E. Balas and W. Pulleyblank, "The perfectly matchable subgraph polytope of a bipartitie graph", *Networks*, to appear.
[2] J. Edmonds and R. Giles, "A min-max relation for submodular functions on graphs", *Ann. Discrete Math.* **1**, 185–204 (1977).
[3] F.R. Giles and W.R. Pulleyblank, "Total dual integrality and integer polyhedra", *Linear Algebra Appl.* **25**, 191–196 (1979).
[4] A.J. Hoffman, "A generalization of max flow-min cut", *Math. Programming* **6**, 352–359 (1974).
[5] W. Pulleyblank, "Polyhedral combinatorics", *Proc. XI. International Symposium on Mathematical Programming*, Bonn, W. Germany (1982), to appear.
[6] A. Schrijver, "On total dual integrality", *Linear Algebra Appl.* **38**, 27–32 (1981).
[7] A. Schrijver, "Proving total dual integrality with cross-free families—a general framework", Report AE 5/82, Inst. Act. & Econ., Univ. van Amsterdam, Amsterdam (1982).
[8] A. Schrijver, "Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions", *Proc. Silver Jubilee Conference on Combinatorics*, University of Waterloo (1982), to appear.
[9] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, Springer, Berlin (1970).