MATH 650 : Mathematical Modeling

Spring, 2019 - Written Assignment #1

Due by 11:59 p.m. EST on Tuesday, June 11th, 2019

Instructions:

- The problems on this assignment involve concepts, solution methods, and applications of first order differential equations, as introduced in Modules 1 to 4. Thus it is a *comprehensive* and *summative* assessment of your understanding of this material.
- Feel free to discuss the problems with one another, and to make use of any other resources which help you to work your way through them. However, your submissions must be your own, and must reference any source (human or otherwise) from whom you received help.
- Presentation is important. Please write your solutions in clear sentences which convey your reasoning. (See the Sample Solutions for a handy guide.) Remember that I can't know what you're thinking...I can only know what you tell me. Handwritten solutions are just fine, but they must be well-organized and legible, with the problems in numerical order. (I will send the LaTeX file if requested.)
- A few of the problems use specific Maple files which will be made available. (I'm assuming you can do simple Maple commands such as *plot* on your own.)
- There are four 'choice' questions on the assignment. A complete assignment is 140 marks. If you wish, you may submit ONE extra problem, either the optional problem 18 or another one, but no more than that will be graded. The marks for each problem are given on the following page.
- **I suggest you spend no more than 18 hours in total on this assignment, including writing up your solutions.** While you might not complete all the problems you'd like to, give it your best try...but don't worry about it beyond that time span. Partially completed problems are acceptable.

  (**Note:** This will be the longest assignment you will do in MATH 650; it covers five weeks of material, whereas each of the other two written assignments cover about three plus weeks each. This is mainly because the material in Modules 1 to 4 constitutes a self-contained block of material focussed on first order DEs, and hence this is a natural break-point for WA1.)
- All assignments will be submitted electronically in PDF format using the assignment drop box in LEARN.

Available Marks for Each Problem

Compulsory Problems	Marks	Choice Problems	Marks
1	10		
2	12		
3	10		
1 of 4 or 5	12	1 of 4 or 5	12
6	6		
1 of 7 or 8	8	1 of 7 or 8	8
9	8		
1 of 10 or 11	10	1 of 10 or 11	10
12	8		
1 of 13 or 14	12	1 of 13 or 14	12
15	10		
16	15		
17	12		
		18 (optional)	10
Presentation	7		
	140		

Solution Methods and Theoretical Considerations

1. **a.** Consider the DE $\dfrac{dy}{dt} + \dfrac{a}{t}\,y = t^{a-1}$, where a is a constant.

i. Find the general solution $y_1(t)$ of the given DE for $a = 2$.

ii. Find the general solution $y_2(t)$ of the given DE for $a = -2$.

b. Describe the behaviour of each of these solutions as $t \to \infty$.

c. Find an initial condition $y_2(1) = y_0$ for $y_2(t)$ such that, as $t \to \infty$, $y_1(t) \to y_2(t)$.

d. It appears that for $a \neq 0$, the solutions of the given DE consist of power functions t^a and t^{-a}. Determine whether this is also true if $a = 0$.

2. **a.** Explain how you know that there exists a unique solution of the IVP $\dfrac{dy}{dt} = 2\,t\,y^2$, $y(t_0) = y_0$ for every point (t_0, y_0) in the plane.

b. Solve the IVP $\dfrac{dy}{dt} = 2\,t\,y^2$, $y(0) = y_0$, and state any equilibrium solutions.

c. Show that, for the solutions in part b.,

i. every solution is symmetric about the $y-$axis;

ii. if $y_0 > 0$, the solution has a finite domain bounded by two vertical asymptotes;

iii. if $y_0 < 0$, the solution is a negative function defined for all real t, and is asymptotic to an equilibrium solution as $|t| \to \infty$.

d. Illustrate how the solutions depend on the initial condition y_0 by sketching the solutions for $y_0 = -2,\ -1,\ 0,\ \frac{1}{4},\ 1$.

3. For each of the given DEs, state whether the DE is linear, separable, exact, homogeneous, or a Bernoulli equation, or reducible to a linear or separable DE. Where appropriate, state the substitution required to reduce or solve the DE. Where a DE is of more than one type, state all possibilities. DO NOT SOLVE.

a. $\sqrt{\dfrac{y}{x}} + \cos x + \sqrt{\dfrac{x}{y}}\,\dfrac{dy}{dx} = 0$

b. $\dfrac{dy}{dt} + ty = y$

c. $\dfrac{dy}{dt} = \dfrac{ty^2 - \sin t \cos t}{(1 - t^2)y}$

d. $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{x^2 + y^2}}$

e. $y'' + 4y' = t^2$

⋆Do ONE of problems 4. or 5.

4. **a.** Find the integral curves of the exact DE $2x - y + (2y - x)\,\dfrac{dy}{dx} = 0$. (Easy.)

b. Show that the integral curve through $(1,1)$ defines exactly two explicit functions $y_1(x)$ and $y_2(x)$, and state the interval on which these functions are defined.

c. For the IVP $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$, $y(1) = 1$, the Existence-Uniqueness Theorem predicts possible trouble where $y = \frac{x}{2}$. Since $y_1(x)$ passes through $(1,1)$, it contains the solution of this IVP. What portion of $y_1(x)$ is this solution?

d. The integral curve in b. is actually a 'tilted' ellipse $(x - \frac{y}{2})^2 + \frac{3}{4}y^2 = 1$, with axes $y = x$ and $y = -x$. Sketch this ellipse, indicating the explicit solution through $(1,1)$. What is the slope of the ellipse as it crosses $y = \frac{x}{2}$?

5. **a.** Show that the general solution of the DE $x^2\,\dfrac{dy}{dx} = y^2$ is $y(x) = \dfrac{x}{Cx + 1}$.

b. For EACH of the ICs (i) $y(4) = -4$ and (ii) $y(1) = 2$:

- substitute the given IC to find the unique solution of the IVP;
- state the domain of validity (remember that it must include the initial point); and
- sketch the solution, labelling any vertical and/or horizontal asymptotes.

c. The general solution reveals that there are many solutions through $(0,0)$. Show that every solution through $(0,0)$ has slope $y'(0) = 1$, and this common tangent line is itself a solution of the DE. Add this tangent line, and some other typical solutions through $(0,0)$, to your sketch in b.

d. Does the existence of multiple differentiable solutions through $(0,0)$ violate the Existence-Uniqueness Theorem? Explain why, or why not.

Newton's Law of Temperature Change

6. Suppose a pot of soup has been simmering on the stove at 100 °C. For storage in the fridge, it should not be warmer than 20 °C. Initially, we cool it in a sink full of 5 °C water (running the water to keep it cold).

a. Let $t = 0$ when the soup pot was removed from the stove. If the soup cools to 60 °C after 10 minutes, what is the value of the rate constant k? (You may use the known solution from Module 1.)

b. With the soup at 60 °C, we remove the pot from the sink and set it on the counter to finish cooling. If the room temperature is 15 °C, how much longer will it be until we can put it in the fridge?

⋆Do ONE of problems 7. or 8.

A Universal Growth Law

7. If the energy required to both maintain existing cells and create new cells is taken into account, then it can be shown that the mass m of a living organism grows according to the IVP

$$\frac{dm}{dt} = am^{\frac{3}{4}}\left(1 - \left(\frac{m}{M}\right)^{\frac{1}{4}}\right), \quad m(0) = m_0\,,$$

where a and M are positive constants with $[M] = \mathcal{M}$, and $[a]^4 = \mathcal{M}\mathcal{T}^{-4}$.

a. Give a physical interpretation of the constant M. [HINT: What has happened to the organism if $\dfrac{dm}{dt} = 0\,$?]

b. Show that, in terms of a new variable $y = 1 - \left(\dfrac{m}{M}\right)^{\frac{1}{4}}$, the IVP becomes

$$\frac{dy}{dt} = -k\,y\,, \;\; y(0) = y_0\,,$$

where $k = \dfrac{a}{4M^{\frac{1}{4}}}$, and $y_0 = 1 - \left(\dfrac{m_0}{M}\right)^{\frac{1}{4}}$.

c. Write down the well-known solution of the IVP in part **b.**, and hence find the solution of the IVP in **a.**.

d. Given numeric values $M = 16$, $m_0 = 1$, and $a = 2$ for the constants, use Maple to plot the solution $m(t)$ for $0 \le t \le 25$, and estimate the time t^* at which the organism reaches half its ultimate size.

e. Explain how you could use the result of part **c.** to get a more accurate estimate of t^*.

Reference: (Problems 7 and 8 iii) *Topics in Mathematical Modeling*, by K. K. Tung, Princeton University Press, 2007, pages 48-49 and 74-75.

Exponential Decay

8. i. Read carefully the preamble to problem 12 of section 2.2 of your text (section 2.3 of Edition 1) on the use of carbon-14 (^{14}C) for dating archeological artifacts. Then answer each of parts a), b), and c) in that question.

ii. Show that the variables $y = \dfrac{Q}{Q_0}$ and $\tau = r\,t$ are dimensionless, and that the equivalent dimensionless IVP is $\dfrac{dy}{d\tau} = -y\,, \;\; y(0) = 1\,.$

iii. Search the web for "The Great hall and Round Table in Winchester" and read about this amazing oak table and its link to the legend of King Arthur and his Knights. In 1976, one of several tests on the table revealed that it contained 91.02% of the amount of ^{14}C in living wood. Use the solution of the IVP in part **ii** to determine the value of τ when $y = 0.9102$. Then convert your answer back to years. Given that King Arthur lived in the 5^{th} century AD, could the table have been his famous Round Table?

Comments:

1. There appears to be some disagreement over the exact value of the half-life of ^{14}C. While your text uses $T_h = 5730$ years, in *Differential Equations: A Modeling Perspective*, 2nd ed., J. Wiley & Sons, 2004, R. Borelli and C. Coleman cite the "internationally agreed upon value" as 5568 ± 30 years. Think about how that would affect your estimate.

2. The use of ^{14}C depends on an accurate value for the current ratio of ^{14}C in the atmosphere. Recently, this has changed dramatically due to industrial smoke and nuclear weapons testing. Carbon dating is most accurate for artifacts between 200 and 70 000 years old; for more recent items, or much older items, other radioactive elements can be used. (See the reference in 1. above for more details.)

Mixing A Toxic Gas

9. A coal miner is trapped in a cave-in after an explosion. He lies in a small chamber (volume $50\,\text{m}^3$) into which air containing $100\,\text{g/m}^3$ of carbon monoxide (CO) is seeping at a rate of $0.001\,\text{m}^3$ per minute, and from which the well-mixed CO and air is leaking at the same rate.

a. State an IVP for the quantity $Q(t)$ g of CO in the cave for $t \geq 0$, assuming the chamber was free of CO at the time of the cave-in.

b. Solve the IVP in a.

c. Given that CO at a concentration of 1200 ppm (or $\approx 1.4\,\text{g/m}^3$) is lethal to human beings, how long do his rescuers have to reach him?

⋆Do ONE of problems 10. or 11.

Toricelli's Law

10. In Module 1, Section 1, Lecture B iii a, the video shows the development of Toricelli's Law. One of the assumptions made is that the stream of water exiting a asmall hole of cross-sectional area a in the bottom of a tank retains that cross-section. If you've ever watched the flow of water from a tap, you'll know that's not the case - the column of water actually narrows as it exits the tap. This motivates the introduction of a dimensionless contraction coefficient α, so that the volume of water leaving the tank per unit time through the hole is assumed to be $\alpha\, a\sqrt{2gh}$, rather than just $a\sqrt{2gh}$, where h is the depth of water in the tank.

a. Suppose a cylindrical tank with constant cross-sectional area A has a small hole of area a at the bottom. If water is pumped into the tank at a constant rate r m^3 per minute, justify this DE for the depth $h(t)$ of water in the tank:

$$\frac{dh}{dt} = \frac{r - \alpha\, a\sqrt{2gh}}{A}.$$

b. Find the equilibrium solution h_e of this DE, and give a physical interpretation.

c. If the tank has height H, state a condition for the equilibrium solution to be physically meaningful. (Ask yourself what happens if r is too large.)

d. At $t = 0$, the pump is turned off (i.e., $r = 0$) just as $h = 3\,\mathrm{m}$. If $A = \pi\,\mathrm{m}^2$, $a = 0.01\pi\,\mathrm{m}^2$, $\alpha = 0.6$ for water, and $g = 9.8\,\mathrm{m\,s^{-2}}$, how long will it take the tank to empty?

Pond Pollution

11. At $t = 0$, a well-circulated pond with 10^6 litres of water contains a bacterial pollutant at a concentration of 0.01 kg per litre. Bacteria-free water drains into the pond at a rate of 2400 litres per day, and pure water evaporates from the pond surface (leaving the bacteria behind) at a rate of 1200 litres per day.

a. Find the volume $V(t)$ of fluid in the pond for $t \geq 0$.

b. Find the quantity Q_0 of bacterial pollutant in the pond at time $t = 0$, and explain why it does not change.

c. Show that the concentration $C(t) = \dfrac{Q_0}{V(t)}$ of bacterial pollutant decreases very slowly by determining how long it takes to drop to 0.001 kg per litre.

d. You realize that you forgot to allow for the fact that the bacteria in the pond also reproduce at a rate $r\,Q(t)$ kg per day. Find the quantity $Q(t)$ of bacteria for $t \geq 0$, using $Q(0) = Q_0$. Then show that if $r \geq 0.0012$ per day, the concentration $C(t) = \dfrac{Q(t)}{V(t)}$ never decreases (i.e., $\dfrac{dC}{dt} \geq 0$ for all $t \geq 0$).

Vertical Motion

12. The intricately rough surface of a grain of pollen, its size, and other features, result in motion subject to viscous damping.

Thus, assuming displacement positive upward, a grain of pollen of mass m falls from rest according to the IVP

$$m\frac{dv}{dt} = -mg - \gamma v\,, \quad v(0) = 0$$

with known solution

$$v(t) = -\frac{mg}{\gamma}(1 - e^{-\frac{\gamma}{m}t})$$

If we observe that a grain of pollen of known mass falls to earth from a height of 1 m after 25 s, can we find the drag coefficient γ from this data? Let's see.

a. Find the displacement $s(t)$ by solving the IVP

$$\frac{ds}{dt} = -\frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right), \; s(0) = 1\,.$$

b. Show that, in terms of the ratio $x = \frac{\gamma}{m}$, setting $s(t) = 0$ gives the equation

$$0 = -\frac{g}{x^2}\left(e^{-xt} + xt - \frac{x^2}{g} - 1\right)$$

In particular, with $t = 25\,\mathrm{s}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$, we see that $s(25) = 0$ is equivalent to

$$\frac{1}{9.8}x^2 - 25x + 1 = e^{-25x}$$

c. Obviously one solution of this equation is $x = 0$ (not too useful). Use Maple plots to narrow down the possible solution to around $x = 245$. Since this clearly occurs at the greater positive root of the quadratic, find its roots, verifying that $x = 244.96\,\mathrm{s^{-1}}$ to two decimals.

d. Now comes the fun part. We will use data for the pollen of *Amaranthus palmeri*, assuming a spherical grain with diameter 30 μm and density $1435\,\mathrm{kg\,m^{-3}}$. Find the mass m, and use the solution from part c. to find the drag coefficient γ.

e. The researchers found that for a grain of pollen with the properties of part d., the terminal velocity had magnitude of about $4\,\mathrm{cm\,s^{-1}}$. Do your results from c. and d. confirm that value?

Reference: (Problem12 d.) *www.researchgate.net/publication/*232695757

$\star$Do ONE of problems 13. or 14.

13. **a.** During the 'free-fall' portion of her descent, a sky-diver of mass $m = 75$ kg (including equipment) has terminal speed $|v_T| = 50\,\mathrm{m\ s^{-1}}$. Assuming positive displacement **downward**, an appropriate IVP is

$$m\,\frac{dv}{dt} = mg - \beta v^2\,, \quad v(0) = 0\,.$$

State the role of each term, find the dimensions of the positive constant β, and estimate its numerical value. (Use $g = 10\,\mathrm{m\ s^{-2}}$.)

b. For any positive values of m, g, and β, show that $u = \dfrac{v}{a}$ and $\tau = \dfrac{t}{a/g}$ are dimensionless variables, where $a = \sqrt{\dfrac{mg}{\beta}}$. Hence derive the equivalent dimensionless IVP

$$\frac{du}{d\tau} = 1 - u^2\,, \quad u(0) = 0\,, \ \text{for} \ \ 0 \le u \le 1\,.$$

c. Solve the DE in part b. using partial fractions. Hence find the solution of the IVP in part a. , and confirm that

$$\lim_{t\to\infty} |v(t)| = 50\,\mathrm{m\ s^{-1}}\,,$$

i.e., the terminal speed is the limit of $|v(t)|$ as $t \to \infty$.

d. Once the parachute opens, clearly the drag coefficient will change. Suppose that this happens at $t = 10\,\mathrm{s}$, and the new DE for her velocity during the second stage of her descent is

$$m\,\frac{dv}{dt} = mg - \gamma v^2\,.$$

Find a value for the positive constant γ such that her new terminal velocity is 5 $\mathrm{m\ s^{-1}}$, a safe landing speed.

e. Will the transition be smooth? (Think about how the drag force will change at $t = 10\,\mathrm{s}$.) What might help smooth this 'jerk'?

Archimedes' Law in Motion!

14. A rubber balloon filled with helium rises vertically after being released from rest. Three forces act on the balloon: gravity, a resistive drag force proportional to the square of its speed, and the buoyant force due to the fact that helium is much less dense than air. Here are the relevant data.

- The empty balloon has a mass of 10 g.
- The balloon filled with helium has volume 0.01 $\mathrm{m^3}$.

- The density of helium is 0.2 kg m^{-3}.
- The density of air is 1.3 kg m^{-3}.
- Gravitational acceleration is 9.8 m s^{-2}.
- The constant of proportionality for the drag force is $\gamma = 0.25\,\mathrm{N\ s^2\ m^{-2}}$.

a. Formulate an IVP for the velocity $v(t)$ of the balloon, assuming displacement positive upward from the point of release at time $t = 0$. DO NOT SOLVE.

b. Determine the terminal velocity of the balloon as it rises, if one exists.

c. Suppose a pinhole leak develops at some time $t_1 > 0$, when the velocity is v_1. The leak allows 1 $\mathrm{cm^3\,s^{-1}}$ of the helium to escape, with velocity $u = v - 0.002\,\mathrm{m\ s^{-1}}$.

i. Justify a new model for $v(t)$ given by

$$m\,\frac{dv}{dt} + 0.002\,\frac{dm}{dt} = -mg - \beta v^2 + B(t)\,,$$

where $B(t)$ is the (now variable) buoyant force, and $\beta = 0.01\,\mathrm{N\ s^2\ m^{-2}}$ is the new drag coefficient for the deflated balloon. (Use the general variable mass model of Module 2.)

ii. Find the volume $V(t)$, the mass $m(t)$, and the buoyant force $B(t)$ under these circumstances, and determine the time $t^\star$ when the balloon is emptied of helium.

iii. WITHOUT CONSULTING ANY DEs, describe in your own words the motion of the balloon on three intervals:

1. $0 \le t \le t_1$ 2. $t_1 \le t \le t^\star$ 3. $t > t^\star$

iv. Will the balloon have a terminal velocity in the interval $t > t^\star$? If so, what will it be?

Epidemic Models

15. Read carefully through the preamble on page 89 of your text (page 93 of Edition 1, 91 of Edition 3), and problem 20 (6 in Edition 3) just below. Don't solve problem 6, as it is just logistic growth (with $r = \alpha$ and $K = 1$), which we have studied in detail. However, do note the conclusion of this simple epidemic model, namely that eventually everyone becomes infectious.

Suppose local health care providers are able to isolate a fraction βy of the infectious individuals per unit time, giving a new IVP

$$\frac{dy}{dt} = \alpha\, y\,(1 - y) - \beta\, y\,, \quad y(0) = y_0\,.$$

a. Show that, if $\alpha > \beta$, this model is equivalent to logistic growth (find the new r and K), but that now not everyone eventually becomes infectious. DO NOT SOLVE the DE; use your prior knowledge of logistic growth and illustrate with two sketches, one showing a typical solution and any equilibria for $\beta = 0$, the other for $0 < \beta < \alpha$,

b. Show that, if $\alpha \leq \beta$, the epidemic dies out eventually. (Just examine the sign of $\dfrac{dy}{dt}$.) Illustrate your conclusions with two sketches of typical solutions and any equilibria, one for $\alpha = \beta$, and one for $\alpha < \beta$.

c. Explain why your answers to parts a. and b. make sense in terms of the roles of the parameters α and β as they affect the growth and removal of infectives.

Population Growth with Harvesting

16. **a.** A population $p(t)$ of mice in a barn grows exponentially with intrinsic growth rate $r = 0.1\,\text{days}^{-1}$, but is 'harvested' by the resident tomcat at a constant rate of $h > 0$ mice per day.

i. Justify the IVP $\dfrac{dp}{dt} = 0.1\,p - h\,,\ \ p(0) = p_0$ for this situation, explaining the role of each term.

ii. Using a qualitative analysis based on the signs of $\dfrac{dp}{dt}$ and $\dfrac{d^2p}{dt^2}$, sketch typical solutions (including any equilibrium solutions), and sketch the phase line.

iii. Hence show that if $h \leq 0.1\,p_0$, the mice survive, but if $h > 0.1\,p_0$, they are exterminated.

b. Suppose the barn's resources are limited and will support at most K mice, giving a revised model $\dfrac{dp}{dt} = 0.1\,p\,(1 - \dfrac{p}{K}) - h\,,\ \ p(0) = p_0\,.$

i. If $h = 0$ (no tomcat), what type of growth results? Illustrate your answer with a sketch of typical solutions, for $p_0 = \frac{3}{2}K,\ K,\ \frac{1}{2}K,$ and $\frac{1}{10}K$. Show clearly any equilibrium solutions and their stability, and any changes in concavity.

ii. Suppose $K = 200$ mice. Show that the model can be rewritten as

$$\frac{dp}{dt} = f(p) = -0.0005\left((p-100)^2 + 2000(h-5)\right), \quad p(0) = p_0\,.$$

Hence explain why, if the tomcat kills more than 5 mice per day, the mice will be exterminated regardless of the value of p_0.

[HINT: Examine the sign of $\dfrac{dp}{dt}$.]

iii. What if the tomcat is not quite so deadly, and only kills 15 mice every 4 days (i.e., $h = 3.75$ mice per day)? Show that in this case,the model can be rewritten as $\dfrac{dp}{dt} = f(p) = -0.0005\,(p-50)(p-150)\,,\quad p(0) = p_0\,.$ (ctd)

Sketch the graph of $f(p)$ for $p \geq 0$, and use qualitative analysis to sketch typical solutons for $p_0 = 40,\ 50,\ 75,\ 100,\ 150,\ 175\,.$ Sketch the phase line, and give the stability of any equilibria. State a sufficient condition on p_0 for the mice to survive in this case.

Euler's Method

17. a. Use the first two commands on the Maple worksheet **MMTEulerRK4** to plot piecewise approximations to the solution $y = \phi(t)$ of the IVP

$$\frac{dy}{dt} = (1-y)^2,\ \ y(0) = -5\,,$$

on the interval $0 \leq t \leq 2$, with $h = 0.4$ (i.e., 5 steps).

b. The plot in part a. is not very satisfying. Try it again, with *output=information* instead of *output=plot*. What do you observe about the numeric values of the approximations?

c. Try changing the stepsize h by doubling the number of steps from the default (5) to 10, using *numsteps*. What do you observe?

d. Now repeat parts a., b., and c. for the interval $0 \leq t \leq 1$. What do you observe?

e. Use the command *dfieldplot* to plot a direction field for the given DE on $-3.5 \leq t \leq 3.5$, $-5 \leq y \leq 2$. Use this plot, and your knowledge of how these methods obtain the approximations in parts a. to d., to explain your observations.

f. We obtained a crude upper bound for the error in Euler's Method after N steps of size h, namely

$$|E_N| \leq |t_N - t_0|\,\frac{Mh}{2}\,,$$

where M is an upper bound on the size of the second derivative $|\phi''|$ of the solution.

i. Show that the solution of the separable IVP in part a. is

$$y = \phi(t) = 1 - \frac{6}{6\,t+1}\,.$$

ii. To see just how crude our error bound for E_N is, show that on the interval $0 \leq t \leq 1$, with $h = 0.2$ (hence $N = 5$), it gives the bound $|E_N| \leq 43.2$. Then compare that to the actual error from your output in part d. with *output=information.*

g. Play with the tool *EulerTutor* (see the final command in **MMTEulerRK4**) to figure out how it works. Then consider the IVP $\dfrac{dy}{dt} = y^2 - t,\; y(0) = 0.4$. Explore the Euler's Method solution using 8 steps (i.e., $N = 8$) on the intervals $[0,1]$, $[0,2]$, $[0,4]$. Make a table showing the interval, the number of steps N, the step-size h and the absolute error E. By examining the graph of the true solution, try to explain why a relatively smaller h on $[0,1]$ gives much greater error than a much larger h on $[0,4]$. A direction field plot may help.

The Radius of the Universe (*Optional*)

18. Under certain simplifying assumptions, Einstein's field equations of general relativity reduce to the single nonlinear second order DE

$$2\,R\,R'' + R'^2 + k\,c^2 = 0, \quad R(t_0) = R_0 > 0, \quad R'(t_0) = v_0 > 0\,,$$

where R is the "radius of the universe", c is the speed of light, and k is a constant which is related to the geometry of the universe. Note that the independent variable, time t, does not appear in this DE.

a. Show that the DE can be rewritten as $[R\,(R')^2 + k\,c^2 R]' = 0$.

[Hint: Multiply the original DE by R'.]

Hence show that the 'reduced order' solution for R' is given implicitly by

$$R\,(R')^2 + k\,c^2 R = R_0\,v_0^2 + k\,c^2 R_0\,.$$

b. For a Euclidean universe (zero curvature) the constant $k = 0$. Show that, in this case, the DE in part a. has solution

$$R^{3/2} = R_0^{3/2} \pm 1.5\,v_0\sqrt{R_0}\,(t - t_0)\,.$$

c. Explain why this solution offers two possibilities as $t \to \infty$: a "big crunch" (where $R(t) \to 0$ at some finite $T > t_0$), and a "big bang".

Reference: *Differential Equations: A Modeling Perspective*, 2nd ed., J. Wiley & Sons, 2004, R. Borelli and C. Coleman, page 251.