Preface

A discrete quantum walk is determined by a unitary matrix U, the *transition matrix* of the walk. If the initial state of the system is given by a vector z, then the state of the system at time k is $U^k z$. The problem is to choose U and z so that we can do something useful, and indeed we can—Grover showed how an implementation of this setup could be used to enable quantum computers to search a database faster than any known classical algorithm.

The framework we have just described is impossibly general, a quantum computer can conveniently implement only a small subset of the set of unitary matrices. There is also a mathematical difficulty, in that it may be impossible to derive useful predictions of the behaviour of the walk without imposing some structure on U.

As we have described it, the transition matrix U is an operator on the complex inner product space $\mathbb{C}^d$. However, for the reasons just given, much of the work on discrete quantum walks considers the case where U is an operator on the space of complex functions on the arcs (ordered pairs of adjacent vertices) of a graph X . Physically meaningful questions must be expressed in terms of the absolute values of the entries of the powers U^k. Thus we might ask if, for a given initial state z, is there an integer k such that the absolute values of the entries of U^k are close to being equal?

Then goal of our work on this topic has been to attempt to relate the properties of the walk to the properties of the underlying graph, and this book is both an introduction to the topic and a report on our progress.

We start our treatment with the most famous topic, Grover's search algorithm. We offer two approaches, but in both cases we find that the transition matrix arises as a product $U = RC$, where R and C are unitary matrices with simple structure and are defined in terms of an underlying graph. In fact R and C are both involutions, and the algebra they generate is a matrix representation of the dihedral group. We make use of the fact to determine the spectral decomposition of U, in terms of the underlying graph. (If the graph is k-regular on n vertices, U is of order $nk \times nk$, so we have reduced the scale of the problem.) We then apply the resulting theory to the study of properties of our walks, and determine useful parameters. Of course, each time we identify a parameter of a walk, we have introduced a possibly new graph parameter, and many interesting questions raise their heads.

In the second part of the book we relax our assumptions that R and C are involutions. We find that, to properly specify the resulting walks, we must specify a linear ordering on the arcs leaving a vertex. As any graph theorist is aware, embeddings of graphs in an orientable surface are specified by cyclic orderings of the arcs leaving a vertex. Hence we offer a detailed treatment of graph embeddings and graph covers. Following this we consider walks based on shunts and walks on the line. We close the book with a treatment of what we call vertex-face walks, which are explicitly derived from embeddings of graphs in orientable surfaces.

We note that this book is based on the Ph.D. thesis of the second author https://uwspace.uwaterloo.ca/handle/10012/13952. The intended audience is mathematicians, particularly those who might be interested in new graph theoretical problems arising from the study of discrete quantum walks. The book by Portugal [58] provides a complementary view. We do not think any knowledge of physics is required to profit from this work; the required background is linear algebra (spectral decomposition) and some field theory. We have tried to keep things self-contained, but G&R [34] may prove a useful backup.

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