Controlling Graphs

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Outline

1. Control Theory
   - Linear Systems
   - Controllability and Observability
2. Graph Theory
   - Walk Matrices
   - Generating Functions
   - Controllable Pairs
3. Physics
   - Laplacians
   - Algebra
   - The Unitary Group

Getting Started

We consider a system whose state at time n is a vector x_n in $\mathbb{F}^v$, and which evolves under the constraint

$$x_{n+1} = Ax_n + Bu_n$$
$$y_n = Cx_n$$

Here A is $d \times d$, while B is $d \times e$ and C is $f \times d$. Usually e and f will be 0 or 1. The sequence $(u_n)_{n\geq 0}$ is chosen by us, it is our control of the system. The vectors y_n allow for the fact that we may not have explicit knowledge of the state, just some kind of summary of it.

An Example

We consider a crude model of a rod, heated at one end. As combinatorialists, we use a discrete model. The rod consists of five pieces, the temperature of the i-th piece at time n is $x_i(n)$ and the evolution is governed by the recurrence

$$x_i(n+1) = \frac{1}{3}(x_{i+1}(n) + x_i(n) + x_{i-1}n). \qquad (n = 1, 2, 3)$$

and $x_4(n+1) = \frac{1}{2}(x_3(n) + x_4(n))$. We control the temperature of the zeroth piece, so $x_0(n) = u(n)$.

. . . ctd.

We thus have

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \\ x_4(n+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ x_4(n) \end{pmatrix} + u(n) \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and it would reasonable to suppose that

$$c = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Outline

Controllability

The first question we ask is: which temperature distributions can we obtain?

Solving the Recurrence

In general:

$$x_1 = Ax_0 + u_0 b$$
$$x_2 = A^2 x_0 + u_0 Ab + u_1 b$$
$$x_3 = A^3 x_0 + u_0 A^2 b + u_1 Ab + u_2 b$$

From this we see that x_n is the sum of $A^n x_0$ and a linear combination of the vectors $A^r b$ for $r = 0, \ldots, n-1$. Using the Cayley-Hamilton theorem we see that if $n \geq d$ then our linear combination lies in the row space of

$$\begin{pmatrix} b & Ab & \ldots A^{d-1} b \end{pmatrix}.$$

Hence the state at time n is the sum of $A^n x_0$ and a vector in the column space of this matrix.

The Controllability Matrix

Definition

The matrix

$$\begin{pmatrix} B & AB & \dots & A^{d-1}B \end{pmatrix}$$

is the controllability matrix of our system. The system is controllable if its rows are linearly independent.

Observability

A second question we can ask is whether, from the sequence (y_n), we can determine the state vector.

The Observability Matrix

Definition

The observability matrix is

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{pmatrix};$$

the system is observable if its columns are linearly independent.

If the system is observable, then we can compute the state from the outputs (as you might have expected).

Outline

Subsets

Definition

Suppose $S \subseteq V(X)$ and the characteristic vector of S is b. If $v = |V(X)|$ then the walk matrix of the pair (X, S) is the matrix

$$W_S := \begin{pmatrix} b & Ab & \dots & A^{v-1}b \end{pmatrix}.$$

We say (X, S) is controllable if its walk matrix is invertible.

We will that X itself is controllable if the pair $(X, V(X))$ is controllable.

The Control Theory Analog

The linear system that best corresponds to our graph theory would be

$$x_{n+1} = Ax_n$$
$$y_n = b^T x_n$$

Questions about observability will translate to combinatorially significant questions; controllability questions tend not to.

Automorphisms

Theorem

If (X, S) is controllable, then any automorphism of X that fixes S as a set is the identity.

Proof.

Automorphisms are the permutation matrices P that commute with the adjacency matrix A. An automorphism fixes S as a set if and only if $Pb = b$. So

$$\begin{aligned} PW_S &= \begin{pmatrix} Pb & PAb & \dots & PA^{v-1}b \end{pmatrix} \\ &= \begin{pmatrix} Pb & APb & \dots & A^{v-1}Pb \end{pmatrix} \\ &= W_S. \end{aligned}$$

If W_S is invertible, it follows that $P = I$. □

Equitable Partitions

Theorem

If (X, S) is controllable and π is an equitable partition such that S is a union of the cells of π, then π is the discrete partition (with all cells of size one).

Simple Eigenvalues

Theorem

If there is a subset S of $V(X)$ such that (X, S) is controllable, then all eigenvalues of X are simple.

Proof of Simplicity

Proof.

Assume (X, S) is controllable and suppose that $E_1, \dots, E_r$ are the orthogonal projections onto the distinct eigenspaces of A. (So $r \le v$, and we want to show that $r = v$.)

Now if the θ_r is the eigenvalue associated with E_r, we have

$$A^m b = \sum_r \theta_r^m E_r b$$

and the vectors $E_r b$ are orthogonal. Hence $\mathrm{rk}(W_S)$ is equal to the number of non-zero vectors of the form $E_\theta b$. □

A Conjecture

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Almost all graphs are controllable.

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The evidence in support is strong:

- Up to 9 vertices, the proportion of controllable graphs is increasing.
- We can randomly sample graphs on n when n is large.

Outline

Walks

The walk matrix W_S is invertible if and only if $W_S^T W_S$ is invertible. Each entry of the latter matrix has the form

$$b^T A^r b$$

which is equal to the number of walks of length r in X that start and finish at a vertex in S. We naturally encode the sequence $(b^T A^r b)_{r \geq 0}$ as a generating function

$$\sum_{r \geq 0} b^T A^r b\, t^r = b^T (I - tA)^{-1} b.$$

The Transfer Matrix

If

$$X(t) := \sum_r t^r x_r, \quad U(t) := \sum_r t^r u_r$$

then we find that

$$X(t) = C(I - tA)^{-1} x_0 + tC(I - tA)^{-1} BU(t).$$

In control theory, the matrix of rational functions

$$C(zI - A)^{-1} B$$

is the transfer matrix of the system.

Isomorphism

Definition

Let b_1 and b_2 be vectors. Two pairs (X_1, b_1) and (X_2, b_2) are isomorphic if there is an orthogonal matrix L such that

$$LA_1L^T = A_2, \qquad Lb_1 = b_2$$

So

$$LW_1 = W_2$$

which ensures that controllability is an invariant. Further

$$W_2^TW_2 = W_1^TL^TLW_1 = W_1^TW_1$$

which provides a second invariant.

Isomorphism and Generating Functions

Theorem (Godsil)

Two pairs (X_1, b_1) *and* (X_2, b_2) *are isomorphic if and only if* A_1 *and* A_2 *are similar and*

$$b_1^T(I - tA_1)^{-1}b_1 = b_2^T(I - tA)^{-1}b_2.$$

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This implies an old result of Johnson and Newman: if X and Y are cospectral with cospectral complements, there is an orthogonal matrix L such that

$$L^T A(X) L = A(Y), \quad L^T A(\overline{X}) L = A(\overline{Y})$$

A Corollary

Corollary

If (X_1, b_1) *and* (X_2, b_2) *are controllable and*

$$b_1^T (I - tA_1)^{-1} b_1 = b_2^T (I - tA)^{-1} b_2$$

then (X_1, b_1) *and* (X_2, b_2) *are isomorphic.*

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then (X_1, b_1) and (X_2, b_2) are isomorphic.

Thus if X and Y are controllable graphs and

$$\mathbf{1}^T A(X)^r \mathbf{1} = \mathbf{1}^T A(Y)^r \mathbf{1}$$

for all r, then X and Y are cospectral with cospectral complements.

Near Automorphisms

Theorem

Let S_1 and S_2 be subsets of $V(X)$ with characteristic vectors b_1 and b_2. Then (X, S_1) and (X, S_2) are isomorphic if and only there is a rational symmetric orthogonal matrix Q such that:

- $QA = AQ$.
- $Qb_1 = b_2$.

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It is comparatively easy to show that the condition of the theorem is sufficient, the surprise is that it is necessary. For the latter, a generous hint is that $Q = W_2 W_1^{-1}$.

Cones

Definition

If $S \subseteq V(X)$, then the cone $\widehat{X}$ of X relative to S is the graph we get by adding a new vertex and declaring it to be adjacent to each vertex in S.

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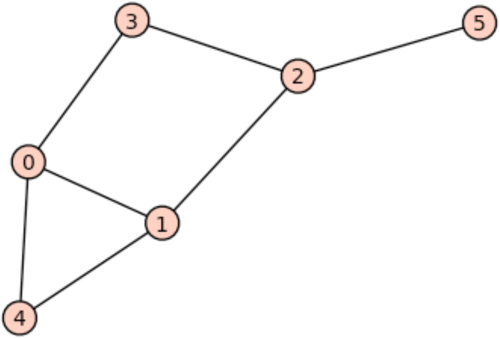
Theorem

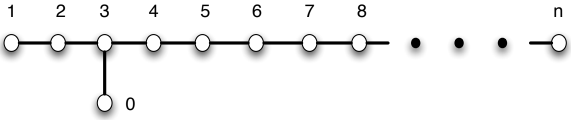
The pairs (X, S) and (Y, T) are isomorphic if and only if X is cospectral to Y and the cone of X relative to S is cospectral with the cone of Y relative to T.

Outline

An Example



More Examples



A Sequence

6	7	9	10	15	19	21	22	25	27	30	31
34	37	39	42	45	46	49	51	54	55	57	61
66	67	69	70	75	79	81	82	85	87	90	91
94	97	99	102	105	106	109	111	114	115	117	121

Vertices and Cones

Theorem

The pair (X, S) is controllable if and only if $(\widehat{X}, \{0\})$ is controllable.

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If X is a path and u is an end-vertex, then (X, u) is controllable.

Outline

Adding Edges

Suppose $L = D - A$ is the Laplacian of X and $i, j \in V(X)$ are not adjacent. If we set

$$H_{i,j} := (e_i - e_j)(e_i - e_j)^T$$

then $L + H_{i,j}$ is the Laplacian of the graph we get by adding the edge ij to X. If we denote this graph by Y and set $h = e_i - ej$, then

$$\frac{\phi(L(Y), t)}{\phi(L(X), t)} = 1 + \sum_{\lambda} \frac{h^T F_\lambda h}{t - \lambda}$$

where the F_λ's are the projections onto the eigenspaces of $L(X)$.

Controllable Laplacians

Definition

We define the pair $(X, \{i, j\})$ to be Laplacian controllable if the rows of the controllability matrix of L and

$$\begin{pmatrix} \mathbf{1} & e_i - e_j \end{pmatrix}$$

are linearly independent.

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are linearly independent.

Lemma

If $(X, \{i, j\})$ is Laplacian controllable, then the only automorphism of X that fixes the set $\{i, j\}$ is the identity.

Outline

Generating Matrices

Theorem (Godsil)

Let X be a graph with adjacency matrix A and let S be a subset of $V(X)$ with characteristic vector b. The following claims are equivalent:

- (X, S) *is controllable.*

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Let X be a graph with adjacency matrix A and let S be a subset of $V(X)$ with characteristic vector b. The following claims are equivalent:

- (X, S) *is controllable.*
- *The matrices A and bb^T generate the algebra of all $n \times n$ matrices.*
- *The matrices $A^i bb^T A^j$ ($0 \leq i, j < n$) are a basis for the space of $n \times n$ matrices.*

A Conjecture

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Almost all graphs are controllable.

Irreducibility

Lemma

If $\phi(X, t)$ is irreducible over $\mathbb{Q}$, then X is controllable.

Lemma

If $u \in V(X)$ and $\phi(X, t)$ and $\phi(X \setminus u, t)$ are coprime, then (X, u) is controllable.

Outline

Quantum Walks

If A is the adjacency matrix of X, then

$$H_X(t) = \exp(iAt)$$

is the transition matrix of a quantum walk on X.

Unitary Generators

Theorem

If (X, S) is controllable and b is the characteristic vector os S, the matrices

$$\exp(iAt), \quad \exp(ibb^T t)$$

generate a dense subgroup of the group of all unitary matrices.

Extensions

The key to the proof of the previous theorem is that if (X, S) is controllable, then A and bb^T together generate the algebra of all matrices.

If X itself is controllable, then A and J generate all matrices. If X is connected and D is its matrix of degrees, then J lies in the matrix algebra generated by A and D. Hence A and D generate all matrices.

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The key to the proof of the previous theorem is that if (X, S) is controllable, then A and bb^T together generate the algebra of all matrices.

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Does it follow that $\exp(iAt)$ and $\exp(iDt)$ generate a dense subgroup of the unitary group? (The difficulty is to determine the Lie algebra generated by A and D.)