Are Almost All Graphs Cospectral?

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Outline

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The Characteristic Polynomial

Definition

Let G be a graph with adjacency matrix A. The characteristic polynomial $\phi(G,t)$ of G is the characteristic polynomial of A:

$$\phi(G,t) := \det(tI - A).$$

Examples

The characteristic polynomials of K_1, K_2 and P_3 are respectively:

$$t, \quad t^2 - 1, \quad t^3 - 2t.$$

Walks

Lemma

If A is the adjacency matrix of G, then $(A^r)_{i,j}$ is the number of walks in G from vertex i to vertex j with length r. □

Closed Walks

A walk in G is closed if its first and last vertices are equal. The number of closed walks in G with length r is

$$\sum_{i \in V(G)} (A^r)_{i,i} = \operatorname{tr}(A^r).$$

Short Closed Walks

If G has n vertices, e edges and contains exactly t triangles, then

$$\begin{aligned} \operatorname{tr}(A^0) &= n \\ \operatorname{tr}(A^1) &= 0 \\ \operatorname{tr}(A^2) &= 2e \\ \operatorname{tr}(A^3) &= 6t. \end{aligned}$$

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(And then it gets messy!)

A Generating Function

The generating function for the closed walks in G, counted by length, is

$$\sum_{r\geq 0} \mathrm{tr}(A^r)t^r.$$

It is a rational function:

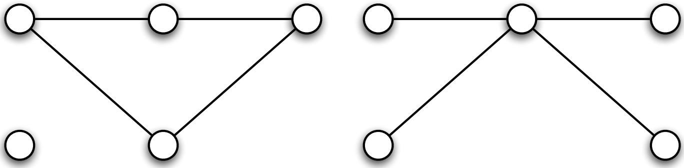
$$\sum_{r\geq 0} \mathrm{tr}(A^r)t^r = \frac{t^{-1}\phi'(G,t^{-1})}{\phi(G,t^{-1})}.$$

A Characterisation

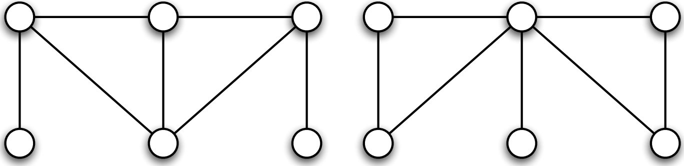
Corollary

Two graphs G and H are cospectral if and only if their generating functions for closed walks are equal.

The Smallest Cospectral Graphs

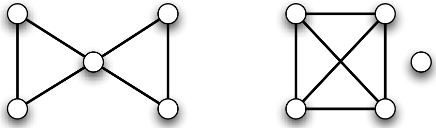


The Smallest Connected Cospectral Graphs



Cospectral, Cospectral Complements?

The graphs $\overline{C_4 \cup K_1}$ and $\overline{K_{1,4}}$ have two and four triangles respectively—they are not cospectral.



Another Generating Function

The number of walks of length r in G is equal to

$$\mathrm{tr}(A^r J) = \mathbf{1}^T A^r \mathbf{1}$$

and thus

$$\sum_{r \geq 0} \mathrm{tr}(A^r J) t^r$$

is the generating function for all walks in G.

Complements and Walks

Theorem

Suppose G and H are cospectral graphs with respective adjacency matrices A and B. Then $\overline{G}$ and $\overline{H}$ are cospectral if and only if the generating functions for all walks in G and in H are equal.

Regular Graphs

If G is a k-regular graph on n vertices then its walk generating function is

$$\frac{n}{1-kt}.$$

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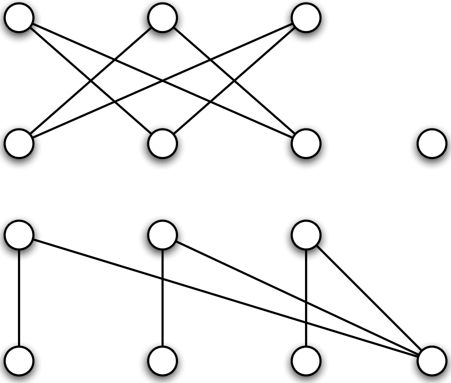
Lemma

Cospectral regular graphs have cospectral complements. □

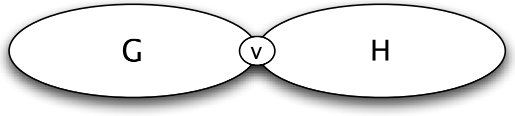
Two Irregular Graphs



Outline

0-Sums

The 0-sum of two graphs G and H is got by identifying a vertex in G with a vertex in H:



Spectrum of a 0-Sum

If we create the 0-sum F by merging v in G with v in H, then

$$\phi(F) = \phi(G)\phi(H \setminus v) + \phi(G \setminus v)\phi(H) - t\phi(G \setminus v)\phi(H \setminus v).$$

Example

If $G = K_2$ and $K = K_2$ then their 0-sum F is P_3, whence

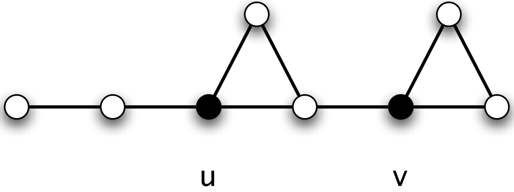
$$\phi(P_3, t) = (t^2 - 1)t + t(t^2 - 1) - t(t^2) = t^3 - 2t.$$

Corollary

If we hold G and its vertex v fixed, then the characteristic polynomial of the 0-sum of G and H is determined by the characteristic polynomials of H and $H \setminus v$.

Deleting Vertices

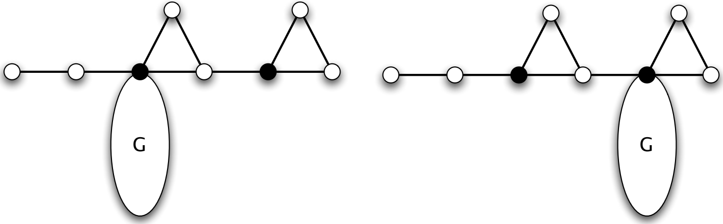
If H is the graph



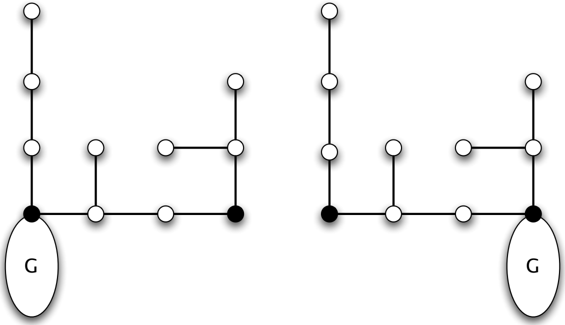
then $H \setminus u$ and $H \setminus v$ are isomorphic...

A Cospectral Pair

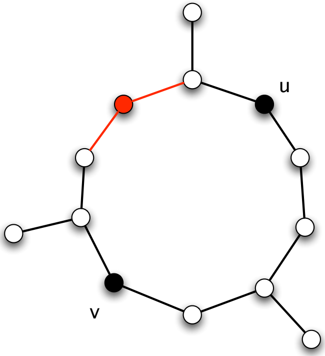
. . . and thus we obtain a pair of cospectral graphs:



Another Pair



A Hint



A Theorem

Theorem (Schwenk. . . Godsil & McKay)

Almost all trees are cospectral. . .

A Theorem

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Almost all trees are cospectral. . . with cospectral complements.

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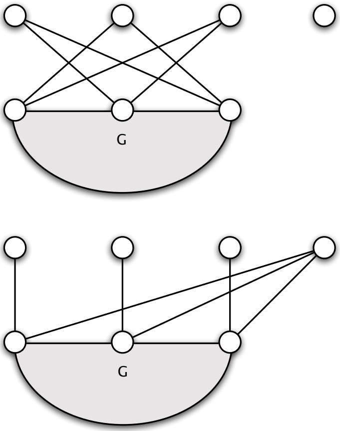
Yet Another Construction



If

$$K := \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then $K\mathbf{1} = \mathbf{1}$ and $K^2 = I$, whence K is orthogonal, and

$$KM = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = J_4 - M.$$

$$\begin{pmatrix} K & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M & 0 \\ M^T & A_1 & B_1 \\ 0 & B_1^T & A_2 \end{pmatrix} \begin{pmatrix} K & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & J-M & 0 \\ (J-M)^T & A_1 & B_1 \\ 0 & B_1^T & A_2 \end{pmatrix}$$

Therefore. . .

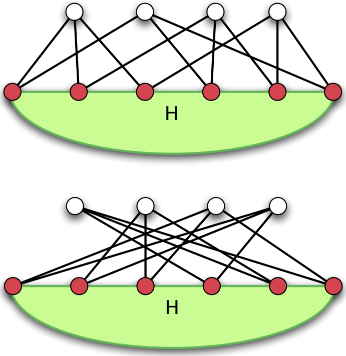
Theorem

Switching related graphs are cospectral, with cospectral complements.

A Problem

Is it true that almost all graphs are determined by their spectrum?

A Related Example



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A Subspace

Let G be a graph on n vertices with adjacency matrix A. Define U to be the subspace of $\mathbb{R}^n$ spanned by the vectors $A^r\mathbf{1}$, for all non-negative integers r.

Automorphisms

Theorem

If the permutation matrix P is in $\mathrm{Aut}(G)$, then $Pu = u$ for all u in U.

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Proof.

If P is a permutation matrix, $P\mathbf{1} = \mathbf{1}$. If $P \in \mathrm{Aut}(G)$, then $PA = AP$ and so, for all r

$$PA^r\mathbf{1} = A^rP\mathbf{1} = A^r\mathbf{1}.$$

□

1-Rank

Definition

The 1-rank of G is the dimension of U.

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Lemma

The $\mathbf{1}$*-rank of* G *is less than or equal to the number of orbits of* $\operatorname{Aut}(G)$ *on the vertices of* G*.*

1-Full Graphs

Definition

A graph G on n vertices is $\mathbf{1}$-full if its $\mathbf{1}$-rank is n.

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Definition

A graph G on n vertices is 1-full if its 1-rank is n.

Corollary

A 1-full graph is asymmetric.

Also. . .

Theorem (Godsil & McKay)

A 1-full graph is vertex reconstructible.

A Question

Is it true that almost all graphs are 1-full?

Outline

A Basis of Matrices

Theorem

Let G be a graph on n vertices. If G is $\mathbf{1}$-full, the matrices

$$A^i J A^j, \qquad 0 \le i, j < n$$

form a basis for $\mathrm{Mat}_{n\times n}(\mathbb{R})$.

The Proof

Proof.

For $i = 0, \ldots, n-1$, set $u_i = A^i \mathbf{1}$. Then $A^i J A^j = u_i u_j^T$. The vectors $u_0, \ldots, u_{n-1}$ are linearly independent, and so any non-zero linear combination of the matrices can be written as

$$u_0 v_0^T + \cdots + u_{n-1} v_{n-1}^T$$

where none of the vectors $v_0, \ldots, v_{n-1}$ are zero. Since the u_i's are linearly independent, this sum cannot be zero. □

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Walk Equivalent

Definition

Two graphs G and H are walk equivalent if their generating functions for walks are equal.

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(Thus any two k-regular graphs on the same number of vertices are walk equivalent.)

Walk-Equivalent 1-Full Graphs

Theorem

If G and H are walk equivalent graphs and G is $\mathbf{1}$-full, then G and H are cospectral with cospectral complements.

An Endomorphism

Assume A and B are the adjacency matrices of G and H respectively. Since the matrices $A^i J A^j$ (where $0 \le i, j < n$) form a basis for $\mathcal{M} = \mathrm{Mat}_{n\times n}(\mathbb{R})$, there is a unique linear map $\Phi : \mathcal{M} \to \mathcal{M}$ such that

$$\Phi(A^i J A^j) = B^i J B^j.$$

Products

Let w_r denote the number of walks of length r in G. Then

$$A^i J A^j \, A^k J A^\ell = w_{j+k} A^i J A^\ell$$

and consequently

$$\begin{aligned}\Phi(A^i J A^j \, A^k J A^\ell) &= w_{j+k} \Phi(A^i J A^\ell) \\ &= w_{j+k} B^i J B^\ell \\ &= B^i J B^j \, B^k J B^\ell\end{aligned}$$

Isomorphisms

It follows that Φ is a homomorphism (and not just a linear map). Since $\mathcal{M}$ is a simple algebra, Φ is an isomorphism. By the Noether-Skolem theorem it follows that there is an invertible matrix L such that

$$\Phi(M) = L^{-1}ML$$

for all matrices M.

Conclusion

So we have

$$L^{-1}AL = B,$$

whence G and H are cospectral.

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Since $\Phi(J) = J$, we have $L^{-1}JL = J$, whence $\overline{G}$ and $\overline{H}$ are cospectral.