Erdős-Ko-Rado Theorems

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Collaborators

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Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

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Intersecting Families

Definition

A family of subsets $\mathcal{F}$ of some set is intersecting if any two members of $\mathcal{F}$ have at least one point in common.

EKR

Theorem

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1. $|\mathcal{F}| \leq \binom{v-1}{k-1}$.
2. *If equality holds, $\mathcal{F}$ consists of the the k-subsets that contain i, for some i in V.*

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Cocliques

Definition

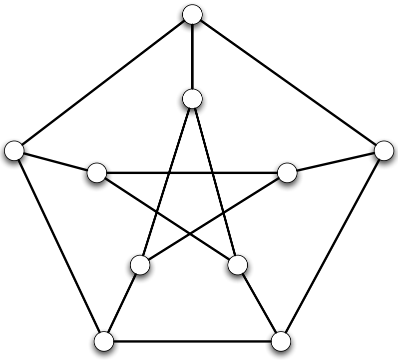
A coclique in a graph is a set of vertices, such that no two vertices in the set are adjacent. The maximum size of a coclique in a graph X is $\alpha(X)$.

A Graph

Definition

The Kneser graph $K_{v:k}$ is the graph with the k-subsets of a v-set as its vertices, where two k-subsets are adjacent if they are disjoint as sets.

$K_{5:2}$



EKR for Graphs

Theorem

We have $\alpha(K_{v:k}) = \binom{v-1}{k-1}$ and a coclique of maximum size consists of the k-subsets that contain i, for some i.

Other Graphs

q-Kneser: The vertices are the k-dimensional subspaces of a vector space of dimension v over $GF(q)$; subspaces are adjacent if their intersection is the zero subspace.

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Derangements: The vertices are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

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Derangements: The vertices are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

Partitions: Vertices are the partitions of a set of size n^2 consisting of n cells of size n; two partitions are adjacent if their meet is the discrete partition.

Cocliques

q-Kneser: The subspaces that contain a given 1-dimensional subspace.

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Partitions: The partitions with i and j in the same cell.

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Quote

It claims to be fully automatic, but actually you have to push this little button here.
—Gentleman John Killian

A Positive Semidefinite Matrix

Let X be a k-regular graph on v vertices with adjacency matrix A and let τ be the least eigenvalue of A. We define

$$M := (A - \tau I) - \frac{k - \tau}{v} J.$$

Eigenvalues

We have

$$M\mathbf{1} = (k-\tau)\mathbf{1} - \frac{k-\tau}{v}J\mathbf{1} = (k-\tau)\mathbf{1} - (k-\tau)\mathbf{1} = 0.$$

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If $Az = \theta z$ and $\mathbf{1}^T z = 0$, then

$$Mz = (\theta-\tau)z - \frac{k-\tau}{v}Jz = (\theta-\tau)z.$$

So all eigenvalues of M are non-negative and consequently $M \succcurlyeq 0$.

Inequalities

Let S be a coclique in X with characteristic vector x. Then $x^T Ax = 0$ and, since M is positive semidefinite, $x^T Mx \geq 0$. Consequently

$$\begin{aligned} 0 &\leq x^T Ax - \tau x^T x - \frac{k-\tau}{v} x^T Jx \\ &= 0 - \tau|S| - \frac{k-\tau}{v}|S|^2. \end{aligned}$$

Delsarte-Hoffman

Theorem

If X is a k-regular graph on v vertices with least eigenvalue τ, then

$$\alpha(X) \leq \frac{v}{1 - \frac{k}{\tau}}.$$

This is the ratio bound for cocliques, due to Delsarte and Hoffman.

EKR bound

The Kneser graph $K_{v:k}$ has valency

$$\binom{v-k}{k}$$

and least eigenvalue

$$-\binom{v-k-1}{k-1}.$$

EKR1

So

$$\alpha(K_{v:k}) \le \frac{\binom{v}{k}}{1+\frac{\binom{v-k}{k}}{\binom{v-k-1}{k-1}}} = \frac{\binom{v}{k}}{1+\frac{v-k}{k}} = \binom{v-1}{k-1}.$$

q-Kneser

Consider the q-Kneser graph. This has $\begin{bmatrix} v \\ k \end{bmatrix}$ vertices, valency

$$q^{k^2} \begin{bmatrix} v-k \\ k \end{bmatrix}$$

and its least eigenvalue is

$$-q^{k(k-1)} \begin{bmatrix} v-k-1 \\ k-1 \end{bmatrix}$$

EKR2

The ratio bound is

$$\begin{bmatrix} v-1 \\ k-1 \end{bmatrix}$$

which is realized by the k-subspaces that contain a given 1-dimensional subspace.

The Derangement Graph

The vertices of the derangement graph $D(n)$ are the permutations of $1, \dots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

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The vertices of the derangement graph $D(n)$ are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

cocliques: The set $S_{i,j}$ of permutations that map i to j is a coclique, of size $(n-1)!$.

cliques: Latin squares are cliques. In particular, if G is a regular subgroup of $\mathrm{Sym}(n)$, then the elements of G form a clique of size n, as do its cosets. (This implies that $\alpha(D(n)) = (n-1)!$.)

Eigenvalues of $D(n)$

???

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Remarks

1. If S is a coclique with characteristic vector x and $|S| = v/(1-\frac{k}{\tau})$ then $x^T M x = 0$.

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2. If $M \succcurlyeq 0$ and $x^T M x = 0$, then $Mx = 0$. (Proof: $M = U^T U$ and $x^T U^T U x = 0$ if and only if $Ux = 0$.)

Remarks

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2. If $M \succcurlyeq 0$ and $x^TMx = 0$, then $Mx = 0$. (Proof: $M = U^TU$ and $x^TU^TUx = 0$ if and only if $Ux = 0$.)
3. Hence if $y := x - \frac{|S|}{v}\mathbf{1}$, then $My = 0$.

Eigenvectors

Theorem

If X is a k-regular graph on v vertices with least eigenvalue τ and x is the characteristic vector of a coclique with size $v/(1-\frac{k}{\tau})$, then $x - \frac{|S|}{v}\mathbf{1}$ is an eigenvector for $A(X)$, with eigenvalue τ.

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Proof.

If $y := x - \frac{|S|}{v}\mathbf{1}$, then

$$0 = My = (A - \tau I)y - \frac{k-\tau}{v}Jy = (A-\tau I)y.$$

□

Eigenspaces

Let W be the $\binom{v}{k} \times v$ matrix whose rows are the characteristic vectors of the k-subsets of $\{1, \ldots, v\}$. Then the eigenspace for the least eigenvalue of $K_{v:k}$ consists of the vectors in $\text{col}(W)$ that are orthogonal to $\mathbf{1}$.

Eigenspaces

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Corollary

If x is the characteristic vector of a coclique of size $\binom{v-1}{k-1}$ in $K_{v:k}$, then $x \in \operatorname{col}(W)$.

Our Problem

Prove that if Wh is a 01-vector, then $h = e_i$ for some i.

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A Polytope

Suppose x is the characteristic vector of a coclique S in $K_{v:k}$ with size $\binom{v-1}{k-1}$. Then

$$x = Wh$$

for some h. If we view the rows of W as points in $\mathbb{R}^v$, they form the vertices of a convex polytope and the support of x is a face.

Faces

Theorem

A face of the polytope generated by the rows of W consists of the k-subsets that contain a given subset S, and are contained in a given subset T. All faces arise in this way.

Proving EKR, I

By the theorem, the support of Wh consists of the k-subsets α such that

$$S \subseteq \alpha \subseteq T$$

for some sets S and T. If $S \neq \emptyset$, we are done.

Proving EKR, II

So assume $S = \emptyset$. Then the support of Wh consists of all k-subsets of T. Since the support is an intersecting family, $|T| \leq 2k-1$ and consequently our family has size

$$\binom{2k-1}{k} = \binom{2k-1}{k-1}.$$

But $v \geq 2k+1$ and

$$\binom{2k-1}{k-1} < \binom{2k}{k-1} = \binom{v-1}{k-1}.$$

Thus if $S = \emptyset$, the support of Wh is not an intersecting family of maximal size.

A Second Proof

If $\alpha \in S$ and β is a k-subset disjoint from S, then $(Wh)_\beta = 0$. Let N be the submatrix formed by the rows of W that are indexed by subsets in the complement of α. Then

$$Nh = 0$$

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$$Nh = 0$$

Further we can write N in the form

$$N = \begin{pmatrix} 0 & N_1 \end{pmatrix}$$

where the initial zero columns are indexed by the elements of α.

Rank

The rows of N_1 are indexed by the k-subsets disjoint from α, the columns by the $v-k$ points not in α. So N_1 is $W_{v-k,k}$ and, if $v-k>k$, its columns are linearly independent.

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Hence if

$$\begin{pmatrix} 0 & N_1 \end{pmatrix} h = 0,$$

then α contains the support of h.

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Babbage to Tennyson

Sir, in your otherwise beautiful poem (The Vision of Sin) there is a verse which reads:

Every moment dies a man,
every moment one is born.

Obviously this cannot be true and I suggest that in the next edition you have it read:

Every moment dies a man,
every moment one-and-one-sixteenth is born.

Even this value is slightly in error but should be sufficiently accurate for the purposes of poetry.

The Bound

The cosets of a regular subgroup of $\mathrm{Sym}(n)$ form a partition of the vertices into $(n-1)!$ cliques of size n, and since any coclique contains at most one vertex from each clique, it follows that

$$\alpha(D(n)) \le (n-1)!.$$

Clique-Coclique

Theorem

If the graph X has a set of cliques of the same size that cover each vertex the same number of times, then $\alpha(X)\omega(X) \leq |V(X)|$.

Proof

$\mathcal{C}$: the set of cliques in our clique cover

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If x is the characteristic vector of a coclique, then $x^T N \le \mathbf{1}^T$ and hence $x^T N\mathbf{1} \le |\mathcal{C}|$. On the other hand

$$x^T N\mathbf{1} = \nu x^T\mathbf{1} = \nu|S|$$

and so $|S| \le |\mathcal{C}|/\nu$. Since $|\mathcal{C}|\omega(X) \ge \nu|V(X)|$, the result follows.

Equality

Suppose we have a uniform clique cover and equality holds in the clique-coclique bound. Let S and C respectively be a coclique and clique of maximum size with characteristic vectors x_S and x_T. Then $|S \cap C| = 1$ and the vectors

$$x_S - \frac{|S|}{|V(X)|}\mathbf{1}, \qquad x_C - \frac{|C|}{|V(X)|}\mathbf{1}$$

are orthogonal.

Bad News

- Each eigenspace of the derangement graph is a sum of irreducible modules for $\mathrm{Sym}(n)$. These modules are indexed by integer partitions and each occurs exactly once.

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Good News

- We can find cliques whose characteristic vectors span the orthogonal complement of the module associated to the partition $(n-1,1)$.

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- We can find cliques whose characteristic vectors span the orthogonal complement of the module associated to the partition $(n-1,1)$.
- By the clique-coclique bound, this implies that the characteristic vector of a coclique of size $(n-1)!$ must lie in the module associated to the partition $(n-1,1)$.
- From this it follows ($\mathrm{rk}(N_1)$ argument, or use perfect matching polytope) that a coclique of size $(n-1)!$ must be one of the sets $S_{i,j}$.

Problems

1. Perfect matchings in K_{2m}.

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2. Partitions graphs with $n \geq 4$.