Quantum Coloring Problems

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Fort Collins, November, 2011: In memory of Bob Liebler

Outline

Cosmology

Quote

Hydrogen is a colorless, odorless gas which given sufficient time, turns into people. (Henry Hiebert)

Axioms

Quote

"The axioms of quantum physics are not as strict as those of mathematics"

Outline

An orthogonality graph, and a problem

Definition

We define $\Omega(d)$ to be the graph with the unit vectors in $\mathbb{R}^d$ as its vertices, where two vertices are adjacent if and only if they are orthogonal.

Problem

What is $\chi(\Omega(d))$?

Cliques in $\Omega(d)$

Since each orthonormal basis for $\mathbb{R}^d$ forms a clique in $\Omega(d)$, we have

$$\chi(\Omega(d)) \geq d.$$

A finite subgraph of $\Omega(d)$

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3. $\alpha(\Phi(d)) \leq \frac{2^d}{d}$ and thus $\chi(\Phi(d)) \geq d$; hence if $\chi(\Phi(d)) = d$, then d is a power of two.

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4. $\omega(\Phi(d)) \leq d$, equality holds if and only if a Hadamard matrix exists.

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4. $\omega(\Phi(d)) \leq d$, equality holds if and only if a Hadamard matrix exists.
5. If $\chi(\Phi(d)) = d$ and there is a $d \times d$ Hadamard matrix, then d is a power of two.

The chromatic number of $\Phi(d)$ increases exponentially

Theorem (Frankl and Rödl)

There is a constant c such that $0 < c < 2$ and if $4|d$ and d is large enough, then $\alpha(\Phi(d)) < c^d$.

Outline

Coloring planes

Definition

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Theorem (Carter and Vogt, Hales and Straus)

The proper colorings of $\mathcal{P}(\mathbb{F})$ correspond to the non-trivial non-Archimedean valuations of $\mathbb{F}$.

Planes and spheres

Every coloring of $\Omega(3)$ gives a coloring of the projective plane, but the converse does not hold. But no coloring of the plane lists to a sphere coloring:

Corollary

$\chi(\Omega(3)) > 3.$

Outline

1. Physics 101
2. The Unit Sphere
   - Coloring the Sphere
   - Projective Planes
   - Gleason
   - Equiangular Lines
3. The Unitary Group
   - Quantum Colorings
   - Mutually Unbiased Bases
   - Partitions

Gleason's theorem

Definition

Let $\Omega(d)$ denote the graph whose vertices are the unit vectors in $\mathbb{R}^d$, where two unit vectors are adjacent if they are orthogonal. A frame function is a non-negative function on unit vectors that sums to 1 on each orthonormal basis.

Theorem (Gleason, 1957)

If $d \geq 3$ and f is a frame function, then there is a positive semidefinite matrix M such that $\operatorname{tr}(M) = 1$ and $f(x) = x^T M x$ for all x.

No d-colorings

Corollary

If $d \geq 3$ then $\chi(\Omega(d)) > d$.

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Proof.

Suppose $\Omega(d)$ is d-colorable and let S be a color class in a d-coloring. Then each orthonormal basis must contain a vertex in S, and therefore the characteristic vector of S is a frame function.

No d-colorings

Corollary

If $d \geq 3$ *then* $\chi(\Omega(d)) > d$.

Proof.

Suppose $\Omega(d)$ is d-colorable and let S be a color class in a d-coloring. Then each orthonormal basis must contain a vertex in S, and therefore the characteristic vector of S is a frame function. But this characteristic function is not continuous. □

Applying compactness

Theorem (Kochen and Specker)

Assume $d \geq 3$. There is a finite subgraph of $\Omega(d)$ whose vertex set is a union of orthonormal bases, such that no coclique contains a vertex in each orthonormal basis.

Outline

Complex lines

A line in $\mathbb{C}^d$ can be represented by a unit vector that spans it. If x spans a line then

$$P = (x^* x)^{-1} x x^*$$

represents orthogonal projection onto the line spanned by x.

Angles between complex lines

The angle between the lines spanned by unit vectors x and y is determined by

$$|\langle x, y\rangle| = |x^* y|.$$

If P and Q are the projections xx^* and yy^*, then

$$\operatorname{tr}(PQ) = \operatorname{tr}(xx^* yy^*) = \operatorname{tr}(y^* xx^* y) = |\langle x, y\rangle|^2.$$

Linear combinations of projections

Suppose we have m lines in $\mathbb{C}^d$ such that the angle between any pair of lines is the same. Let $P_1, \ldots, P_m$ be the corresponding projections. If

$$0 = \sum_r c_r P_r$$

then, if $\operatorname{tr}(P_r P_s) = a^2$ when $r \neq s$,

$$0 = \sum_r c_r \operatorname{tr}(P_k P_r) = c_k(1 - a^2) + a^2 \sum c_r.$$

Hence the coefficients c_r are all equal and it follows they are all zero.

A bound on the size of a set of equiangular lines

Lemma

If $P_1, \ldots, P_m$ are the orthogonal projections onto a set of equiangular lines in $\mathbb{C}^d$, then they form a linearly independent subset of the vector space of $d \times d$ Hermitian matrices. Hence $m \leq d^2$.

If equality holds, the angle is determined

Theorem

If we have a set of d^2 equiangular lines in $\mathbb{C}^d$, then $a^2 = (d+1)^{-1}$.

Proof.

Suppose $\mathcal{L}$ is an equiangular set of m lines in $\mathbb{C}^d$, with associated projections $P_1, \ldots, P_m$. If $m = d^2$ then there are scalars c_i such that $I = \sum_r c_r P_r$ and therefore

$$1 = \operatorname{tr}(P_k) = (1 - a^2)c_k + a^2 \sum_r c_r.$$

So the scalars c_r are all equal and, since $\operatorname{tr}(I) = d$, we have $c_r = d/m$. Substituting this into the above equation yields the value stated for a^2. □

An question about chromatic number

Let $X(d)$ be the graph on lines in $\mathbb{C}^d$, where lines given by projections P and Q are adjacent if $\operatorname{tr}(PQ) = (d+1)^{-1}$. Then $\omega(X(d)) \leq d^2$.

Problem

What is the chromatic number of $X(d)$?

What can we construct?

- Sets of d^2 lines that are equiangular to machine precision have been constructed up to $d = 67$ (Scott and Grassl 2009).

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- Sets of d^2 lines that are equiangular to machine precision have been constructed up to $d = 67$ (Scott and Grassl 2009).
- Equiangular sets with size d^2 exist when $d \in \{2, \ldots, 15, 19, 24, 35, 48\}$ (Scott and Grassl 2009).
- In $\mathbb{R}^d$ we can get sets of size at most $\binom{d+1}{2}$ and, if $d > 3$, then d is odd and $d + 2$ is a perfect square. Examples are known only for $d = 2, 3, 7, 23$.

Outline

A unitary Cayley graph

Definition

Let cD denote the set of $d \times d$ unitary matrices for which all diagonal entries are zero. A graph Y as a quantum d-coloring if there is a graph homomorphism from Y into the Cayley graph $X(U(d), \mathcal{D})$.

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$\Phi(d)$ has a quantum d-coloring if d is a power of two.

Embedding the symmetric group

View the symmetric group $\mathrm{Sym}(d)$ as a group of $d \times d$ permutation matrices.

- $\mathrm{Sym}(d) \leq U(d, \mathbb{C})$ and two elements σ and τ of $\mathrm{Sym}(d)$ are adjacent in $X(U(d), \mathcal{D})$ if and only if $\tau\sigma^{-1}$ is a derangement.

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- A regular subgroup of $\mathrm{Sym}(d)$ forms a clique of size d.
- The map that sends a permutation σ to 1σ is a proper d-coloring of the image of $\mathrm{Sym}(d)$.

Outline

Unbiased bases

Definition

Two orthonormal bases $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ of $\mathbb{C}^d$ are unbiased if

$$|\langle x_r, y_s \rangle|$$

is independent of r and s. (If it is, then it must be equal to $1/\sqrt{d}$.)

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is independent of r and s. (If it is, then it must be equal to $1/\sqrt{d}$.)

If U_1 and U_2 are unitary matrices, their columns are unbiased if and only if all entries of $U_1^* U_2$ have the same absolute value, that is, if the matrix $U_1^* U_2$ is flat.

A Cayley graph for the unitary group

Let $\mathcal{F}$ denote the set of flat matrices in $U(d, \mathbb{C})$. Then a set of mutually unbiased bases for $\mathbb{C}^d$ is a clique in the Cayley graph $X(U(d), \mathcal{F})$.

How large can a set of mutually unbiased bases be?

If U is a flat unitary matrix and D, E are diagonal matrices of order $d \times d$, then

$$\operatorname{tr}(DU^{-1}EU) = \operatorname{tr}(D)\operatorname{tr}(E)$$

If $\mathcal{D}$ denotes the algebra of all diagonal matrices, it follows that

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Corollary

The columns of the unitary matrices $U_1, \ldots, U_m$ are mutually unbiased if and only if for all r and s (with $r \neq s$)

$$U_r^{-1}\mathcal{D}U_r \cap U_s^{-1}\mathcal{D}U_s = \{cI : c \in \mathbb{C}\}.$$

Hence we can have at most $d+1$ mutually unbiased matrices in $\mathbb{C}^d$.

The basic question?

Question

For which values of d can we construct a mutually unbiased set of $d+1$ orthonormal bases of $\mathbb{C}^d$?

Some partial answers

- There are mutually unbiased bases of size $d+1$ if d is a prime power.

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- There are mutually unbiased bases of size $d+1$ if d is a prime power.
- There is always a set of size three.
- If $d = 2d_0$ where d_0 is odd, we do not know how to do better than three.

Constructions from projective planes

All known examples of sets of $d+1$ mutually unbiased bases in $\mathbb{C}^d$ can be constructed from either:

- A $(d, d, d, 1)$-relative difference set in an abelian group of order d^2.

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All known examples of sets of $d+1$ mutually unbiased bases in $\mathbb{C}^d$ can be constructed from either:

- A $(d, d, d, 1)$-relative difference set in an abelian group of order d^2.
- A symplectic spread in a vector space of even dimension: a set of q^d symmetric $d \times d$ matrices such that the difference of any two distinct matrices is invertible.

Outline

A graph on set partitions

Definition

Let V be a set of size d^2. Define $\mathcal{P}(d)$ to be the graph whose vertices are the partitions of V into d cells of size d, where two such partitions are adjacent if each cell of the first partition contains a point from each of the d cells of the second partition.

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We can represent each partition π by a $d^2 \times d$ matrix $M(\pi)$ whose columns are the characteristic vectors of its cells. Then $\pi \sim \rho$ in $\mathcal{P}(d)$ if and only if

$$M(\pi)^T M(\rho) = J_d.$$

$\mathcal{P}(3)$

Example

Assume $d = 3$. Then $\mathcal{P}(3)$ has 280 vertices and is regular with valency 36. There are 70 partitions which have 1 and 2 in the same cell, these form a coclique of maximal size (and all cocliques of size 70 are equivalent to this).

Coloring partitions

Meagher and Stevens:

$$\chi(\mathcal{P}(d)) \leq \binom{d+1}{2}.$$

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(The upper bound is tight if $d = 3$. Nothing more is known.)

Cliques in $\mathcal{P}(k)$

Lemma

The cliques of size k in $\mathcal{P}(d)$ correspond to orthogonal arrays with k rows and entries from $\{1, \ldots, d\}$.

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Corollary

$\omega(\mathcal{P}(d)) = d + 1$ *if and only if there is an affine plane of order d.*

The End(s)