Uniform Mixing and Continuous Quantum Walks

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Outline

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"a vector in $\mathbb{R}^3$ is not the same thing as the list of its components. The vector has a . . . meaning."

Outline

Transition Operators

Definition

If X is a graph with adjacency matrix A, we define the transition operator $U(t)$ by

$$U(t) = \exp(itA).$$

It is a unitary matrix.

An Example: P_2

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$$U(t) = \cos(t)I + i\sin(t)A = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

Composite Systems

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$$U_X(t) \otimes U_Y(t).$$
- The Cartesian product of X and Y has adjacency matrix $A_X \otimes I + I \otimes A_Y$. Since $A_X \otimes I$ and $I \otimes A_Y$ commute,
$$U_{X \square Y}(t) = U_X(t) \otimes U_Y(t).$$

An Example: Cartesian Product

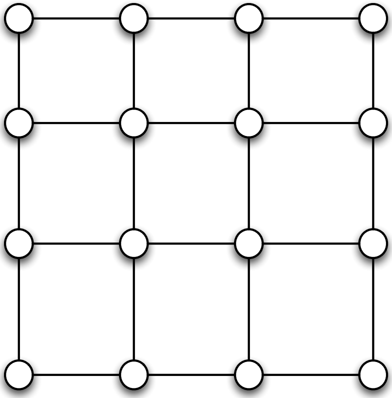

Figure: $P_4 \square P_4$

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Definition

The mixing matrix $M_X(t)$ for a walk is $U(t) \circ \overline{U(t)}$.

We note that $\overline{U(t)} = U(-t)$.

The Mixing Matrix: K_2

$$M_{K_2}(t) = \begin{pmatrix} \cos^2(t) & \sin^2(t) \\ \sin^2(t) & \cos^2(t) \end{pmatrix}.$$

What We Observe

If the initial state of our system is given by the standard basis vector e_a, then the row $e_a^T M(t)$ describes a probability density. If we measure the system at time t using the standard basis, then $M(t)_{a,b}$ is the probability that, on measurement, the state of the system is e_b.

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So at each time t a continuous quantum walk gives rise to probability densities. The interesting densities are the extreme cases: concentrated at a vertex, or uniform.

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We say that uniform mixing occurs at time t on the graph X if $U_X(t)$ is flat or, equivalently if $M_X(t) = |V(X)|^{-1}J$.

d-Cubes: Mixing

As

$$M_{K_2}(t) = \begin{pmatrix} \cos^2(t) & \sin^2(t) \\ \sin^2(t) & \cos^2(t) \end{pmatrix},$$

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The d-cube Q_d is the Cartesian product of d copies of P_2 and therefore we have uniform mixing on Q_d (at time $\pi/4$).

The Question

Which graphs admit uniform mixing?

Outline

Spectral Decomposition

Let A be the adjacency matrix of X, let $\theta_1, \ldots, \theta_m$ be the distinct eigenvalues of A, and let E_r be the matrix that represents orthogonal projection onto the eigenspace associated with θ_r.

Then

$$A = \sum_r \theta_r E_r$$

and, more generally, if f is a function defined on the eigenvalues of A, then

$$f(A) = \sum_r f(\theta_r) E_r.$$

Complete Graphs

If $X = K_n$, then we have the spectral decomposition

$$A = (n-1)\left(\frac{1}{n}J\right) + (-1)\left(I - \frac{1}{n}J\right)$$

and therefore

$$\begin{aligned} U(t) &= e^{i(n-1)t}\left(\frac{1}{n}J\right) + e^{-it}\left(I - \frac{1}{n}J\right) \\ &= e^{-it}\left(I - \frac{1-e^{int}}{n}J\right) \end{aligned}$$

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- We have uniform mixing on K_3 at time $2\pi/9$ (and hence on Cartesian powers of K_3).
- If $n \geq 5$, uniform mixing does not take place on K_n.

Outline

Transition Matrices of Bipartite Graphs

If X is bipartite, then we can write A in the form

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

and it follows that there are real matrices C_1, C_2, K (functions of t) such that

$$U(t) = \begin{pmatrix} C_1 & iK \\ iK^T & C_2 \end{pmatrix} = \begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} C_1 & -K \\ K^T & C_2 \end{pmatrix} \begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix}$$

Uniform Mixing on Bipartite Graphs

Suppose we have uniform mixing on a bipartite graph at time t. Then $U(t)$ is flat and consequently the matrix

$$\begin{pmatrix} C_1 & -K \\ K^T & C_2 \end{pmatrix}$$

is a flat real orthogonal matrix—it's a Hadamard matrix!

Lemma

If we have uniform mixing a bipartite graph X on n vertices then $n = 2$ or $4|n$. If X is also regular, then n is the sum of two integer squares.

Gelfond-Schneider

Theorem (Gelfond-Schneider)

If α and β are algebraic numbers and $\alpha \neq 0, 1$ and α^β is algebraic, then β is rational.

Eigenvalues of $U(t)$

If $\theta_1, \ldots, \theta_m$ are the distinct eigenvalues of A, then the eigenvalues of $U(t)$ are

$$e^{it\theta_r}, \qquad (r = 1, \ldots, m)$$

where

$$e^{it\theta_s} = \left(e^{it\theta_r}\right)^{\theta_s/\theta_r}.$$

Theorem (N. Mullin)

If the entries of $U(t)$ are all algebraic numbers, then the ratios of the eigenvalues of A are rational.

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2. If $U_{C_8}(t)$ is flat, its entries are algebraic numbers.
3. $2/\sqrt{2}$ is not rational.

The Conclusion

No even cycle of length greater than four admits uniform mixing.

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Defining Type-II Matrices

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Definition

An $n \times n$ Schur-invertible matrix W is a type-II matrix if

$$WW^{(-)T} = nI.$$

Examples of Type-II Matrices

- Hadamard matrices.
- Flat unitary matrices.
- Kronecker products of type-II matrices.
- If $t \neq 0$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & t & -t \\ 1 & -1 & -t & t \end{pmatrix}.$$

- $tI + J$ for two choices of t (Potts model).
- http://arxiv.org/pdf/0707.1836.pdf (Chan and Godsil).

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Type-II Matrices from SRGs

Suppose X is strongly regular with adjacency matrix A and let $\overline{A} = J - I - A$. Then we know that $U(t)$ is a linear combination of I, A and $\overline{A}$.

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Ada Chan and I determined which type-II matrices can be expressed as such linear combinations. We needed to find x and y such that

$$\begin{aligned} nI &= (I + xA + y\overline{A})(I + x^{-1}A + y^{-1}\overline{A}) \\ &= I + (x + x^{-1})A + (y + y^{-1})\overline{A} + (xy^{-1} + x^{-1}y)A\overline{A}. \end{aligned}$$

We can express the RHS as a linear combination of I, A, $\overline{A}$, where the coefficients are polynomials in x, x^{-1}, y, and y^{-1}.

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Note: a flat unitary of the form $I + xA + y\overline{A}$ need not be a transition matrix.

A Theorem

Theorem (Godsil, Mullin, Roy)

If X is strongly regular, it has uniform mixing if and only if either

(a) *X is the Paley graph on nine vertices.*

(b) *X comes from a regular symmetric Hadamard matrix with constant diagonal.*

For (b), we could start with Kronecker powers of

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

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Theorem (Haagerup)

If p is a prime, there are only finitely many cyclic p-roots.

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- The set of type-II matrices in the Bose-Mesner algebra of an association scheme is an algebraic variety defined by polynomials with integer coefficients.
- If such a variety is finite, the coordinates of any point in it are algebraic numbers.
- If C_p admits uniform mixing, the ratio of its eigenvalues must be rational.
- C_3 is the only prime cycle that admits uniform mixing.

Outline

I know how you feel right now...

... but there are a number of problems:

1. Which odd cycles admit uniform mixing?
2. If uniform mixing occurs on a graph, does it follow that the ratios of its eigenvalues are rational?
3. If uniform mixing occurs on X, does it follows that X is regular?
4. Mullin conjectures that if $n \geq 5$, a Cayley graph for $\mathbb{Z}_n^d$ cannot admit uniform mixing.
5. For Cartesian powers of $K_{1,3}$, although uniform mixing does not occur, we can choose t so one column of $U(t)$ is flat (H. Zhan). Find other examples of this behaviour.
6. ϵ-uniform mixing: it does occur on prime cycles (N. Mullin), where else?

The End(s)