Periodic Graphs

Chris Godsil

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Outline

1 Periodicity & State Transfer

2 Some Results

3 Some Questions

Unitary Operators

Suppose X is a graph with adjacency matrix A.

Definition

We define the operator $H_X(t)$ by

$$H_X(t) := \exp(iAt).$$

An Example

We have

$$H_{K_2}(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}$$

Note that $H_X(t)$ is symmetric, because A is, and unitary because

$$H_X(t)^* = \exp(-iAt) = H_X(t)^{-1}.$$

Probability Distributions

If H is unitary, the Schur product

$$H \circ \overline{H}$$

is doubly stochastic. Hence each row determines a probability density. (It determines a continuous quantum walk.)

State Transfer

Definition

We say that perfect state transfer from the vertex u to the vertex v occurs at time τ if

$$|(H_X(\tau))_{u,v}| = 1.$$

Example:

$$H_{K_2}(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

thus we have perfect state transfer between the end vertices of K_2 at time $\pi/2$.

More Examples

Since

$$H_{X \square Y}(t) = H_X(t) \otimes H_Y(t)$$

it follows that if perfect state transfer from u to v in X occurs at time τ, then we also have perfect state transfer from (u,u) to (v,v) in $X \square X$ at time τ.

So we get perfect state transfer between antipodal vertices in the d-cube Q_d at time $\pi/2$.

Squaring

If perfect state transfer from 1 to 2 occurs at time τ, then

$$H_X(\tau) = \begin{pmatrix} 0 & \gamma & 0 & \dots & 0 \\ ? & 0 & ? & \dots & ? \\ \vdots & \vdots & & Q & \\ ? & 0 & & & \end{pmatrix}$$

where $|ga| = 1$. Consequently $|(H_X(\tau))_{2,1}| = 1$ and

$$H_X(2\tau) = \begin{pmatrix} \gamma^2 & 0 & 0 & \dots & 0 \\ 0 & \gamma^2 & 0 & \dots & 0 \\ \vdots & \vdots & & Q & \\ 0 & 0 & & & \end{pmatrix}$$

Periodicity

Definition

We say that X is periodic at the vertex u with period τ if $|(H_X(\tau))_{u,u}| = 1$.

Lemma

If perfect state transfer from u to v occurs at time τ, then X is periodic at u and v.

Spectral Decomposition

We have

$$A = \sum_{\theta} \theta E_\theta$$

where θ runs over the distinct eigenvalues of A and the matrices E_θ represent orthogonal projection onto the eigenspaces of A.

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where θ runs over the distinct eigenvalues of A and the matrices E_θ represent orthogonal projection onto the eigenspaces of A. Further if f is a function on the eigenvalues of A, then

$$f(A) = \sum_\theta f(\theta) E_\theta$$

and therefore

$$H_X(t) = \sum_\theta \exp(i\theta t) E_\theta.$$

Integer Eigenvalues

Lemma

If the eigenvalues of X are integers, it is periodic with period 2π.

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In fact:

Theorem

If X is a connected regular graph, then X is periodic if and only if its eigenvalues are integers.

Vertex-Transitive Graphs

Theorem

If X is vertex transitive and perfect state transfer occurs at time τ, then

$$H_X(\tau) = \gamma \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix} = \gamma T$$

where $|\gamma| = 1$ and T lies in the center of $\mathrm{Aut}(X)$.

Antipodal Vertices

If X has diameter d, we say that vertices at distance d are antipodal. In all examples we have where perfect state transfer takes place, the vertices involved are antipodal.

- Is antipodality necessary?
- If $|V(X)| \geq 3$, can we get perfect state transfer between adjacent vertices?

Antipodal Vertices

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Efficiency

What is the minimum number of edges in a graph where perfect state transfer takes place between two vertices at distance d? (Beat 2^d.)

Cubelike Graphs

Suppose X is a Cayley graph for $\mathbb{Z}_2^d$ with connection set $\{c_1, \ldots, c_m\}$ and set $s = c_1 + \cdots + c_m$. If $s \neq 0$, we get perfect state transfer from 0 to s at time $\pi/2$. Can perfect state transfer occur if $s = 0$?

Mixing

We say that perfect mixing occurs at time τ if, for all vertices u and v in X,

$$|(H_X(\tau))_{u,v}| = \frac{1}{\sqrt{|V(X)|}}.$$

(For example K_2 of Q_d at time $\pi/4$. What can we usefully say about graphs where perfect mixing occurs?