Type-II Matrices

Chris Godsil

August 2, 2005

Outline

1. Matrices
   - Definitions
   - Examples: Unitary
   - Examples: Combinatorial
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2. Link Invariants
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Outline

Schur Product

Definition

If A and B are $m \times n$ matrices, their Schur product $A \circ B$ is the $m \times n$ matrix given by

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}.$$

Inverses

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- If no entry of A is zero, there is a unique matrix $A^{(-)}$ such that
$$A \circ A^{(-)} = J;$$
we call $A^{(-)}$ the Schur inverse of A.

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So if W is a type-II matrix then

$$W^{-1} = \frac{1}{v} W^{(-)T}.$$

Outline

The Cyclic Spin Model

Example

Choose θ so that θ^2 is a primitive complex v-th root of 1, and let W be the $v \times v$ matrix given by

$$W_{i,j} := \theta^{(i-j)^2}, \qquad 0 \leq i, j < v.$$

spin

Check:

$$
\begin{aligned}
(WW^{(-)T})_{i,j} &= \sum_r \theta^{(i-r)^2-(j-r)^2} \\
&= \theta^{i^2-j^2} \sum_r \theta^{2(j-i)r} \\
&= v\delta_{i,j}.
\end{aligned}
$$

Flat Matrices

Definition

A complex matrix M is flat if its entries all have the same absolute value.

Two Thirds

Theorem

If W is a $v \times v$ matrix over $\mathbb{C}$, then any two of the following statements imply the third:

(a) *W is type II.*

(b) *W is flat.*

(c) *W is unitary.*

Quantum Physics

Definition

Two orthogonal bases $x_1, \dots, x_v$ and $y_1, \dots, y_v$ of $\mathbb{C}^v$ are unbiased if all inner products $\langle x_i, y_j \rangle$ have the same absolute value. Two unitary matrices X and Y are unbiased if X^*Y is flat.

continued

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- Unitary matrices X and Y are unbiased if and only if the (unitary) matrices I and X^*Y are unbiased.

continued

- If X and Y are unitary matrices, then X^*Y is unitary. So X are Y are unbiased if and only if X^*Y is a flat type-II matrix.
- Unitary matrices X and Y are unbiased if and only if the (unitary) matrices I and X^*Y are unbiased.
- Hence each flat type-II matrix determines an unbiased pair of bases.

A Problem

A Question

What is the maximum size of a set of mutually unbiased bases in $\mathbb{C}^v$?

What We Know

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(b) This bound can be realized if v is a prime power.

(c) In general, the best we can do is three. :-(

Outline

The Potts Model

Let J be the $v \times v$ matrix with all entries equal to 1 and set

$$W := (\gamma - 1)I + J.$$

Then $J^2 = vJ$ and so

$$WW^{(-)T} = (2 - \gamma - \gamma^{-1})I + (v - 2 + \gamma + \gamma^{-1})J.$$

Hence W is type II if and only if

$$\gamma^2 + (v - 2)\gamma + 1 = 0.$$

spin

Symmetric Designs

Definition

For today's purposes, a symmetric design is given by a $v \times v$ 01-matrix N such that, for suitable integers k and λ,

$$NJ = N^T J = kJ, \quad NN^T = (k-\lambda)I + \lambda J.$$

The Fano Plane

Example

$$N = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Generalized Potts

If

$$W := (\gamma - 1)N + J$$

then

$$WW^{(-)T} = (2 - \gamma - \gamma^{-1})(k - \lambda)I + (k(\gamma + \gamma^{-1} - 2) + v)J$$

and therefore W is type II if and only if

$$(k - \lambda)(\gamma - 1)^2 + v(\gamma - 1) + v = 0.$$

Outline

Equiangular Lines

Definition

Let $x_1, \ldots, x_n$ be a set of unit vectors in $\mathbb{C}^d$. The lines spanned by these vectors are equiangular if there is a scalar a such that if $i \neq j$, then

$$|\langle x_i, x_j \rangle|^2 = a.$$

A Bound

Lemma

Suppose the lines spanned by $x_1, \ldots, x_n$ are equiangular and the matrices X_i are defined by $X_i = x_i x_i^*$. Then the matrices X_i are linearly independent elements of the space of Hermitian matrices.

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Corollary

$n \leq d^2$.

The Construction

Theorem

Suppose $n = d^2$ *and* $x_1, \ldots, x_n$ *spans a set of equiangular lines in* $\mathbb{C}^d$. *Let* G *be the Gram matrix of this set of vectors. Then* $G^2 = dG$ *and if*

$$\gamma^2 + (d+2)\gamma + a^2 = 0,$$

then $(\gamma - 1)I + G$ *is a type-II matrix.*

Outline

The Nomura Algebra

Let W be a complex $v \times v$ Schur invertible matrix. Then $W_{i/j}$ is the vector in $\mathbb{C}^v$ given by:

$$(W_{i/j})_r := \frac{W_{r,i}}{W_{r,j}}.$$

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Definition

The Nomura Algebra $\mathcal{N}_W$ of a Schur-invertible matrix is the set of complex matrices M such that each vector $W_{i/j}$ is an eigenvector for M.

An Example

Consider the cyclic spin model:

$$W_{i,j} = \theta^{(i-j)^2}$$

(where θ^2 is a primitive complex v-th root of 1). Then

$$(W_{i/j})_r = \theta^{i^2-j^2}\, \theta^{2(i-j)r}$$

and so the vectors $W_{i/j}$ are (essentially) the columns of a Vandermonde matrix.

Equivalence

If W is type II and P_1, P_2 are permutation matrices and D_1, D_2 are invertible diagonal matrices, then

$$P_1 D_1 W D_2 P_2$$

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is type II. We say that it is equivalent to W.

Theorem

If W and W' are equivalent type-II matrices, there is a permutation matrix P such that

$$\mathcal{N}_{W'} = P^T \mathcal{N}_W P.$$

Nontriviality

- $I \in \mathcal{N}_W$

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- $I \in \mathcal{N}_W$
- $J \in \mathcal{N}_W$ if and only if W is a type-II matrix.

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Definition

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Each spin model determines a link invariant.

Examples

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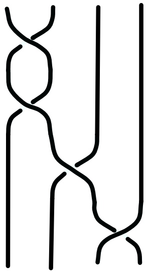
- The Potts model. potts

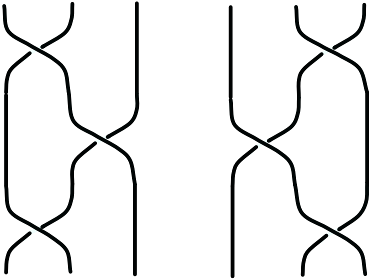
Outline

One Braid



$\sigma_1^2 \sigma_2 \sigma_3^{-1}$

Two Braids



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$

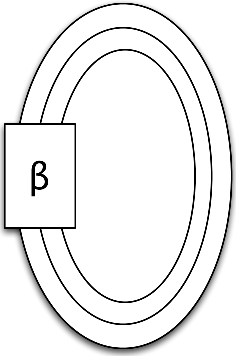
Generators and Relations

Definition

The braid group B_n on n strands is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, subject to the relations:

- If $|i - j| > 1$, then $\sigma_i \sigma_j = \sigma_j \sigma_i$.
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Links



Markov Moves

Suppose α and β are braids on n strands. Then the following operations do not change the isotopy class of the closure of β:

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Markov II: $\beta \rightarrow \beta\sigma_n$,

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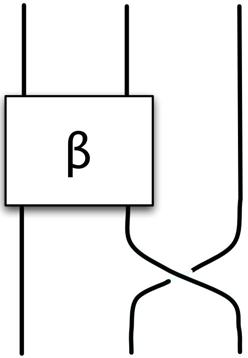

Link Invariants

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- Given the first Markov move, we see that for a braid invariant to give us a link invariant, it must be constant on conjugacy classes in the braid group.

Trace

Let V be a complex vector space of finite dimension and choose invertible elements X and Y in $\mathrm{End(V)}$ such that $XYX = YXY$. Then we have a homomorphism, ρ say, from B_3 into $\mathrm{End}(V)$ such that

$$\rho(\sigma_1) = X, \qquad \rho(\sigma_2) = Y.$$

A Markov Trace

If $\beta \in B_3$, then $\mathrm{tr}(\rho(\beta))$ only depends on the conjugacy class of β.
If we are lucky, this will be a link invariant.

Spin Models

Suppose W is a spin model of order $v \times v$. Let $V = \mathrm{Mat}_{d\times d}(\mathbb{C})$ and if $M \in V$, define

$$X(M) = \frac{1}{\sqrt{v}} WM, \qquad Y(M) = W^{(-)} \circ M.$$

Then $XYX = YXY$, and we are lucky.

Outline

A Transform

Suppose W is a $v \times v$ type-II matrix. If $M \in \mathcal{N}_W$, let $\Theta(M)$ be the $v \times v$ matrix such that $\Theta(M)_{i,j}$ is the eigenvalue of M on $W_{i/j}$. If $M, N \in \mathcal{N}_W$ then:

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- $\Theta(MN) = \Theta(M) \circ \Theta(N)$.
- $\Theta(M) \in \mathcal{N}_{W^T}$.
- $\Theta^2(M) = vM^T$.

Schur Closure

Theorem (Jaeger, Nomura)

If W is a type-II matrix then $\mathcal{N}_W$ is Schur-closed.

Outline

Axioms

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- $A_i^T \in \mathcal{A}$, for all i.
- $A_i A_j \in \operatorname{span}(\mathcal{A})$.
- $A_i A_j = A_j A_i$, for all i and j.

Discreteness

There are only finitely many association schemes on v vertices.

Outline

A Question

If $\mathcal{A}$ is an association scheme, can $\mathbb{C}[\mathcal{A}]$ contain infinitely many type-II matrices? (If the dimension of the scheme is three, then it contains at most six.)

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We have the following classes of spin models:

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- A class of examples with Nomura algebra equal to the Bose-Mesner algebra of distance-regular antipodal double cover of a complete bipartite graph. (Found by Nomura.)

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- A class of examples with Nomura algebra equal to the Bose-Mesner algebra of distance-regular antipodal double cover of a complete bipartite graph. (Found by Nomura.)
- Products of the above.

Problem

Find new non-trivial examples of type-II matrices W such that $\dim(\mathcal{N}_W) \geq 3$.

We do not have any examples that are not spin models!