# Quantum Colouring and Derangements

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# Outline

1. Colourings and Derangements
   - Colourings
   - Derangements
2. Quantum Colourings
   - Projections
   - Rank-1 Colourings
   - Colouring Erdős-Rényi graphs
   - Derangements of index $k$
   - Permutations

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   - Permutations

# Sources


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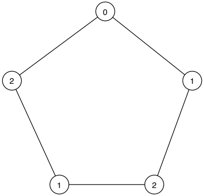
# Graph colouring

## Definition

A $c$-colouring of a graph $X$ is a function $\varphi$ from $V(X)$ to $\{1,\ldots,c\}$ such that if $u$ and $v$ are adjacent vertices in $X$, then $\varphi(u) \neq \varphi(v)$.



# The matrix of a colouring

## Definition

The matrix $M(\varphi)$ of a $c$-colouring $\varphi$ of a graph $X$ is the $|V(X)| \times c$ matrix such that

$$(M(\varphi))_{a,i} = \begin{cases} 1, & \varphi(a) = i; \\ 0, & \text{otherwise.} \end{cases}$$

# For $C_5$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Properties of the colouring matrix

We use $e_u$ to denote the standard basis vector indexed by the vertex $u$.

Suppose $M$ is the matrix of a $c$-colouring of $X$. Then

1. $M\mathbf{1} = \mathbf{1}$.
2. If $a$ and $b$ are adjacent in $X$, then $e_a^T M$ and $e_b^T M$ are orthogonal.

It follows that the map $a \mapsto e_a^T M$ is an orthogonal representation of $X$ in $\mathbb{R}^c$.

# Orthogonality graphs

### Definition

Assume $S$ is a subset of a real or complex inner product space. The orthogonality graph based on $S$ is the graph with vertex set $S$, with two elements of $S$ adjacent if they are orthogonal.

For us, $S$ will usually be either the unit vectors in $\mathbb{C}^d$ or $\mathbb{R}^d$, or the set of $d \times d$ projections (real or complex).

# Graph homomorphisms

### Definition

Let $X$ and $Y$ be graphs. A map $\varphi$ from $V(X)$ to $V(Y)$ is a graph homomorphism if it maps adjacent vertices in $X$ to adjacent vertices in $Y$. We write $X \to Y$ to denote that there is a homomorphism from $X$ to $Y$. If $X \to Y$ and $Y \to X$, then $X$ and $Y$ are homomorphically equivalent.

A graph $X$ is $c$-colourable if and only $X \to K_c$. If $X$ and $Y$ are homomorphically equivalent, then $\chi(X) = \chi(Y)$.

# Orthogonal rank

## Definition

We use $\Omega(d)$ to denote the graph with the unit vectors in $\mathbb{C}^d$ as vertices, with two unit vectors adjacent if and only if they are orthogonal. The least value of $d$ such that $X$ admits a homomorphism into $\Omega(d)$ is the orthogonal rank of $X$, denoted $\xi(X)$.

# Flat orthogonal rank

**Definition**

A vector (or matrix) is flat if all its entries have the same absolute value. We use $\xi^\flat(X)$ to denote the least $d$ such that there is a homomorphism from $X$ into the subgraph $\Omega^\flat(d)$ induced by the flat unit vectors.

Summary: $\xi(X)$, $\xi^\flat(X)$.

# Some easy bounds

We use $\chi(X)$ to denote the chromatic number of $X$.

**Lemma**

*For any graph $X$,*

$$\xi(X) \leq \xi^\flat(X) \leq \chi(X).$$

(The second inequality needs proof, be patient.)

# Outline

# A graph on permutations

## Definition

A derangement is a permutation with no fixed points. The vertices of the derangement graph $\mathcal{D}(n)$ are the elements of the symmetric group $\mathrm{Sym}(m)$, two permutations $\sigma$ and $\tau$ are adjacent in $\mathcal{D}(n)$ if $\tau\sigma^{-1}$ is a derangement.

Thus $\mathcal{D}(n)$ has $n!$ vertices and valency $\lfloor n!/e \rfloor$.

# $\mathcal{D}(n)$ as an orthogonality graph

If $A$ and $B$ are $m \times n$ complex matrices, the map

$$(A, B) \mapsto \operatorname{tr}(A^* B)$$

is an inner product whose value we will denote by $\langle A, B \rangle$. If we represent permutations $\sigma$ and $\tau$ of $\operatorname{Sym}(n)$ by $n \times n$ permutation matrices, $S$ and $T$ respectively, then $\operatorname{tr}(S^{-1}T)$ is the number of points fixed by $\sigma^{-1}\tau$ and therefore $\sigma^{-1}\tau$ is a derangement if and only if

$$\langle S, T \rangle = 0.$$

# Cliques in $\mathcal{D}(n)$

If $\sigma$ and $\tau$ are two elements of $\mathrm{Sym}(n)$, then $\tau\sigma^{-1}$ is a derangement if and only if the $2 \times n$ matrix

$$\begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ \tau_1 & \dots & \tau_2 \end{pmatrix}$$

has different entries in each column. Hence an $n$-clique in $\mathcal{D}(n)$ corresponds to an $n \times n$ Latin square, and there are no cliques in $\mathcal{D}(n)$ with more than $n$ vertices.

# $\chi(\mathcal{D}(n))$

The map that assigns the value $\sigma_1$ to a permutation $\sigma$ from $\mathrm{Sym}(n)$ is an $n$-colouring of $\mathcal{D}(n)$. Accordingly:

**Lemma**

$\omega(\mathcal{D}(n)) = \chi(\mathcal{D}(n)) = n.$

# More complex derangements. . .

### Definition

We use $U(d)$ to denote the group of $d \times d$ unitary matrices; the subgroup consisting of the real matrices in $U(d)$ is the orthogonal group $O(d)$. A unitary derangement is a unitary matrix with all diagonal entries zero.

# ...and a more complex graph

## Definition

The unitary derangement graph $\mathcal{UD}(n)$ has the elements of $U(n)$ as its vertices, and two matrices $A$ and $B$ are adjacent if and only if $BA^{-1}$ is a unitary derangement.

The derangement graph $\mathcal{D}(n)$ is an induced subgraph of $\mathcal{UD}(n)$.

# Outline

# A review of projections

A square complex matrix $P$ is a projection if $P = P^2 = P^*$.

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1. If $P$ and $Q$ are projections then $\operatorname{tr}(PQ) \geq 0$ and $\operatorname{tr}(PQ) = 0 \iff PQ = 0$.
2. If $P_1, \ldots, P_r$ are $d \times d$ projections and $\sum_i P_i = I_d$, then $P_i P_j = 0$ if $i \neq j$.
3. If $P$ is a $d \times d$ projection with rank $r$, there is a $d \times r$ matrix $U$ such that $P = UU^*$ and $U^*U = I_r$.

# Replacing 0 and 1 by bigger projections

## Definition

A graph $X$ has a quantum $c$-colouring if there is a $|V(X)| \times c$ matrix $M$ with entries $d \times d$ projections such that

- Each row of $M$ sums to $I_d$.
- If $a$ and $b$ are adjacent vertices of $X$, then $M_{a,i}M_{b,i} = 0$ for $i = 1 \ldots, c$.

The least $c$ for which a quantum $c$-colouring of $X$ exists is the quantum chromatic number of $X$, denoted $\chi_q(X)$.

# $\chi$ versus $\chi_q$

Since $0$ and $1$ are the $1 \times 1$ projections, a classical $c$-colouring is a quantum $c$-colouring with $d = 1$ and so $\chi_q(X) \leq \chi(X)$.

# Quantum colourings and orthogonality

## Lemma

*Let $X$ be a graph and let $M$ be a $|V(X)| \times c$ matrix of $d \times d$ projections. The matrix $M$ defines a quantum $c$-colouring of a graph if whenever $a$ and $b$ are adjacent vertices in $X$, then $\sum_i M_{a,i} M_{b,i} = 0$.*

# Proof of the lemma

**Proof.**

Clearly the stated condition is necessary. For sufficiency, if $\sum_i M_{a,i}M_{b,i} = 0$, then

$$0 = \sum_i \operatorname{tr}(M_{a,i}M_{b,i}).$$

As projections are positive semidefinite, $\operatorname{tr}(M_{a,i}M_{b,i}) \geq 0$, and therefore this implies that $\operatorname{tr}(M_{a,i}M_{b,i}) = 0$ for each $i$, and hence that $M_{a,i}M_{b,i} = 0$. □

# Quantum colourings with rank $r$

**Lemma**

*If there is a quantum $c$-colouring of $X$, there is a quantum $c$-colouring where all projections have the same rank.* □

A rank-$r$ quantum $c$-colouring is a quantum $c$-colouring where all projections have rank $r$. The least integer $c$ such that $X$ admits a rank-$r$ quantum $c$-colouring is denoted $\chi_q^{(r)}$.

For a rank-$r$ $c$-colouring using $d \times d$ projections, $rc = d$. If $r = 1$, then $c = d$.

# Outline

# Derangements and rank-1 colourings

**Theorem**

*A graph has a rank-1 quantum $d$-colouring if and only if it admits a homomorphism into $\mathcal{UD}(d)$.*

# Proof

**Proof.**

- A rank-1 projection is equal to $xx^*$ for some unit vector $x$. Vectors $x$ and $y$ are orthogonal if and only $xx^*yy^* = 0$.

□

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- Each row of a rank-1 quantum $d$-colouring corresponds to an orthonormal basis of $\mathbb{C}^d$, hence to a unitary matrix.
- If $U_a$ and $U_b$ are the unitary matrices correspondng to the $a$- and $b$-rows of the colouring and $a \sim b$, then $U_a^* U_b$ is a unitary derangement.

□

# More homomorphisms

**Theorem**

$$K_n \to \Omega^\flat(n) \to \mathcal{UD}(n) \to \Omega(n)$$

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## Corollary

$$\xi(X) \leq \chi_q^{(1)}(X) \leq \xi^\flat(X) \leq \chi(X).$$

# Proofs

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$\Omega^\flat(n) \to \mathcal{UD}(n)$: Let $W$ be a flat unitary. If $z \in \Omega^\flat(n)$, let $D_z$ be the diagonal matrix with $(D_z)_{i,i} = z_i$. Then $\sqrt{n}D_zW$ is a unitary matrix (easy) and if $y$ and $z$ are vectors in $\Omega^\flat(n)$, we have that $(D_zW)^*D_zW$ is a derangement if $y$ and $z$ are orthogonal (harder). So $\Omega^\flat(n) \to \mathcal{UD}(n)$.

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$\mathcal{UD}(n) \to \Omega(n)$: if $M, N \in U(d)$, then $\langle Me_1, Ne_1\rangle = (M^*N)_{1,1}$, and $Me_1$ and $Ne_1$ are orthogonal if $M^*N$ is a derangement. Hence $\mathcal{UD}(n) \to \Omega(n)$.

# 2-colourings, 3-colourings

**Lemma**

$\chi_q(X) = 2 \iff \chi(X) = 2.$

The proof of this left as an exercise. For the next result, the key is that derangements in $U(3)$ are monomial matrices, and in consequence, $\mathcal{UD}(3)$ and $\mathcal{D}(3)$ are homomorphically equivalent.

**Lemma**

$\chi_q^{(1)} = 3 \iff \chi(X) = 3.$

# Quantum Latin squares

## Definition

A quantum Latin square is a square matrix with rank-1 projections of order $d \times d$ as entries, such that each row and each column sums to $I_d$.

If $L$ is an $n \times n$ Latin square with entries $1, \ldots, n$ and (for each $i$) we replace by the matrix $E_{i,i} = e_i e_i^T$, the result is a quantum Latin square.

# Quantum Latin squares from unitary matrices

If $A$ is an $n \times n$ unitary matrix with columns $a_1, \ldots, a_n$, then the matrices

$$a_1 a_1^*, \ldots, a_n a_n^*$$

are a set of $n$ rank-1 projections with sum $I_d$. If $B$ is a second $n \times n$ unitary matrix with columns $b_1, \ldots, b_n$ and corresponding projections

$$b_1 b_1^*, \ldots, b_n b_n^*,$$

then $a_r a_r^* b_r b_r^* = 0$ for each $r$ if and only if $A^* B$ is a unitary derangement.

# Cliques and quantum Latin squares

**Theorem**

*Cliques of size $n$ in $\mathcal{UD}(n)$ correspond to quantum Latin squares of order $n \times n$.*

# Outline

# Erdős-Rényi graphs

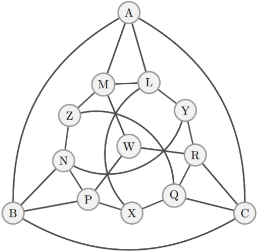
**Definition**

Let $\mathbb{F} = GF(q)$ be a finite field of odd order. The vertices of the Erdős-Rényi graph $ER(q)$ are the 1-dimensional subspaces of $V(3, \mathbb{F})$; subspaces $\langle x \rangle$ and $\langle y \rangle$ are adjacent if $x^T y = 0$.

The Erdős-Rényi graphs have loops, in fact there are exactly $q + 1$ vertices with loops in them. We will work with the graph obtained by deleting the loops; this graph is not regular, there are $q + 1$ vertices with valency $q$, the remainder have valency $q + 1$. In total there are $q^2 + q + 1$ vertices.

# $ER(3)$: a picture

1



# $ER(3)$: vectors

The 13 columns of the following matrix span the 13 1-dimensional subspaces of $V(3,3)$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

If we view the entries of the above matrix as real numbers, the orthogonality graph $F$ of the 13 real vectors is isomorphic to $ER(3)$ with the loops deleted. Clearly $\xi(F) \le 3$.
The computer tells us that $\chi(F) = 4$ and consequently $\chi_q^{(1)}(F) = 4$. Further, $\chi_q(ER(3)) = 4$ (but this is not trivial).

# Help from the quaternions

Cameron et al. prove the following, using properties of the quaternions.

**Lemma**

*There is a homomorphism from $S_{\mathbb{R}}(4)$ into the subgraph of $\mathcal{UD}(4)$ induced by the real orthogonal matrices.* □

**Corollary**

*If $\xi_{\mathbb{R}}(x) \leq 4$, then $\chi_q^{(1)}(X) \leq 4$.* □

# The cone over $ER(13)$

The cone $\hat{X}$ of a graph $X$ is the graph we get by adding one new vertex to $X$, and joining it each of the old vertices.

- $\chi(\hat{X}) = \chi(X) + 1$.
- $\xi(\hat{X}) \le \xi(X) + 1$: use a standard basis vector.
- The cone over $ER(3)$ has an orthogonal embedding in $\mathbb{R}^4$; as $\Omega_{\mathbb{R}}(4)$ is homomorphically equivalent to $\mathcal{OD}(4)$ the rank-1 quantum chromatic number of the cone is four, and its chromatic number is five.

# Outline

# Generalizing derangements

**Definition**

A unitary matrix $M$ is a unitary derangement of index $k$ if it has order $mk \times mk$ and

$$M \circ (I_m \otimes J_k) = 0.$$

(Thus it has $m$ diagonal blocks of zeros, each of order $k \times k$.)

What we have been calling a unitary derangement is now a unitary derangement of index $1$.

# Another Cayley graph

Since the set of $mk \times mk$ unitary derangements of index $k$ is closed under conjugate transpose and does not contain the identity, we can use it as the connection set for a Cayley graph, which we denote by $\mathcal{UD}_k(m)$.

**Theorem**

*A graph $X$ has a rank $k$ quantum $m$-colouring if and only if there is a homomorphism $X \to \mathcal{UD}_k(m)$.*

# Grassmann graphs

**Definition**

The Grassmann graph $Gr(d,k)$ is the graph with the $k$-dimensional subspaces of $\mathbb{C}^d$ as vertices, with two subspaces adjacent if they are orthogonal.

We can view $Gr(d,k)$ as a continuous analog of the Kneser graph $K_{d:k}$, and then homomorphisms into $Gr(d,k)$ will be related to fractional colourings.

**Theorem**

*There is a homomorphism $\mathcal{UD}_k(m) \to Gr(mk,k)$.*

# Outline

# Not just colourings: quantum permutations

**Definition**

A quantum permutation is a square matrix with entries $d \times d$ projections, such that the entries in a given row, or column, sum to $I_d$.

When $d = 1$, we have our usual permutation matrices. (Quantum permutations are also referred to as "magic unitary" matrices.)

**Lemma**

*A matrix is a quantum permutation of order $n \times n$ if and only if it is the matrix of a quantum $n$-colouring of $K_n$.*

# The End(s)