A Bestiary of Strongly Regular Graphs

How do we get SRGs & DRGs?

- algebraic / group theoretic constructions –
- combinatorial structures –
- geometry –

① Paley Graphs:

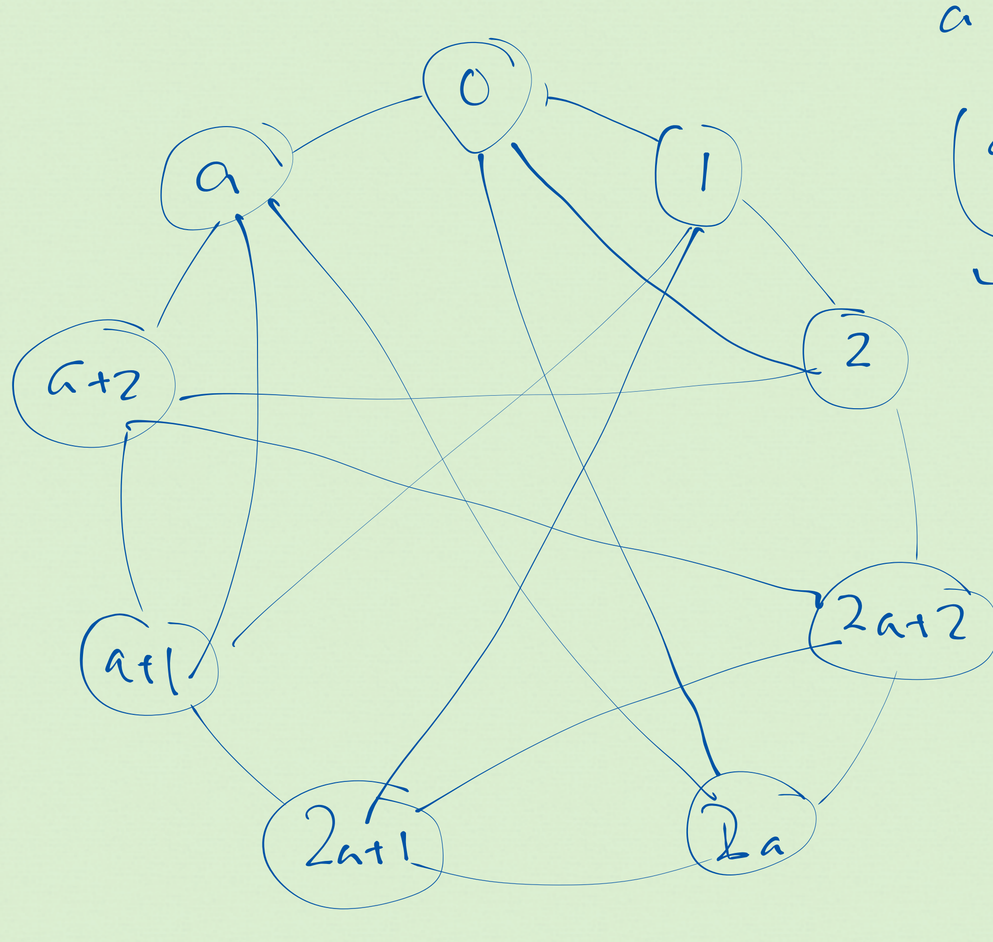
$q \equiv 1 \mod 4$, prime power $\qquad$ ex. $q = 9$

Vxs: elements of $GF(q)$

$a \sim b$: $(b-a)$ is a nonzero square in $GF(q)$ $\qquad$ $(2a)^2 = 4a^2 = 1a^2$

$GF(9) = \mathbb{F}_3(a)$ where $a^2+1=0$

x	0	1	2	a	2a	a+1	a+2	2a+1	2a+2
x^2	0	1	1	2	1	2a	. . . . a		



$\left(9,\ \frac{q-1}{2},\ \frac{q-5}{4},\ \frac{q-1}{4}\right)$

Latin Square Graphs

Latin Square:

for n.

$$L = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & \boxed{1} & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Vxs: (i,j) $\quad i,j = 1,\dots,n$

$(i,j) \sim (k,l)$ if same row $i=k$

or same col $j = l$

or $L(i,j) = L(k,l)$. (same entry)

Orthogonal array (k,n)

row #	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
col #	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
entry	1	2	3	4	2	3	4	1	3	4	1	2	4	1	2	3

$k \times n^2$ array entries from $[n]$ s.t.

the n^2 ordered pairs defined by any two rows

are all distinct.

Graph: vxs: n^2 cols an $OA(k,n)$

$a \sim b$: same ent in some coord.

Exercise: is SRG with parameters $(n^2, (n-1)k, n-2+(k-1)(k-2), k(k-1))$.

Generalized Quadrangles

pt, line incidence structure

$\mathcal{P}, \mathcal{L}$ sets and $I : \mathcal{P} \times \mathcal{L} \to \{0,1\}$.

points, lines

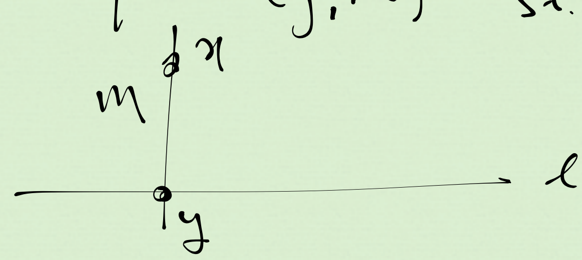
GQ(s,t).

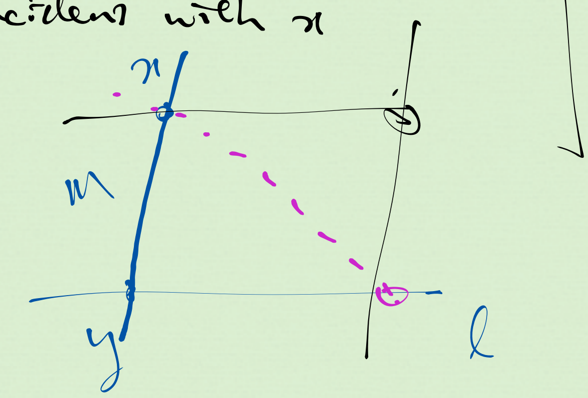
i) $t+1$ lines through each pt

ii) $s+1$ pts sit on each line

iii) $\forall\, x \in \mathcal{P}$ and $\ell \in \mathcal{B}$ not incident with x

$\exists$ unique (y,m) s.t.



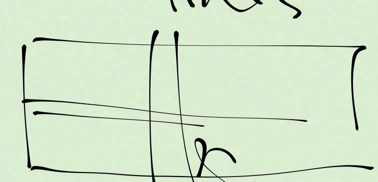


Pt graph: vxs: $\mathcal{P}$

$p \sim q$: $\exists\, \ell$ s.t. p, q incident to ℓ

Exercise 2: point graph of GQ(s,t) is SRG with params

$((s+1)(st+1), s(t+1), s-1, t+1)$.



$= N$

$I(p,\ell)$.

$A = NN^T + \underline{\quad} I$

Point-line incidence graph

vxs: $\mathcal{P} \cup \mathcal{L}$

$p \sim \ell$ if p, ℓ incident

- bipartite gr on $(s+1)(st+1) + (t+1)(st+1)$
- diameter 4; girth 8.
- if $s=t$, is DRG

$$\begin{bmatrix} 0 & N^T \\ N & 0 \end{bmatrix}$$

Ex: Find intersection array of pt-line incidence graph of GQ(s,s)

How to construct GQs?

$W(q)$: Consider $PG(3,q)$, q prime power

$V = \mathbb{F}_q^4$

pts are 1-dim subspaces of V

lines " 2-dim "

planes " 3-dim "

$$H = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

A subspace S of V is totally isotropic if $u^T H v = 0$ $\forall u,v \in S$.

V has $(q^2+1)(q+1)$ totally isotropic 1-dim subspaces & 2-dim subspaces

these are the pts and lines of $GQ(q,q)$.

Smallest example: $(15,6,1,3)$ $GQ(2,2)$

$W(3)$ is $(40,12,2,4)$

dual graph: switch roles of $\mathcal{P}$ & $\mathcal{L}$.

Known params: $(1,q)$

$(q-1, q+1)$

(q,q) ← $W(q)$ & dual

(q,q^2) $GQ(2,4)$

(q^2,q^3)

Open Problem: The following are open for all q, but no graphs are known:

(q, q^2-q) and (q, q^2-q-1)

Does there exist $GQ(4,11)$ or $GQ(4,12)$?

$GQ(4,11) \Rightarrow (225, 48, 3, 12)$ open parameter set

$GQ(4,16)$?

? $GQ(5,25)$

$4(3\cdot 6+1)$ 76, 21;

Distance-Regular Graphs

Johnson Graphs: $[1, \ldots, v] = V$

$(5,2)$

Kneser(v,k)

$a \sim b \Leftrightarrow a \cap b \neq \emptyset$

vxs: k-subsets of V

$a \sim b$: $|a \cap b| = k-1$

$\Rightarrow$ distance-transitive $\hookrightarrow$ $d(a,b) = \ell$

$\Leftrightarrow$ $|a \cap b| = k - \ell$.

$\Rightarrow$ Johnson Scheme

Cayley graph: $X(G, C)$, C closed under inverses and $1 \notin C$.

vxs: G

$x \sim y$: $yx^{-1} \in C$ $\qquad y - x \in C$

Hypercube:

Cayley gr: $X(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$ $\qquad i \rightarrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} =: e_i$

elementary basis

$\Rightarrow$ Hamming Scheme (q,d): diameter n. $X(\mathbb{Z}_q^d, \{e_1, \ldots, e_d\})$

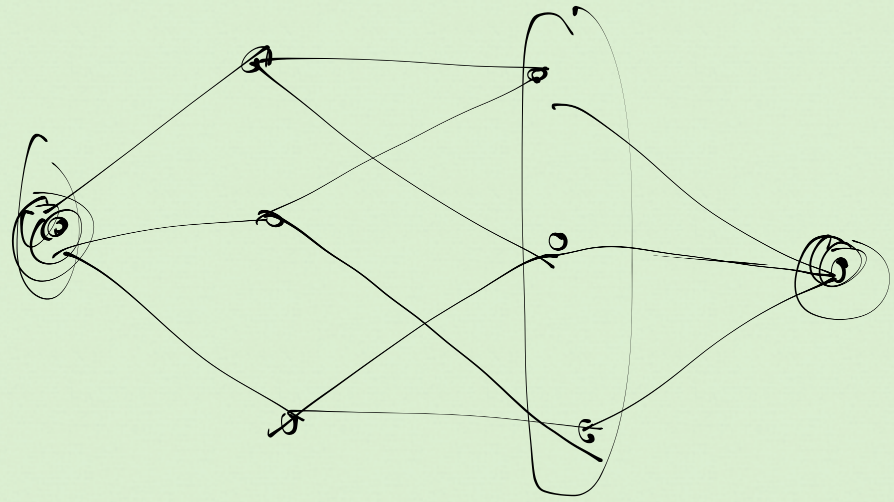
Cube is antipodal: being max distance or $=$ is an equiv relation.

also bipartite

$\{A_1, \ldots, A_d\}$: adj mxes of $G_1, \ldots, G_d$

distance mxes

if one of them is not connected $\Rightarrow$ G_1 is antipodal or bip



Q_3

Halved cube: distance 2 graph has 2 iso components

folded cube: identify each vx with its antipode.

Clebsch graph is halved 5 dim cube

Halved of 3-cube : K_4

Folded 3-cube: K_4

Krein parameters

Recall mtx algebra of a DRG has 2 bases:

$A_0 = I, A_1, \dots, A_d$ and $E_0 = \frac{1}{n}J, E_1, \dots, E_d$.

These are bases of a Schur-closed mtx algebra.

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

$p_{ij}^k \geq 0$, integers.

and

$$E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{d} q_{ij}^k E_k$$

Krein parameters

$\boxed{E_i \circ E_j}$ is principal submtx of $E_i \otimes E_j \succcurlyeq 0$

$\Rightarrow E_i \circ E_j \succcurlyeq 0 \Rightarrow q_{ij}^k \geq 0$

- rules out existence of certain DRGs by intersection array.

Formally self-dual: there is an ordering of idempotents s.t. $p_{ij}^k = q_{ij}^k$ $\forall i, j, k \in \{0, \dots, d\}$

Spherical t-design:

finite subset S of unit sphere in $\mathbb{R}^m$ s.t.

avg of any poly of deg $\le t$ in m variables

= avg on the sphere.

That is $p(x_1, \dots, x_m)$ with deg $\le t$

$$\int_{S_m} p(x_1, \dots, x_m)\, d\bar{x} = \frac{1}{|S|} \sum_{\bar{x} \in S} p(x_1, \dots, x_m).$$

S is spherical 1-design $\iff \sum_{s \in S} s = 0$

$\rightarrow Z = [s_1, \dots, s_{|S|}]$

S is spherical 2-design $\iff ZZ^T = \alpha I$ for some $\alpha \neq 0$ and S is a spherical 1-design.

Lem: $Z^T Z$ has 2 different off-diagonal entries

$\implies Z^T Z = aI + bA + cB$

where A, B are adj. mtces of SRGs.

Matrix algebras

Cellular algebras:

A subalgebra of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ which is

i) closed under $\circ$

ii) closed under transposition and conjugation; and

iii) contains J and I

Given matrices $M_0, \ldots, M_\ell$, we can consider the cellular closure $\langle M_0, \ldots, M_\ell \rangle$ the smallest cellular algebra containing them.

Ex: For any graph X, we can consider $\langle A(X) \rangle$, the Bose-Mesner algebra.

If X is DRG, then this cellular algebra has bases

$A_0, \ldots, A_d$ and $E_0, \ldots, E_d$

A cellular algebra has a unique basis $\mathcal{A}$ of 01 matrices s.t.

1) $\sum_{A \in \mathcal{A}} = J$

2) $\sum_{A \in \mathcal{A}' \subseteq \mathcal{A}} A = I$

3) for $A \in \mathcal{A}$, $A^T \in \mathcal{A}$.

$\Rightarrow$ 4) $A_i A_j \in \langle \mathcal{A} \rangle$ and so

$$A_i A_j = \sum_k p_{ij}^k A_k$$

for some constants p_{ij}^k

Ex: If X is DRG, then

$\underline{p_{1i}^{i-1} = b_{i-1}}$, $\underline{p_{1i}^{i} = a_i}$ and $\underline{p_{1i}^{i+1} = c_{i+1}}$

$\therefore$ can compute all p_{ij}^k from a_i, b_i, c_i

Cospectrality

If X, Y are DRGs with the same intersection array, then
$\{b_0, \dots b_{d-1}, c_1, \dots c_d\}$

$A(X)$, $A(Y)$ cospectral

$L(X)$, $L(Y)$ cospectral

$Q(X)$, $Q(Y)$ cospectral

In fact, for any $\alpha_0, \dots, \alpha_d \in \mathbb{R}$,

$$M_X = \sum_{i=0}^{d} \alpha_i A_i(X)$$

are cospectral.

$$M_Y = \sum_{i=0}^{d} \alpha_i A_i(Y)$$

Pf: $A_i E_j = Q_{ji} E_j$ jth eigenvalue of A_i

Write M_X in $\{E_0, \dots, E_d\}$
(and do the same for M_Y)

$$M_X = \sum_{i=0}^{d} \left(\sum_{j=0}^{d} \alpha_j Q_{ij} \right) E_i$$

One can see that M_Y will have the same coefficients.

Does this generalize? Yes

Caveat : only works if X, Y are DRGs.

Cospectrality of A does not imply anything for $\langle A \rangle$ in general.

Exercise: The Wells graph (Armanios-Wells) on 32 vxs with $\{5,4,1,1;1,1,4,5\}$ has 22 cospectral mates, which are not DRG. Can you construct them?

Also mxes in the BM algebra which distinguish them from the Wells graph.?

Weak and strong isomorphism

Let W and W' be two cellular algebras with bases $\mathcal{A}$ and $\mathcal{A}'$.

Strong isomorphism

$\exists$ an ordering of $\mathcal{A}'$ and a permutation mtx P s.t.

$$A_i = P^T A_i' P$$

weak isomorphism

$\exists$ an ordering of $\mathcal{A}'$ s.t.

$$\phi : \mathcal{A} \to \mathcal{A}' \quad : \quad \phi(A_i) = A_i'$$

preserves addition, $\cdot$, $\circ$ and cx conjugation, transposition.

$\Leftrightarrow$ W, W' have same p_{ij}^k's.

Lemma: If W, W' are cellular algebras with same p_{ij}^k's (weakly iso).
then

$$\sum a_i A_i \quad \text{and} \quad \sum a_i A_i'$$

are cospectral for any $\{a_i\}$.

$\Rightarrow$ We obtain the sequence for DRGs with the same intersection array.

Weisfeiler-Leman algorithm $\quad X = (V, E)$

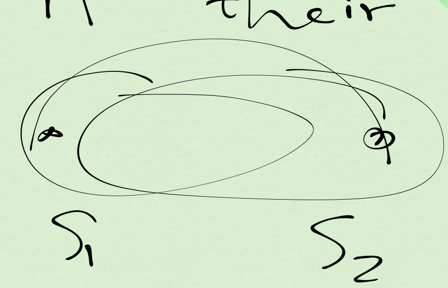
1-dim : color refinement

k-dim : color λ of V^k

For $(v_1, \dots, v_k)$ color with iso class of $X[v_1, \dots, v_k]$

- at every step, append colours of nbr sets.

S_1, S_2 are nbrs if their XOR is an edge.



Next time:

$$\underbrace{\hat{W}^{(k)}}_{\text{k-th extension}} = \langle W^{\otimes k}, \Delta \rangle \qquad n^k \times n^k$$

$$\Delta(x,y) = \begin{cases} 1, & \text{if } x = y = (u, \ldots, u); \\ 0, & \text{else.} \end{cases}$$

Groups acting on graphs

An automorphism of X is a bijection $\phi : V(X) \to V(X)$

s.t. $u \sim v \iff \phi(u) \sim \phi(v)$.

$d(u,v) = d(\phi(u), \phi(v))$

$Aut(X)$ = group of all auts under composition

One can view $Aut(X)$ as a gp of perm matrices which commute with A; P st.

$$P^T A P = A.$$

Also $Aut(X) \leq Sym(V)$ acting on vertices of X.

Transitivity

$G \leq Aut(X)$

G acts transitively on X if $\forall\, u,v \in V(X)$

$\exists\, g \in G$ s.t. $g(u) = v$.

If $Aut(X) \circlearrowright$ transitively on X then X is said to be vertex-transitive.

Ex: K_n, C_n, Petersen graph

an orbit of $G \curvearrowright X$ is $\mathcal{O} \subseteq V$

s.t. $g(u) \in \mathcal{O}$ whenever $u \in \mathcal{O}$ and $g \in G$.

and G acts transitively on $\mathcal{O}$.

X is vx-trans $\iff$ $\mathrm{Aut}(X) \curvearrowright X$ has 1 orbit

If $\mathcal{O}_1, \mathcal{O}_2$ orbits $\Rightarrow$ either disj or equal.

(blocks of imprimitivity)

A part'n $P_1, \dots, P_m$ of V is equitable if

each $v \in P_i$ has b_{ij} nbrs in P_j

where b_{ij} is a constant.

$$A = \begin{array}{c|ccc} & P_1 & \cdots & P_m \\ \hline P_1 & A_{11} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ P_m & A_{m1} & \cdots & A_{mm} \end{array}$$

quotient mtx

$$B = \begin{bmatrix} \text{avg row} \\ \text{sum of } A_{ij} \end{bmatrix}_{i,j}$$

$$= \frac{1}{|P_i|} \mathbb{1}^T A_{ij} \mathbb{1} = b_{ij}$$

Lem: $V_1, \dots, V_m$ orbit partition $\Rightarrow$ equitable.

Recall: Cayley gr $X(G, \mathcal{C})$

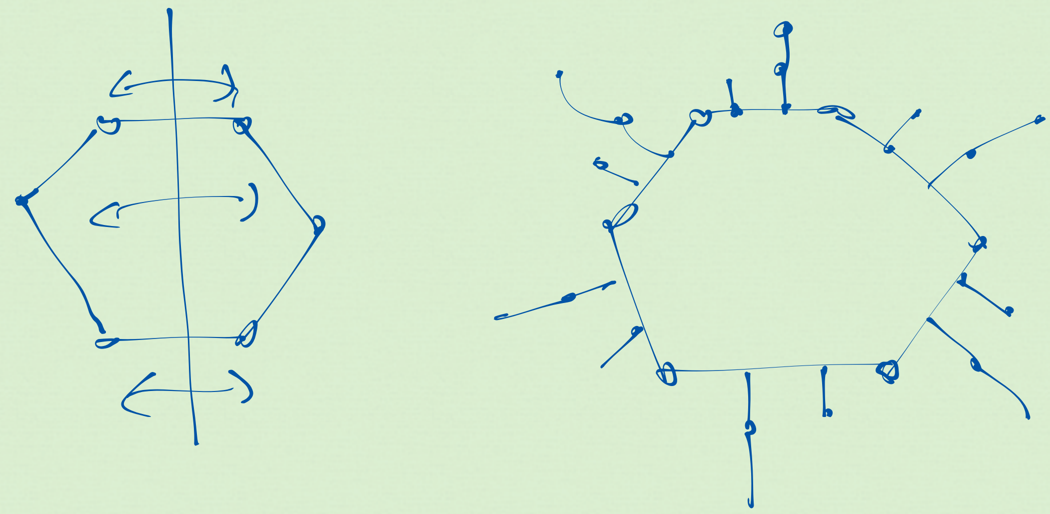
$g \sim h$ if $hg^{-1} \in \mathcal{C}$.

$G \leq \mathrm{Aut}(X(G, \mathcal{C}))$ $\forall\, \mathcal{C}$.

Facts about gps acting on graphs

Thm (Frucht) G a gp, $\exists X$ s.t. $G \cong \mathrm{Aut}(X)$.

Pf: $\mathbb{Z}_6$



Graphical regular repr: $X(G,C)$ s.t. Aut gp is $\cong G$.

Thm (Hetzel-Godsil) All groups G have GRR except generalized dicyclic, abelian of exponent ≥ 3 and 13 small groups.

Groups acting on tuples of vertices

$G \circlearrowright V(X)^k$

$g : (x_1, \dots, x_k) \longmapsto (g(x_1), \dots, g(x_k))$

$k = 2$

orbital: orbits of $G \circlearrowright V^2$

(x,x)

How many orbitals? (How few?)

(x,y) $y \neq x$

$C_i := \{(x,y) : \text{s.t. } d(x,y) = i\}$

orbital refine $C_0, \ldots, C_d$.

If $C_0, \ldots, C_d$ is the orbital partition of

$Aut(X) \Rightarrow X$ is distance-transitive.

Ex: Johnson graphs, Hamming graphs, $W(q)$, Pete

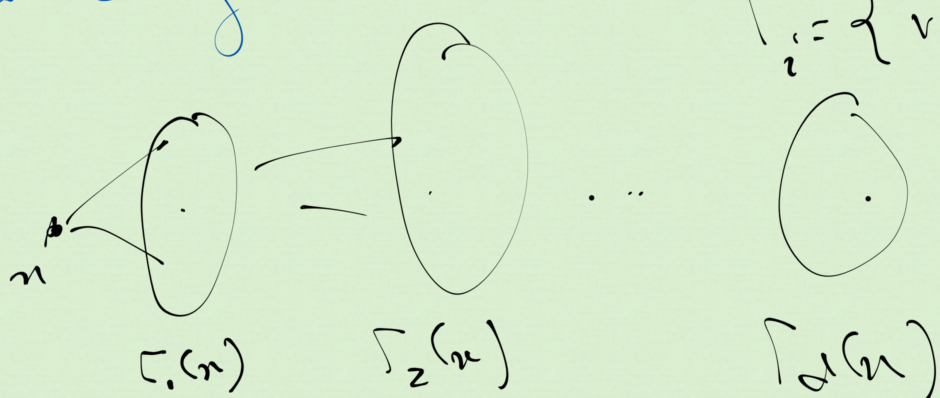
Lemma: vx-trans graph of diam d is

dist-transitive $\Leftrightarrow$ orbits of stabilizer of

$x \in V(X)$ in $Aut(X)$ is $\{\Gamma_i(x)\}_{i=0}^{d}$

for every vx x.

$\Gamma_i := \{v : d(x,v) = i\}$.



$G = Aut(X)_x = \{g \in Aut(X) : g(x) = x\}$.

$\Gamma_0, \Gamma_1, \ldots, \Gamma_d$ is the orbit partn of $G \circlearrowleft X$.

Compute its quotient matrix:

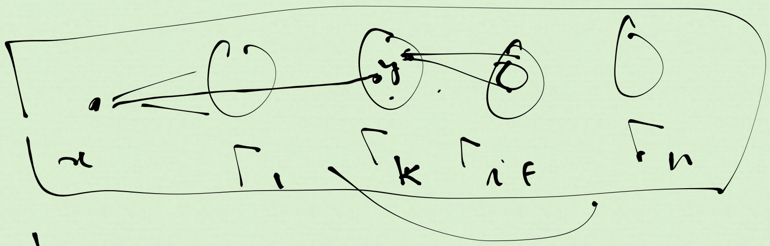
$$A = \begin{array}{c|cccc} & x & \Gamma_1(x) & \cdots & \Gamma_d(x) \\ \hline x & & & & \\ \Gamma_1 & & & & \\ \vdots & & & & \\ \Gamma_d & & & & \end{array}$$

$\vec{L} = [L_{ij}]_{ij}$ the quotient mtx.

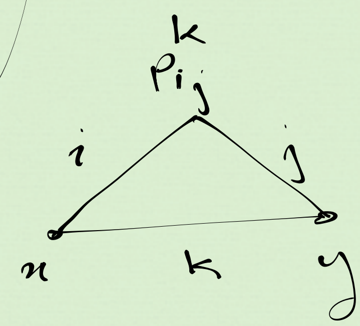
distance-regular $\iff$ $\Gamma_0(x), \dots, \Gamma_d(x)$ is equitable

for every x, with the same quotient matrix.

What are the entries of L.?



L_{ij} = # nbrs of $v \in \Gamma_i(x)$ in $\Gamma_j(x)$.



$$L = \begin{bmatrix} 0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ 0 & & & c_d & a_d \end{bmatrix}$$

$p^k_{k+1,1}$

$\to (L_i)_{j,k} = p^k_{ij}$

Triply transitive graphs

$G \curvearrowright V^3$

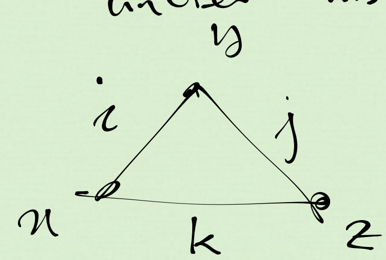
$g : (x_1, x_2, x_3) \mapsto (g(x_1), g(x_2), g(x_3))$

Consider orbits of V^3 under this action.

(x, x, x)

(x, x, y)

(x, y, z)



$D(x,y,z) = (i,j,k)$.

$D(x,y,z) = (d(x,y), d(y,z), d(z,x))$

$R_{i,j,k} = \{(x,y,z) \mid D(x,y,z) = (i,j,k)\}$ ✓

triply transitive = $R_{i,j,k}$ $\forall\, i,j,k$ is the orbit partition of $Aut(X) \curvearrowright V^3$.

$x, y \in V$

$G = \left(\mathrm{Aut}(X)_x \right)_y$

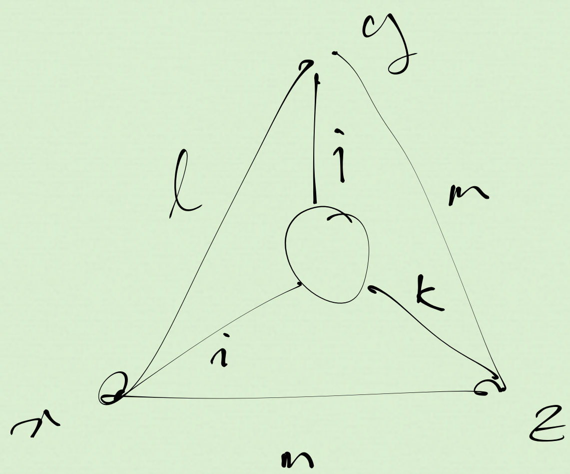
$G \curvearrowright V$ orbits : $X_{ij} = \left\{ z : \begin{array}{l} d(x,z) = i \\ d(y,z) = j \end{array} \right\}$.

triply trans $\iff$ distance-trans

X_{ij} an equitable part'n

quotient matrices depend only on $d(x,y)$?

$$\left(L^{l,k} \right)_{(i,j),(m,n)} = p^{lmn}_{ijk}$$



Grassman scheme

$\mathbb{F} = GF(q)$: vxs: k-dim subspaces of V

$V = \mathbb{F}^{d}$

$W_1 \sim W_2$: W_1, W_2 intersect in a $(k-1)$-dim subspace.

$\# \text{vxs} = \begin{bmatrix} d \\ k \end{bmatrix}_q$