30
Graph Algebras

We consider some of the algebras we can attach to a graph.

30.1 The adjacency algebra

If R is a ring and $A = A(X)$, the ring $R[A]$ of polynomials in A is an algebra (in the moxt general sense). We restrict ourselves to the case where R is a field. If X has diameter d, the dimension of $\mathbb{F}[A]$ is at least $d+1$. If $\mathbb{F} = \mathbb{R}$ then $\dim(\mathbb{F}[A])$ is equal to the degree of the minimal polynomial of A; if $\mathbb{F} = \mathbb{R}$ and $A = A^T$, this is the number of distinct eigenvalues of A.

We note that $\mathbb{R}[A_1]$ and $\mathbb{R}[A_2]$ are isomorphic if and only if X_1 and X_2 are cospectral.

The automorphism group of X lies in the commutant of $\mathbb{F}[A]$.

One question is what we can say about when two adjacency algebras of $\mathbb{F}$ are isomorphic? (The characteristic polynomial is not the right invariant). Note that $\mathbb{F}[A]$ is not semisimple in general.

30.2 Extended adjacency algebras: A and J

We can extend an algebra by adding new elements. We focus on extensions of the adjacency algebra $\mathbb{R}[A]$ of a graph. The key point is that if we add a symmetric matrix M to this algebra, the extended algebra is still closed under transpose and therefore is semisimple.

The first particular case is the algebra $\langle A, J\rangle$. Note that $\langle A, J\rangle = \langle A(\overline{X}), J\rangle$. The automorphism group of X lies in the commutant of $\mathbb{F}[A]$.

If X is connected and regular, then J is a polynomial in A and our "extension" has not changed anything. On the other hand:

30.2.1 Theorem. *Let X be a graph on n vertices. Then $\langle A, J\rangle = \mathrm{Mat}_{n\times n}(\mathbb{R})$ if and only if no eigenvector of X is orthogonal to $\mathbf{1}$.* □

If the stated eigenvector condition holds we say X is *controllable*. If X is controllable, the matrices

$$A^i J A^j, \quad 0 \le i, j \le n-1$$

are a basis for $\mathrm{Mat}_{n\times n}(\mathbb{R})$.

An eigenvalue is a *main eigenvalue* if its eigenspace contains an eigenvector not orthogonal to $\mathbf{1}$. If m is the multiplicity of the eigenvalue θ, then either the eigenspace of θ lies in $\mathbf{1}^{\perp}$ or the subspace of eigenvectors orthogonal to $\mathbf{1}$ has codimension one, and θ is a main eigenvalue. Define the reduced multiplicity of $\nu(\theta)$ to be dimension the intersection of the θ-eigenspace with $\mathbf{1}^{\perp}$.

30.2.2 Lemma. *Let X be a graph with ℓ distinct eigenvalues, and let μ be the number of main eigenvalues of X. Then*

$$\dim(\langle A, J\rangle) = \mu^2 + \ell - \mu. \qquad \square$$

We sketch a proof. The $\langle A, J\rangle$-module generated by $\mathbf{1}$ is irreducible [1]; we denote it by M. The direct sum decomposition

[1] your problem

$$\mathbb{R}^n = M \oplus M^{\perp}$$

is invariant under $\langle A, J\rangle$. Note that $M^{\perp}$ is spanned by the eigenvectors of A in $\mathbf{1}^{\perp}$. The restriction of $\langle A, J\rangle$ to M is isomorphic to $\mathrm{Mat}_{\mu\times\mu}(\mathbb{R})$; its restriction to $M^{\perp}$ is commutative and has $\ell - \mu$ distinct eigenvalues.

30.2.3 Theorem (Johnson & Newman)**.** *Let X_1 and X_2 be graphs. Then X_1 and X_2 are cospectral with cospectral complements if and only if there is an orthogonal matrix L such that*

$$L^{-1}A_1L = A_2, \quad L^{-1}JL = J. \qquad \square$$

So if X_1 and X_2 are controllable graphs on n vertices, then $\langle A_1, J\rangle$ and $\langle A_2, J\rangle$ are isomorphic, but there is an algebra isomorphism L such that $L^{-1}A_1L = A_2$ and $L^{-1}JL = J$ if and only if X_1 and X_2 are cocospectral.[2]

[2] cospectral with cospectral complements

If X_1 and X_2 are controllable, there are cocospectral if and only if

$$\mathbf{1}^T A_1^k \mathbf{1}^T = \mathbf{1}^T A_2^k \mathbf{1}^T$$

for all nonnegartive integers k.

30.3 Extended adjacency algebras: A and Δ

Let Δ denote the diagonal matrix of valencies of X. The algebra $\langle A, \Delta\rangle$ is semisimple and the automorphism group of X lies in its commutant.

If X is connected, then J is a polynomial in A and Δ.[3]. We assume that X is connected, and then $\langle A, J\rangle \le \langle A, \Delta\rangle$.

[3] Exercise!

We have very little to say about the algebra $\langle A, \Delta\rangle$, but there is one result of interest for trees.

30.3.1 Theorem (B. D. McKay)**.** *Let T_1 and T_2 be trees with respective adjacency matrices A_1 and A_2, and valency matrices Δ_1 and Δ_2. If T_1 and T_2 are not isomorphic, there is a polynomial $p(x, y)$ such that*

$$\phi(p(A_1, \Delta_1), t) \neq \phi(p(A_2, \Delta_2), t). \qquad \square$$

We point out that, in general, $p(A, \Delta)$ is not symmetric.

If T_1 and T_2 are isomorphic, there is a permutation matrix P such that

$$P^T A_1 P = A_2, \quad P^T \Delta_1 P = \Delta_2.$$

If T_1 and T_2 are not isomorphic, there is no invertible linear map L such that

$$L^{-1} A_1 L = A_2, \quad L^{-1} \Delta_1 L = \Delta_2.$$

If T_1 and T_2 are controllable, $\langle A_1, \Delta_1 \rangle$ and $\langle A_2, \Delta_2 \rangle$ are isomorphic, because they are both equal to the full matrix algebra.

If $L^{-1} \Delta_1 L = \Delta_2$, then $L^{-1} p(\Delta_1) L = p(\Delta_2)$ for any polynomial p.S This implies that there is a permutation matrix P such that $P^T \Delta_2 P = \Delta_1$; consequently $P^T L^{-1} \Delta_1 LP = \Delta_1$ and therefore LP is block diagonal.

31
Coherent Things

A *coherent algebra* $\mathscr{C}$ is a (finite-dimensional) matrix algebra over a subfield of $\mathbb{C}$ that is

(a) Contains J and is closed under Schur product.

(b) Is closed under transpose and complex conjugation.

From (a) we see that $\mathscr{C}$ is a commutative algebra relative to Schur multiplication. (By default our rings and algebras must have an identity element.) Condition (b) implies that $\mathscr{C}$ is a semisimple matrix algebra.

The Bose-Mesner algebra of an association scheme is a coherent algebra.

Coherent algebras are also known as cellular algebras.

Weisfeiler and Leman introduced *cellular algebras* in 1968 and, in 1970, a paper of Donald Higman introduced *coherent algebras*[1] In 1980?, Higman wrote about coherent algebras. In ??, Graham and Lehrer introduced what they called cellular algebras (in the context of representation theory). Because of this usage, my feeling is that coherent algebras is the better term.

[1] definitions to come

31.1 Coherent Configurations

We start with a simple but very useful result.

31.1.1 Theorem. *If the vector space of matrices $\mathscr{M}$ is closed under Schur product, it has a unique basis of* 01*-matrices.*

Proof. If p is a polynomial

$$p(t) = p_0 t^k + \cdots + p_k$$

and A is a matrix, we define the *Schur polynomial* $p \circ A$ to be

$$p_0 A^{\circ k} + \cdots + p_k J.$$

If λ is an entry of the matrix A, let p_λ be the polynomial that takes the value 1 on λ and 0 on all other entries of A. Then $p \circ A$ is a 01-matrix that lies in $\mathscr{M}$, and it follows that $\mathscr{M}$ is spanned by 01-matrices.

Therefore $\mathscr{M}$ has a basis of 01-matrices, which we must show is unique. We say that a Schur idempotent is *primitive* if it is not zero and cannot be expressed as a sum of two non-zero Schur idempotents. The primitive Schur idempotents span $\mathscr{M}$. The Schur product of two distinct primitive Schur idempotents is zero and, given this, it is easy to show that the primitive Schur idempotents form a basis.

Suppose β is a spanning set for $\mathscr{M}$ consisting of Schur idempotents. Each element of β is a sum of primitive Schur idempotents and if one of the sums is not trivial, then β is not a maximal independent subset, and so it is not a basis. □

Each Schur idempotent is the adjacency matrix of a directed graph, possibly with loops. A set of directed graphs is a *coherent configuration* if it is the set of primitive Schur idempotents of a coherent algebra.

As a coherent algebra contains I, we that I must be a sum of primitive Schur idempotents; these are necessarily diagonal matrices, and determine a partition of the vertices of the coherent configuration. The cells of this partition are known as *fibres*. A coherent algebra (or configuration) is *homogeneous* if I is a primitive Schur idempotent. This leads to the following exercise.

31.1.2 Theorem. *A commutative coherent algebra is homogeneous.* □

Suppose $\mathscr{C}_1$ and $\mathscr{C}_2$ are coherent algebras of $n \times n$ matrices and and L is an invertible matrix such that $M \to L^{-1}ML$ is an algebra isomorphism. Then L must map the diagonal elements of the Schur basis of $\mathscr{C}_1$ to diagonal elements of $\mathscr{C}_2$.[2]. It follows that there is a permutation matrix P such that LP is block diagonal, with blocks corresponding to the fibres of $\mathscr{C}_1$. (The fibre partition is a refinement of the valency partition.)

[2] work out why

31.2 Permutation Groups and Coherent Algebras

Coherent algebras play a significant role in the study of automorphism groups of graphs, and this is the original motivation for the concept.

You should verify the following:

31.2.1 Lemma. *Let A and B by $m \times n$ matrices. If P is an $m \times m$ permutation matrix, then $P(A \circ B) = (PA) \circ (PB)$.* □

31.2.2 Corollary. *If $\mathscr{P}$ is a set of $n \times n$ permutation matrices, the commutant of $\mathscr{P}$ is a coherent algebra. The fibres of this coherent algebra are the orbits of the permutation group generated by $\mathscr{P}$.*

Proof. If A and B commute with a permutation matrix P, then

$$P(A \circ B) = (PA) \circ (PB) = (AP) \circ (BP) = (A \circ B)P.$$

Since the commutant of a set of matrices is a matrix algebra, the commutant of $\mathscr{P}$ is a coherent algebra.

We leave the statement about fibres as an exercise. □

The coherent algebra generated by a set $\mathscr{M}$ of matrices is the smallest coeherent algebra that contains $\mathscr{M}$.

31.2.3 Lemma. *If A is the adjacency matrix of the graph X, then* $\mathrm{Aut}(X)$ *lies in the commutant of the coherent algebra generated by A.* □

Thus if the coherent algebra generated by A is the full matrix algebra (for example, if X is controllable) then $\mathrm{Aut}(X)$ is trivial. (The coherent algebra generated by A can be computed in polynomial time.)

We given an example of a coherent algebra that is not commutative and is not the commutant of a permutation group. Let N be the vertex-block incidence matrix of a 2-(v,k,λ)-design with b blocks. Then the matrices

$$\begin{pmatrix} I_v 0 & \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_b \end{pmatrix}, \begin{pmatrix} J - I_v 0 & \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & J - I_b \end{pmatrix}, \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & J - N \\ J - N^T & 0 \end{pmatrix}$$

are Schur idempotents that sum to J. You may prove that they form a coherent configuration if and only if the design is quasi-symmetric, that is, there are constant a and b such that any two distinct blocks intersect in a or b points.

A quantum permutation of index d is an $n \times n$ matrix whose entries are $d \times d$ projections, such that the entries in each row and each column sum to I_d. The commutant of a set of quantum permutations is a Schur-closed, and hence forms a coherent algebra.

31.3 Isomorphism

Now we get to the messy part. A homomorphism of algebras is a ring homomorphism that commutes with scalar multiplication, and a invertible homomorphism is an isomorphism. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are coherent algebras, it is obvious that an algebra homomorphism from $\mathscr{C}_1$ to $\mathscr{C}_2$ need not preserve the Schur product; if it does we will call it a *coherent homomorphism*.

31.3.1 Theorem. *Let $\mathscr{M}$ and $\mathscr{N}$ be coherent algebras and let Ψ be an algebra homomorphism from $\mathscr{M}$ to $\mathscr{N}$. If Ψ commutes with Schur product and $\Psi(J) \neq 0$, then Ψ is injective.*

Proof. Let $\Psi : \mathscr{M} \to \mathscr{N}$ be an algebra homomorphism and assume $K = \ker(\Psi)$. The K is an ideal of $\mathscr{M}$. If $R \in \mathscr{M}$ and $S \in K$, then

$$\Psi(R \circ S) = \Psi(R) \circ \Psi(S) = 0.$$

This shows that $R \circ S \in K$ if $S \in K$. It follows that K has a basis of Schur idempotents. Suppose S a non-zero Schur idempotent in K. Then $JSJ \in K$, but

$$JSJ = \mathbf{1}(\mathbf{1}^T S\mathbf{1})\mathbf{1}^T = \mathbf{1}^T S\mathbf{1} J.$$

As $\mathbf{1}^T S\mathbf{1} > 0$ this implies that $J \in K$. □

31.3.2 Theorem. *An algebra homomorphism from $\mathscr{C}_1$ to $\mathscr{C}_2$ commutes with Schur product if and only if it maps the primitive Schur idempotents of $\mathscr{C}_1$ to the primitive Schur idempotents of $\mathscr{C}_2$.* □

An automorphism of a matrix algebra is *inner* if it is given by a map $M \mapsto A^{-1}MA$. It is true that any automorphism of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is inner,[3] but this is not true for

$$\mathrm{Mat}_{2\times 2}(\mathbb{C}) \oplus \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

To see this, note that the permutation

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

gives an automorphism that is not inner. I believe it is true that if $\mathscr{C}$ is a semisimple matrix algebra and the commutant of $\mathscr{C}$ is commutative, all automorphisms are inner.

An isomorphism between matrix algebras is *combinatorial* if it is given by a map $M \mapsto P^T MP$ for some permutation matrix P. Combinatorial isomorphisms are necessarily coherent.

[3] Noether-Skolem

31.4 Subalgebras

If $\mathscr{C}$ is a matrix algebra and E is an idempotent in $\mathscr{C}$, then $E\mathscr{C}E$ is subspace of $\mathscr{C}$ that is closed under matrix multiplication. It is not a subalgebra, because it does not contain the identity of $\mathscr{C}$. However the sum

$$E\mathscr{C}E + (I-E)\mathscr{C}(I-E)$$

is a subalgebra. If E is diagonal and 01 and $\mathscr{C}$ is coherent, then $E\mathscr{C}E$ is Schur closed and the subalgebra just given is coherent.

Let $\mathscr{C}$ be the coherent algebra generated by $A(X)$. If $\gamma \in \mathrm{Aut}(X)$, then the map

$$(u,v) \mapsto (u\gamma, v\gamma)$$

is a permutation of $V(X)\times V(X)$, lies in the commutant of $\mathscr{C}\otimes\mathscr{C}$. The subsets

$$\{(u,u) : u \in V\}, \quad \{(u,v) : u \sim v\}, \quad (u,v) : u \neq v, u \not\sim v$$

are unions of orbits of this action of $\mathrm{Aut}(X)$. (If X is complete, the third orbit is empty.)

Define an $n^2 \times n^2$ idempotent F_0 by

$$(F_0)_{(u,v),(u,v)} = \begin{cases} 1, & \text{if } u = v; \\ 0, & \text{otherwise.} \end{cases}$$

Note that F_0 is diagonal. Define F_1 to be the diagonal 01-idempotent with

$$(F_1)_{(u,v),(u,v)} = 1$$

whenever $u \sim v$. Finally set $F_2 = I - F_0 - F_1$. (We point out that we are not using the automorphism group to define these idempotents.)

The coherent algebra generated by $\mathscr{C} \otimes \mathscr{C}$ and F_0 is the *2-extension* of $\mathscr{C}$.

31.5 Jaeger Algebras

Let $\mathscr{M}$ denote $\mathrm{Mat}_{n\times n}(\mathbb{C})$. If $A \in \mathrm{End}(V)$, define operators X_A, Δ_A and Y_A on $V\otimes$ by

$$X_A M = AM, \quad De_A(M) = A \circ M, \quad Y_A(M) = MA^T.$$

If $\mathscr{A}$ is a subalgebra of $\mathscr{M}$, define $\mathscr{J}_3$ to be the algebra generated by the operators X_A and Δ_A for A in $\mathscr{M}$. Define $\mathscr{J}_4(\mathscr{A})$ to be the algebra generated by $\mathscr{J}_3(\mathscr{A})$ and the operators Y_A for A in $\mathscr{A}$. We say that $\mathscr{J}_3$ and $\mathscr{J}_4$ are *Jaeger algebras*.

We need to explain the transpose in the definition of Y_A and the indexing. If A_i and B_i are $n \times n$ matrices (with $i = 0,1$), the map

$$M \mapsto AMB^T$$

is an endomorphism of M and all endomorphisms of M are linear combinations of endomorphisms of this form. Thus we have a map from $\mathrm{End}(V) \otimes \mathrm{End}(V)$ into $\mathrm{End}(\mathscr{M})$. Further

$$A_1 A_2 M B_2^T B_1 = (A_1 A_2) M (B_1 B_2)^T$$

and therefore this map is a homomorphism. Consequently $\mathscr{M}$ is a module over $\mathrm{End}(V) \otimes \mathrm{End}(V)$.

Indexing. Let V a vector space. We define some operators on $V^{\otimes r}$. Assume $X_A(i) \in \mathrm{End}(V)$ and define to be the product operator acting as A on the i-th component and as I on the remaining components. If $1 \le i \le r-1$, let $\Delta_A(i)$ act as Δ_A on the i-th and $(i+1)$-th components[4] and as the identity on the remaining components. Then the algebra generated by the operators $X_A(1)$ for A in $\mathscr{A}$ is $\mathscr{J}_2(\mathscr{A})$ and the algebra generated by

$$\{X_A(1), \Delta_A(1) : A \in \mathscr{A}\}$$

is $\mathscr{J}_3(\mathscr{A})$.[5]

[4] identify $V \otimes V$ with $\mathrm{End}(V)$

[5] the indexing is coming from the theory of braid groups