Problems:

1. Fisher's inequality
2. Number of isomorphism classes of graphs on $e-1$ versus e edges
3. Edge reconstruction

Fisher's inequality

2-(v, b, r, k, λ) design

incidence structure, incidence matrix N

$$NN^T = (r-\lambda)I + J$$

Claim: If $r > \lambda$, then $b \geq v$. $r(k-1) = (v-1)\lambda$

Claim: $rk(N) = v$

Proof: $NN^T = (r-\lambda)I + \lambda J$ is invertible:

(a) eigenvalues: $r-\lambda+\lambda v = r+\lambda(v-1) = r+r(k-1) = rk$ & $r-\lambda^{(v-1)}$

(b) $$x^T N N^T x = (r-\lambda)\overbrace{x^T x}^{x^T I x} + \overbrace{\lambda x^T \underset{\sim}{1}\underset{\sim}{1}^T x}^{x^T J x} = (r-\lambda)\, x^T x + \lambda\,(1^T x)^2$$

psd + pdef = pdef

(c) $$N\underset{\sim}{1} = r\underset{\sim}{1}, \quad N J^T_{v\times b} = r J^T_{v\times b}$$

$$N\left(N - \frac{\lambda}{r} J_{v\times b}\right)^T = (r-\lambda) I + \lambda J - \lambda J = (r-\lambda) I$$

N has a right inverse

Counting graphs

$g(n,e) :=$ number of isomorphism classes of graphs on n vertices with e edges

Claim If $2e \leq \binom{n}{2}$, then $g(n,e-1) \leq g(n,e)$

Tool: induced partitions

$m \times |\rho|$

W : $m \times n$ matrix

ρ : partition of rows, characteristic matrix R

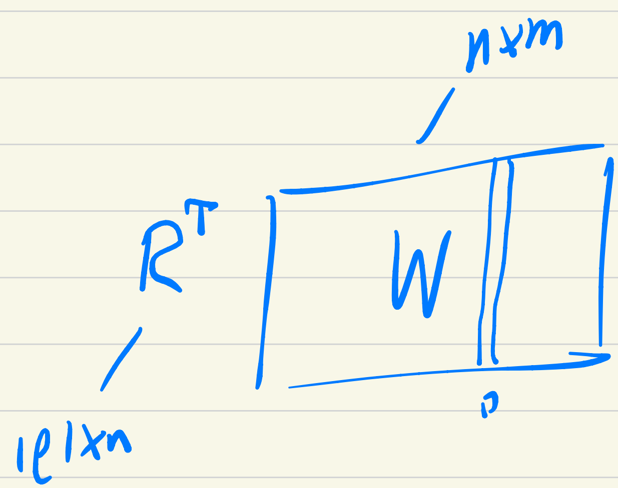
ρ^* : partition of columns of $R^T W$, by equality

often an orbit partition

ρ partition

$|\rho|$ cells

$n \times |\rho|$



e.g. 2-design

Lemma If the rows of W are linearly independent, $|\rho| \leq |\rho^*|$.

Proof. (a) Suppose $\xi^T R^T W = 0$. As rows of W are linearly independent, $\xi^T R^T = 0$.

But rows of R^T are linearly independent, so $\xi = 0$.

Therefore $\mathrm{rk}\,(R^T W) = \rho$.

(b) $\mathrm{rk}\,(R^T W) \leq$ # distinct columns of $R^T W$

$$= |\rho^*|$$

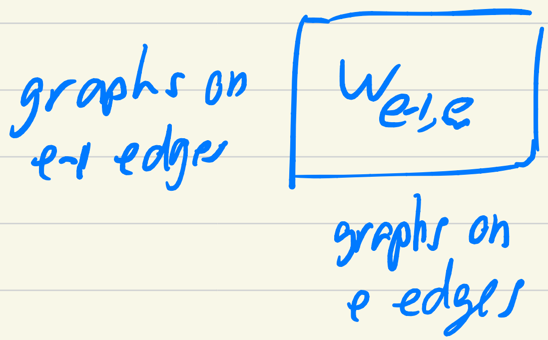
Magic matrix $W_{t,k}(v)$ — $\binom{v}{t} \times \binom{v}{k}$

incidence matrix for t-subsets of a v-set $\{1,\ldots,v\}$ vs. the k-subsets

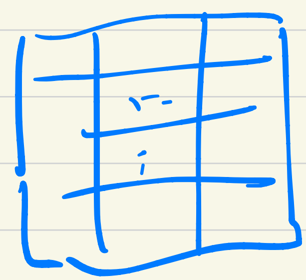
Claim If $2k \le v$ and $t \le k$, then $\mathrm{rk}(W_{t,k}(v)) = \binom{v}{t}$.

Claim If $W_{t,k}\, z = 0$ and $z \neq 0$, then
$$|\mathrm{supp}(z)| \ge 2^{t+1}.$$

Set $N = \binom{n}{2}$, work with $W_{e-1,e}(N)$ $(2e \le N)$



Sym(n) acts on (e-1)-subsets and e-subsets — and orbits are isomorphism classes of graphs.



Lemma Let W be the incidence matrix of an incidence structure $\mathcal{J}$ of points & blocks. If G is a group of automorphisms of $\mathcal{J}$ and ρ is the orbit partition on pts, then the orbit partition on blocks is a refinement of ρ^*.

Proof.

(a) β-column of $R^T W$ gives, for each i, the number of points of β in i-th cell ρ_i of ρ.

(b) $x \in \rho_i$ & $x \in \beta$, $\gamma \in G \Rightarrow x^\gamma \in \beta^\gamma \Rightarrow \beta$ & β^γ columns of $R^T W$ are equal.

in ρ_i

Corollary If rows of W are linearly independent, # pt-orbits $\leq$ # black-orbits.

$$\Rightarrow g(n, e-1) \geq g(n, e)$$

Edge reconstruction

Graph on n vxs is a subset of $E(K_n)$

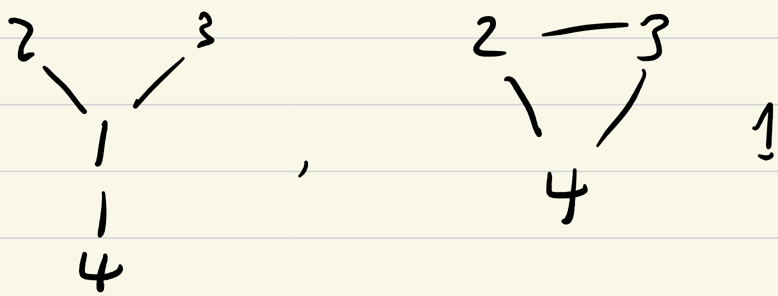
Let X be a graph on n vxs.

If Y is a graph, define $\mu_X(Y)$ to be the number of subgraphs of X isomorphic to Y.

Problem Assume $|V(X)|=n$ & $|E(X)|=e$. Is X determined by the values $\mu_X(Y)$, as Y runs over the graphs on $e-1$ edges?

$(X\setminus e) \cong Y\setminus \beta(e)$

Answer No!



What if $e \geq 4$? Yes, if:

(a) $2|E(X)| \geq \binom{n}{2} + 1$ Lovász

(b) $2^{|E(X)|-1} > n!$ Müller

Define ν_X, a function on graphs on e edges by

$$\nu_X(Y) = \begin{cases} 1, & \text{if } Y \cong X; \\ 0, & \text{otherwise.} \end{cases}$$

(We view it as a column vector.)

Define a function μ_X on graphs on $e-1$ edges, whose value on F is the number of edge-deleted subgraphs of X isomorphic to F.

AIM Prove that if $\mu_X = \mu_Y$, then $X \cong Y$.

Lemma $W_{e-1,e} \cdot |\mathrm{Aut}(X)| v_X = \mu_X$

Proof. The F-entry of $W_{e-1,e} v_X$ is $|\{Y : Y \cong X, F \subseteq Y\}|$

So F-entry of $|\mathrm{Aut}(X)| W_{e-1,e} v_X$ is

$$|\{\alpha \in \mathrm{Sym}(n) : F \subseteq X^{\alpha}\}| = |\{\alpha \in \mathrm{Sym}(n) : F^{\alpha^{-1}} \subseteq X\}| = \mu_X(F) \quad \square$$

Set $z = W_{e-1,e}\left(|\mathrm{Aut}(X)| v_X - |\mathrm{Aut}(Y)| v_Y\right)$

Corollary If $\mu_X = \mu_Y$, then $W_{e-1,e} z = 0$

Now columns of $W_{e-1,e}$ are linearly independent if $2e-1 \geq \binom{n}{2}$ then $z = 0$.

Size of support of v_X is $\frac{n!}{|\mathrm{Aut}(X)|}$. Hence $|\mathrm{supp}(z)| \leq 2n!$ If $z \neq 0$, it is a null $(e-1,e)$-design and $|\mathrm{supp}(z)| \geq 2^e$.

$2^e > 2n!$

k-homogeneous groups

Lemma. If $k \geq 2$, a k-homogeneous group is $(k-1)$-homogeneous.

Lemma Suppose X is connected and not bipartite. If X is edge-transitive, it is vertex-transitive.

Key: incidence matrix B

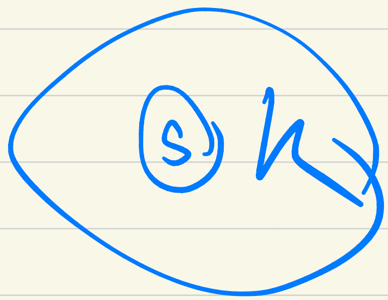
$rk(B) = v - \#$bipartite components

START Sep 22

Working with $W_{t,k}(v)$

Lemma $W_{s,t} W_{t,k} = \binom{k-s}{t-s} W_{s,k}$

$(W_{s,t} W_{t,l})_{\alpha,\beta}$



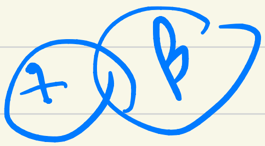
$\Rightarrow \mathrm{row}(W_{s,k}) \subseteq \mathrm{row}(W_{t,k})$

Define $(\overline{W}_{t,k})_{\alpha,\beta} = \begin{cases} 1, & \alpha\cap\beta=\emptyset; \\ 0, & \text{otherwise.} \end{cases}$

Lemma $\overline{W}_{t,k} = \sum_i (-1)^i W_{i,t}^T W_{i,k}$

Proof (of 2^{nd} lemma). Suppose $|\alpha| = t$, $|\beta| = k$

We have



$$\left(W_{i,t}^T W_{i,k}\right)_{\alpha,\beta} = \binom{|\alpha \cap \beta|}{i} \qquad \left(\sum_i (-1)^i W_{i,t}^T W_{i,k}\right)_{\alpha,\beta}$$

and so the (α,β)-entry in the sum is

$$\sum_i (-1)^i \binom{|\alpha \cap \beta|}{i} = \begin{cases} 1, & \alpha \cap \beta = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

This gives the lemma. $\square$

Next steps:

1. $\text{row}(W_{t,k}) = \text{row}(\bar{W}(t,k))$.

2. $W(t, v-t)$ is invertible.

3. rows of $W_{t,k}$ are linearly independent if $k+t \leq v$.

row-space claim

We have $W_{i,k} = \binom{k-i}{t-i}^{-1} W_{i,t} W_{t,k}$ and hence

$$\overline{W}_{t,k} = \sum_i (-1)^i W_{i,t}^T W_{i,k} = \left(\sum_i (-1)^i \binom{k-i}{t-i} W_{i,t}^T W_{i,t}\right) W_{t,k}$$

$$\Rightarrow \mathrm{row}(\overline{W}_{t,k}) \leq \mathrm{row}(W_{t,k})$$

Next

$$W_{i,t}\overline{W}_{t,k} = \binom{v-k-i}{t-i} \overline{W}_{i,k}$$

t-sets disjoint from k-set, on i-set

and so

$$W_{t,k} = \sum_i (-1)^i W_{i,b}^T \overline{W}_{i,k} = \left(\sum_i (-1)^i \binom{v-k-i}{t-i}^{-1} W_{i,t}^T W_{i,t}\right) \overline{W}_{t,k}$$

$$\Rightarrow \mathrm{row}(W_{t,k}) \leq \mathrm{row}(\overline{W}_{t,k})$$

$W_{t,v-t}$ is invertible

- $W_{t,v-t}$ & $\bar{W}_{t,v-t}$ are square, order $\binom{v}{t} \times \binom{v}{t}$
- $\bar{W}_{t,v-t}$ is a permutation matrix, thus invertible
- $\text{row}(W_{t,v-t}) = \text{row}(\bar{W}_{t,v-t})$

Independent rows

$$W_{t,h} W_{h,v-t} = \binom{v-2t}{h-t} W_{t,v-t}$$

Since RHS is invertible, rows of $W_{t,k}$ are linearly independent.

Null designs

If $f \neq 0$ and $W_{t,k}(f) = 0$ we call f a null (t,k)-design. If $W_{t,k}(v) f = 0$, we have:

$$\begin{pmatrix} W_{t-1,k-1}(v-1) & 0 \\ W_{t,k-1}(v-1) & W_{t,k}(v-1) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$

We call f_1 & f_2 the derived and residual designs of f

Lemma If f is a null (t,k)-design with derived design f_1 & residual design f_2, then

(a) f_1 is a null $(t-1,k-1)$-design.

(b) f_2 is a null $(t-1,k)$-design.

Why?

Proof. (a) is immediate. As f is also a null $(t-1,k)$-design

$$\underbrace{W_{t-1,k-1} f_1}_{=0} + W_{t-1,k} f_2 = 0$$

Hence $W_{t-1,k} f_2 = 0$. $\square$

Supports of null designs

$$\mathrm{supp}(f) = \mathrm{supp}(f_1) \cup \mathrm{supp}(f_2)$$

By induction

$$|\mathrm{supp}(f_1)| \geq 2^t, \quad |\mathrm{supp}(f_2)| \geq 2^t$$

$$\Rightarrow |\mathrm{supp}(f)| \geq 2^{t+1}.$$

Exercises:

1. Use Vandermonde identity to prove that

$$W_{t,k} W_{t,k}^T = \sum_{i=0}^{t} \binom{v-2t}{v-k-i} W_{i,t}^T W_{i,t}$$

2. Show that the RHS is positive definite and deduce that the rows of $W_{t,k}$ are linearly independent

3. Given that $\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}\begin{pmatrix} AB & 0 \\ B & BC \end{pmatrix}\begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -ABC \\ B & 0 \end{pmatrix}$,

$$rk \begin{pmatrix} AB & 0 \\ B & BC \end{pmatrix} = rk(B) + rk(ABC)$$

Starting from

$$W_{t,k}(v) = \begin{pmatrix} W_{t-1,k-1}(v-1) & 0 \\ W_{t,k}(v-1) & W_{t,k}(v-1) \end{pmatrix}$$

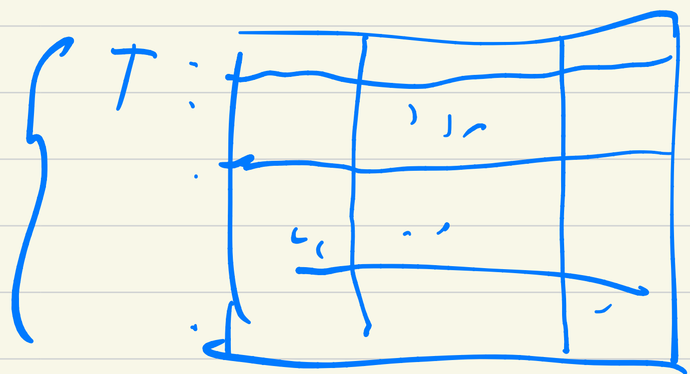
deduce that $\mathrm{rk}(W_{t,k}(v)) = \binom{v}{t}$

4. A set of k-subsets of a v-set is an antidesign if its characteristic vector lies in $\mathrm{row}(W_{t,k})$; the least possible value of t is its strength. Show that $\mathcal{D}$ is a t-design and $\mathcal{D}^*$ is an antidesign with strength at most t, then

$$|\mathcal{D}|\,|\mathcal{D}^*| \geq |\mathcal{D} \cap \mathcal{D}^*| \binom{v}{k}$$

5. Let V be the vector space of dimension v over $GF(q)$. Let $W_{t,k}(V)$ be the incidence matrix for t-subspaces of V versus k-subspaces. Prove that W has full rank.

$(t+1)$



$$z^T W \overset{?}{=} 0$$

$$z_1 \neq 0 \qquad P z_1$$