A diagonalizable $\Leftrightarrow$ χ_A has only simple zeros.

$$A = A^*$$

$$A^2 x = 0$$

$$A^* A x = 0$$

$$x^* A^* A x = 0$$

$Ax = 0$ $\Rightarrow$ no root vectors for 0 with index >1

$$(A - \lambda I)^2 x = 0 \Rightarrow (A - \lambda I) x = 0$$

Companion matrices & cyclic modules.

(we're working with $\mathbb{F}[A]$-modules)

A submodule N of an R-module is cyclic if it is generated by a single element.

i.e. simple

e.g.

(a) any minimal non-zero submodule.

(b) $A \in End(V)$, $v \in V$

$span\{A^r v : r \geq 0\}$. exercise: when is this simple?

$\langle v \rangle_A \quad \langle v \rangle_{F[A]}$

Problem: find cyclic modules with complements.

Assume $A \in \mathrm{End}(V)$, $v \in V$, $v \neq 0$

$U := \mathrm{span}\{v, Av, \dots\}$

Claim U has a basis of the form $v, Av, \dots, A^{d-1}v$, where d is $\deg(\psi_{A,v})$.

The matrix representing A relative to this basis is a companion matrix.

It is determined by ψ.

If $\{v, Av, \ldots, A^{d-1}v\}$ is linearly independent

and $\{v, Av, \ldots, A^{d-1}v, A^d v\}$ is not, then

$$-A^d v = a_1 A^{d-1} v + \cdots + a_d v$$

and

$$\psi(t) = t^d + a_1 t^{d-1} + \cdots + a_d$$

$Az = \lambda z$

$0 = \psi(A) z = \psi(\lambda) z$

$\Rightarrow \psi(\lambda) = 0$

The matrix C_ψ representing A is

$$C_\psi := \left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & -a_d \\ \hline 1 & 0 & \cdots & 0 & -a_{d-1} \\ 0 & 1 & \cdots & 0 & -a_{d-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -a_1 \end{array}\right]$$

companion matrix

I_{d-1}

If

$$W = [v \; Av \cdots A^{d-1}v],$$

then

$$AW = [Av \cdots A^{d-1}v \quad A^d v] = WC_\psi$$

$$AW = WB$$

$$\Updownarrow$$

$col(W)$ is A-inv

$$\left(\begin{array}{c|c} 0 & \vdots \\ \hline I & \vdots \end{array}\right)$$

Note:

(a) C_ψ is invertible $\Leftrightarrow a_d \neq 0$

(b) The eigenvalues of C are the zeros of ψ

(c) $\mathrm{rk}(C_\psi - \lambda I) \geq d-1$ — any eigenvalue has geometric multiplicity equal to one e.g. $A = A(P_4)$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\underbrace{v\ Av \ldots A^3v}_{W}\ A^4v] = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\,=3A^2v - v}$$

$$AW = W \overbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{C_\psi}, \qquad \psi = t^4 - 3t^2 + 1$$

We will see that any square matrix over a field is similar to a block diagonal matrix

$$\begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_k \end{bmatrix}$$

where the diagonal block are companion matrices.

Transposes

We will prove that there is a symmetric matrix Q s.t

$$C_\psi^T = Q^{-1} C_\psi Q$$

For this we need a second basis for $\langle v \rangle_A$.

Define polynomials $\psi_1, \dots, \psi_d$ by declaring $\psi_i(t)$ to be the polynomial part of $t^{-i}\psi(t)$.

So if $d=3$ & $\psi = t^3 + at^2 + bt + c$, then

$\psi_1 = t^2 + at + b$, $\psi_2 = t + a$, $\psi_3 = 1$

We have the recurrence

$$\psi_{i-1}(t) = t\psi_i + a_{d+1-i}$$

If

$$\mathcal{Y} := [\psi_1(A)v, \dots, \psi_d(A)v],$$ "control basis"

then

$$A\mathcal{Y} = \mathcal{Y} C_\psi^T$$

This shows that C_ψ & C_ψ^T are similar.

Define

$$W := [v, \dots, A^{d-1} v],$$

$$Q = \begin{bmatrix} a_{d-1} & a_{d-2} & \cdots & a_1 & 1 \\ a_{d-2} & a_{d-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and observe that

$$Y = WQ$$

$$C_\psi Q = Q C_\psi^T$$

$$\overset{C_\psi}{\begin{bmatrix} 0 & c & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}} \overset{Q}{\begin{bmatrix} a & b & a & 1 \\ b & a & 1 & 0 \\ a & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} -d & 0 & 0 & 0 \\ 0 & b & a & 1 \\ 0 & a & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overset{Q}{\begin{bmatrix} c & b & a & 1 \\ b & a & 1 & 0 \\ a & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \overset{C_\psi^T}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix}} = \begin{bmatrix} -d & 0 & 0 & 0 \\ 0 & b & a & 1 \\ 0 & a & 1 & 0 \\ d & 1 & 0 & 0 \end{bmatrix}$$

Now

$$A\mathcal{W} = \mathcal{W} C_\psi$$

and

$$A\mathcal{Y} = \mathcal{Y} C_\psi^T$$

$$\Rightarrow \quad A\overbrace{\mathcal{W}\Phi} = \overbrace{\mathcal{W}\Phi} C_\psi^T$$

$$A\mathcal{W} = \mathcal{W}.\,\Phi C_\psi^T \Phi^{-1}$$

Hence $\quad C_\psi = \Phi C_\psi^T \Phi^{-1}$ $\qquad$ $C_\psi \Phi$ is symmetric

Frobenius normal form

A matrix C is in Frobenius normal form if

(a) it is block diagonal, with diagonal blocks $C_1, \ldots, C_m$

(b) C_i is the companion matrix of a (monic) polynomial ψ_i

(c) For $i = 1, \ldots, m$ we have $\psi_{i+1} | \psi_i$.

divides

Remarks:

1) The Frobenius normal form is determined by the polynomials $\psi_1, \ldots, \psi_m$.

2) ψ_1 is the minimal polynomial of C.

3) We will prove that every square matrix is similar to a matrix in Frobenius normal form, and this form is unique.

We aim to prove that if $A \in \mathrm{End}(V)$ then V is a direct sum of cyclic A-modules.

Lemma If U is a non-zero cyclic A-module of dimension k, then:

(a) It is contained in a cyclic A-module subspace with a complement.

(b) If $\dim(U)$ is equal to the degree of ψ_A, then U has a complement.

minimal poly. of A

Proof. Assume $\dim \langle u \rangle_A = k$ and set

$$U = [u \ Au \cdots A^{k-1}u]$$

We construct an A^T-module of dimension $\ell \geq k$.

Find w (in $\mathbb{F}^n$) such that

$$w^T A^{k-i} u = \begin{cases} 1, & i=1 \\ 0, & o/w \end{cases}$$

?

and

$$\begin{bmatrix} w^T A^{k-1} \\ w^T A^{k-2} \\ \vdots \\ w^T \end{bmatrix} [u \ Au \cdots A^{k-1}u] = \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & \\ & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Hence $\dim \langle w \rangle_{A^T} \geq k$.

If $l > k$, swap A with A^T and construct a new cyclic A-module with dimension at least l.

Compute the corresponding cyclic A^T-module.

Eventually the A^T-module will have the same dimension as the A-module.

So $U = [u, \dots, A^{k-1}u]$, $W = [(A^T)^{k-1}w \cdots w]$

$W^T U$ is triangular, diagonal entries equal to 1,

hence W is invertible.

Claims:

(a) $\ker(W^T)$ is A-invariant

(b) $\mathrm{col}(U) \cap \ker(W^T) = \langle 0 \rangle$

$\ker(W^T)$ is our complement

(a) $A^T W = WL$ and so $W^T A = L^T W^T$. Hence if $W^T z = 0$ then $W^T A z = 0$

(b) $W^T U$ is invertible.

To complete the proof, suppose

$$\dim(\langle u \rangle_A) = \deg(\psi_A) = k$$

Since $\psi_A = \psi_{A^T}$, we get $k = \ell$ at step 1 and U has a complement.

Theorem Each matrix in $\mathrm{Mat}_{n\times n}(\mathbb{F})$ is similar to a matrix in Frobenius normal form.

~~Proof~~ Assume ψ_A has degree k. As we saw, there is a vector u such that $\psi_{A,u} = \psi$. and so $\dim(\langle u \rangle_A) = k$. There A is similar to a matrix

$$\begin{pmatrix} C & 0 \\ 0 & A_2 \end{pmatrix}$$

where $C = C_\psi$.

As

$$0 = \psi\begin{pmatrix} C & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} \psi(C) & 0 \\ 0 & \psi(A_2) \end{pmatrix}$$

the minimal polynomial of A_2 divides ψ.

Now induct. $\square$

4-10-21

Lemma If two matrices in Frobenius normal form are similar, they are equal.

Proof/ $\begin{pmatrix} L_1 & 0 \\ 0 & D_1 \end{pmatrix}$, $\begin{pmatrix} L_2 & 0 \\ 0 & D_2 \end{pmatrix}$ in FNF

L_1, L_2 companion $\Rightarrow L_1 = L_2$

$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ similar $\overset{?}{\Rightarrow}$ B & C similar

Let ψ_1 be minimal polynomial of D_1. Then

$$\begin{pmatrix} \psi_1(L) & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \psi_1(L) & 0 \\ 0 & \psi_1(D_2) \end{pmatrix}$$

are similar. So $\psi_1(D_2) = 0$.

Now assume that

$$\begin{bmatrix} L & 0 \\ 0 & D_1 \end{bmatrix}, \quad \begin{bmatrix} L & 0 \\ 0 & D_2 \end{bmatrix}$$

are in Frobenius normal form & are similar.

(L not necessarily a companion matrix.)

Use above argument to prove the leading block of D_1 & D_2 are equal....

If φ & ψ are polynomials,

$$\varphi \vee \psi := \operatorname{lcm}(\varphi, \psi), \qquad \varphi \wedge \psi := \gcd(\varphi, \psi)$$

Lemma The matrices

$$\begin{pmatrix} C_\varphi & 0 \\ 0 & C_\psi \end{pmatrix} \qquad \begin{pmatrix} C_{\varphi \vee \psi} & 0 \\ 0 & C_{\varphi \wedge \psi} \end{pmatrix}$$

are similar.

Question: if the matrices

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$$

are similar, are B & C similar?