Eigenvalues & Spectral Decomposition

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Inner product:

$$\langle u, v\rangle := u^* v$$

Adjoint of A:

$$\langle u, Av\rangle = \langle A^* u, v\rangle$$

A is self-adjoint if $A = A^*$ $\quad \langle u, Av\rangle = \langle Au, v\rangle$

Inner product:

$$\langle A, B\rangle = tr(A^* B) = sum(\bar{A} \circ B)$$

$$\|A\| = \langle A, A\rangle^{1/2}$$

$U^{\perp} := \{ v \in V : \langle v, u \rangle = 0 \ \forall u \text{ in } U \}$

$$\left.\begin{array}{ll} \text{(a)} & \dim(U^{\perp}) = \dim(V) - \dim(U) \\ \text{(b)} & U \cap U^{\perp} = \langle 0 \rangle, \quad V = U \oplus U^{\perp} \end{array}\right\} \Rightarrow (U^{\perp})^{\perp} = U$$

$$\boxed{\textbf{Lemma} \ \text{ If } A \text{ fixes } U, \text{ then } A^{*} \text{ fixes } U^{\perp}.}$$

Corollary If A is self-adjoint, any $\mathbb{C}[A]$-module has a complement.

If A is self-adjoint and A fixes U, the restriction of A to U is self-adjoint.

$\langle u, Au \rangle = \langle Au, u \rangle \quad \forall u$ in V

Once we've proved eigenvectors exist:

Corollary If A is $n \times n$ and self-adjoint, $\mathbb{C}^n$ is a direct sum of eigenspaces.

Existence of Eigenvectors

successive rank-1 approx $\to$ rank-k

$\mathbb{F} = \mathbb{R}$

Rank-1 approximations

symmetric rank-1 matrix : zz^*

$$\operatorname{tr}(zz^*) = z^*z = \|z\|^2$$

Problem 1 Given $A = A^*$ and z with norm one, find minimum of $\|A - \lambda zz^*\|$.

Inner product on matrices

Schur product

A, B $m \times n$ matrices

$\langle A, B \rangle = tr(A^* B) = sum(\bar{A} \circ B)$

$\langle A, A \rangle = \sum_{i,j} (\bar{A}_{i,j})^2$

Assume V is an inner product space over $\mathbb{C}$.

If $u, v \in V$ and $\langle v,v \rangle = 1$,

$$\min_{\lambda} \langle u-\lambda v, u-\lambda v \rangle = \langle u,u \rangle - \langle u,v \rangle \langle v,u \rangle$$

Proof/

$$\langle u-\lambda v , u-\lambda v \rangle = \langle u,u \rangle - \lambda \langle v,u \rangle - \bar{\lambda} \langle u,v \rangle + \lambda \bar{\lambda} \qquad (1)$$

RHS is equal to

$$\underbrace{\langle v,u \rangle \langle u,v \rangle - \lambda \langle v,u \rangle - \bar{\lambda} \langle u,v \rangle + \lambda \bar{\lambda}} + (\langle u,u \rangle - \langle u,v \rangle \langle v,u \rangle)$$

$$(\langle v,u \rangle - \bar{\lambda})(\langle u,v \rangle - \lambda)$$

Corollary. If $\|z\| = 1$, then

$$\min_{\lambda} \{ \| A - \lambda z z^* \|^2 \} = \langle A, A \rangle - \langle z z^*, A \rangle \langle A, z z^* \rangle$$

Problem Find $\max_{\|z\|=1} |z^* A z|^2$

characterize z that maximizes this expression

$Q(z) := z^* A z$

$$(z+h)^* A (z+h) = Q(z) + h^* A z + z^* A h + \cancel{h^* A h}$$

$h^* z = 0$, z optimal $\Rightarrow$ $h^* A z + z^* A h = 0$

$h \longmapsto ih$ $\qquad -h^* A z + z^* A h = 0$

So $h^* A z = 0$ if $h^* z = 0$

$\Rightarrow$ $z^\perp \subseteq (Az)^\perp$ $\Rightarrow$ $Az \in \langle z \rangle$

z is an eigenvector!

$Q(z)$ is a continuous function on the compact set $\{z : \|z\| = 1\}$, it realizes its maximum value

$\Rightarrow A$ has an eigenvector.

Spectral decomposition

1) $V = U \oplus U^\perp$ $\quad P$ projection on U

$$0 = \langle Pu, (I-P)v \rangle = \langle u, (P^* - P^*P)v \rangle \quad \forall u,v$$

$$\Rightarrow P^* = P^*P \Rightarrow P = (P^*P)^* = P^*P \Rightarrow P = P^*$$

2) o/n bases exist: choose u_1, choose u_2 in $\langle u_1 \rangle^\perp$, ...

3) $u_1, \dots, u_k$ o/n, $P = \sum_r u_r u_r^*$ is projection on $\operatorname{span}\{u_1, \dots, u_k\}$

$\hat{U} = [u_1, \dots, u_m]$, $\hat{U}^*\hat{U} = I_m$, $\quad P = \hat{U}\hat{U}^*$

Assume A is self-adjoint on $\mathbb{C}^n$,

with eigenspace decomposition

$$\mathbb{C}^n = U_1 \oplus \cdots \oplus U_k.$$

If the eigenvalue of A on U_r is θ_r and the projection on U_r is E_r,

$$A = \sum_r \theta_r E_r$$

$$A = LDL^T, \quad D_r^2 = D_r \ \text{diag}$$

$$= \sum_r \theta_r \underbrace{L D_r L^T}_{E_r}$$

If $u_1, \ldots, u_k$ is an orthonormal basis for U_r

then $E_r = \sum_i u_i u_i^T$ and $AE_r = \sigma_r E_r$.

Consequnences of spectral decomposition

1) $A^k = \sum_r \theta_r^k E_r$ $\qquad$ $E_r E_s = \delta_{r,s} E_r$

2) If f is defined on the spectrum of A,

$$f(A) = \sum_r f(\theta_r) E_r$$

$p_r(\theta_r) = 1$
$p_r(\theta_s) = 0$

(a) E_r is a polynomial in A

(b) $\boxed{(tI - A)^{-1} = \sum_r \frac{1}{t-\theta_r} E_r}$

(c) $\exp(tA) = \sum_r e^{t\theta_r} E_r$

06-10-21

Review of positive semidefinite matrices

A Hermitian matrix M is positive semidefinite $M \succeq 0$
if $z^* M z \geq 0 \ \forall z$. (Positive definite if psd and
$z^* M z = 0 \Rightarrow z = 0$.)

Theorem TFAE:

(a) $M \succeq 0$

matrix of inner products

(b) $M = N^* N$ for some N Cholesky decomposition

(c) $M = N^2$, for some Hermitian N,

(d) M is Hermitian and all eigenvalues of M are non-negative.

1) e.g. $N = [x, y]$ $\quad N^*N = \begin{bmatrix} x^*x & x^*y \\ y^*y & y^*y \end{bmatrix} \succeq 0$

Since $\det(N^*N) \geq 0$, we have $(x^*x)(y^*y) - (y^*x)(x^*y) \geq 0$

ie

$$\|x\|^2 \|y\|^2 \geq \langle y, x\rangle \langle x, y\rangle = |\langle x, y\rangle|^2$$

(Cauchy-Schwarz)

2) If $M \succeq 0$ and $x^*Mx = 0$, then $Mx = 0$

3) If $M \succeq 0$, any principal submatrix is psd.
(So $M_{ii} \geq 0$ for all i.)

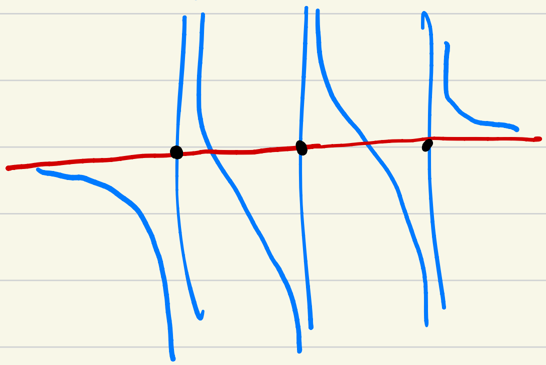
$M \circ I \succeq 0$

Interlacing

$$\frac{\phi(X \setminus i; t)}{\phi(X, t)} = \sum_r \frac{(E_r)_{i,i}}{t - \theta_r}$$

$\psi(t) \qquad \psi'(t) < 0 \quad \forall t$

$$\frac{\phi(X \setminus i; t)}{\phi(X, t)}$$



Walk generating functions

$(A^n)_{u,v}$ # uv-walks, length n

$\left(\sum_{n\geq 0} t^n A^n\right)_{uv}$ generating function

$= (I-tA)^{-1}_{u,v}$ rational function

$= \sum_r \frac{(E_r)_{uv}}{1-t\theta_r}$ partial fraction expansion

Cramer

$$\left((tI-A)^{-1}\right)_{u,u} = \frac{\phi(X \setminus u, t)}{\phi(X, t)}$$

$$(I-tA)^{-1}_{u,u} = \frac{t^{-1}\phi(X \setminus u, t^{-1})}{\phi(X, t^{-1})}$$

$$(tI-A)^{-1}_{u,u} = \sum_r \frac{(E_r)_{u,u}}{t-\theta_r}$$

$$\operatorname{tr}\left((tI-A)^{-1}\right) = \sum_r \frac{\operatorname{tr}(E_r)}{t-\theta_r} = \underbrace{\sum_r \frac{\operatorname{mult}(\theta_r)}{t-\theta_r}}_{\text{equal to?}}$$

Working with generating functions

If S is a set of walks from a graph and $|\sigma|$ denotes the length of the walk σ, the generating function for S is

$$W_S(t) := \sum_{\sigma \in S} t^{|\sigma|}$$

Operating rules?

I) If $S \cap T = \emptyset$, then

$$\boxed{W_{S \cup T}(t) = W_S(t) + W_T(t)}$$ SUM

II) If $|(\sigma,\tau)| = |\sigma| + |\tau|$, then

$$\boxed{W_{S \times T}(t) = W_S(t)\, W_T(t)}$$ PRODUCT

$\underline{\text{III}}$ If σ is a sequence $\sigma_1, \dots, \sigma_m$ of walks from S then $|\sigma| := \sum_i |\sigma_i|$. The set of all sequences is S^*. If S contains no walk of length 0,

$$\boxed{W_{S^*}(t) = \frac{1}{1 - W_S(t)}}$$

SEQUENCE

Choose a in $V(X)$. Set

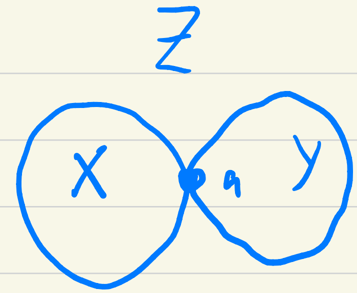
$$S = \{\text{closed walks starting at } a\}$$

$$S^{(1)} = \{\text{closed walks from } a, \text{ returning exactly once}\}$$

- $$W_S(t) = W_{a,a}(t) = (I-tA)^{-1}_{a,a} = \frac{t^{-1}\phi(X \setminus a, t^{-1})}{\phi(X \setminus a, t)}$$

- If $\sigma \in S^{(1)}$, $|\sigma| \geq 2$. Each walk in S is a sequence of walks from $S^{(1)}$.

Characteristic polynomial of 1-sum



$$W_{S(X)} = \frac{t^{-1}\phi(X \setminus a, t^{-1})}{\phi(X, t^{-1})} \; ; \quad W_{S(Y)} = \frac{t^{-1}\phi(Y \setminus a, t)}{\phi(Y, t^{-1})}$$

$$W_{S^{(1)}(Z)} = W_{S^{(1)}}(X) + W_{S^{(1)}}(Y)$$

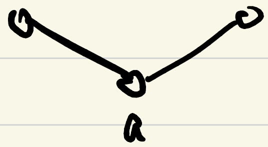
$$1 - W_{S(Z)} = 1 - W_{S(X)} + 1 - W_{S(Y)}$$

$$W_{S^{(1)}} = 1 - W_S^{-1}$$

Theorem $\phi(Z,t) = \phi(X \setminus a, t)\,\phi(Y,t) + \phi(X,t)\,\phi(Y \setminus a, t) - t\,\phi(X \setminus a, t)\,\phi(Y \setminus a, t)$

Or: $$\frac{\phi(Z)}{\phi(Z \setminus a)} = \frac{\phi(Y)}{\phi(Y \setminus a)} + \frac{\phi(X)}{\phi(X \setminus a)} - 1$$

Z



$X = Y = P_2 \qquad \phi(X,t) = t^2 - 1$

$$\phi(Z) = (t^2-1)t + t(t^2-1) - t^3 = t^3 - 2t$$