Tensors

Assume A is $k \times l$ & B is $m \times n$. The **Kronecker product** $A \otimes B$ is the $km \times ln$ block matrix with ij-block equal to $A_{i,j} B$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} a & 2a \\ b & 2b \\ c & 2c \\ 3a & 4a \\ 3b & 4a \\ 3c & 4c \end{bmatrix}$$

Properties

1) $A \otimes B$ is linear in each term

2) If the required products exist

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D)$$

3) If $Ax = \lambda x$, $By = \mu y$, then $(A \otimes B)(x \otimes y) = \lambda\mu (x \otimes y)$

4) If $P : u \otimes v \to v \otimes u \ (\forall v, u)$ then

$P(M \otimes M^T)$ is symmetric.

A map $\mathrm{Mat}_{m\times n}(\mathbb{R}) \to \mathbb{R}^{mn}$

If $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, then

$$\mathrm{vec}(A) = \begin{bmatrix} Ae_1 \\ \vdots \\ Ae_n \end{bmatrix}$$

This is clearly linear.

Lemma $\mathrm{vec}(BMA^T) = (A \otimes B)\,\mathrm{vec}(M)$

Tensor product of modules

M, N modules over ring R.

Construct module $M \otimes N$ as quotient of finitely supported functions in $\mathbb{F}^{M \times N}$ over ideal generated by relations:

$(x, y+z) - (x, y) - (x, z)$, $(w+x, y) - (w, y) - (x, y)$

$(ax, y) - a(x, y)$, $(x, by) - b(x, y)$

Image of (x, y) in quotient is $x \otimes y$

The Kronecker product is the tensor product on vector spaces

Bilinear maps

The tensor product turns questions about bilinear maps on $M \times N$ to questions about linear maps.

Moral: complicate the objects, keep the maps simple:

$$\begin{tikzcd} M \otimes N \arrow[drr, "\hat{\beta}"] & & \\ M \times N \arrow[u] \arrow[rr, "\beta"] & & L \end{tikzcd}$$

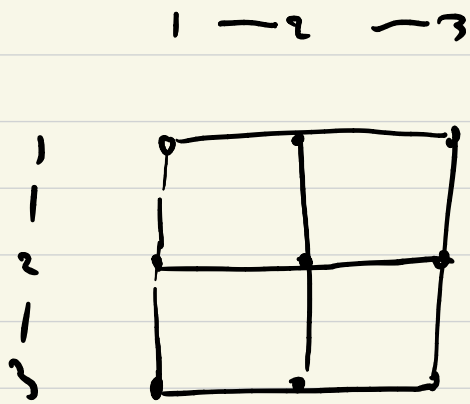
Graph products

Cartesian product $X \square Y$

$$A(X \square Y) = A(X) \otimes I + I \otimes A(Y)$$

$\mathrm{dist}((u,v),(x,y))$

$= \mathrm{dist}(u,x) + \mathrm{dist}(v,y)$

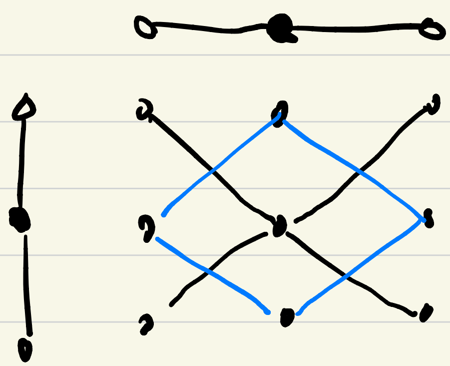


eigenvalues $\lambda_r + \mu_s \quad \forall r, s$

Direct product $X \times Y$

$$A(X \times Y) = A(X) \otimes A(Y)$$

(a)

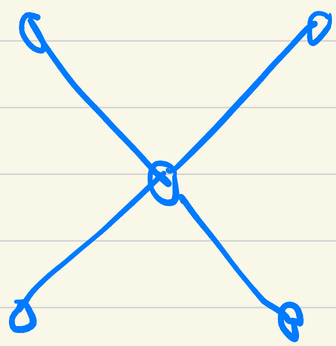


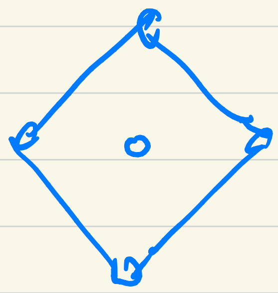
eigenvalues $\lambda_r \mu_s \ \forall r, s$

· X, Y connected
$X \times Y$ connected
unless X & Y both
bipartite

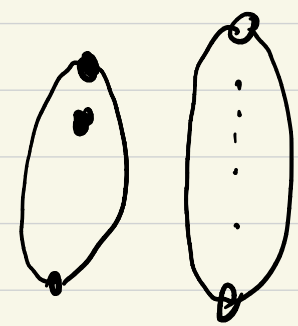
(b) $K_2 \otimes 2K_3 \cong 2C_6 \cong K_2 \otimes C_6$

Cospectral graphs I





What is the spectrum of $K_{m,n}$?



Components of tensor product

$$\begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & C^T \\ C & 0 \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & B^T \otimes C^T \\ 0 & 0 & B^T \otimes C & 0 \\ \hline 0 & B \otimes C^T & 0 & 0 \\ B \otimes C & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 0 & B^T \otimes C & 0 & 0 \\ B \otimes C^T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & B^T \otimes C^T \\ 0 & 0 & B \otimes C & 0 \end{array}\right]$$

$$A(P_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & B^T \otimes B^T \\ B \otimes B & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & B^T \otimes B \\ B \otimes B^T & 0 \end{pmatrix}$$

C $\qquad$ D

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ 1 & & 0 & & \\ 1 & & & & \\ 1 & & & & \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$C^2 = \begin{pmatrix} B^TB \otimes B^TB & 0 \\ 0 & BB^T \otimes BB^T \end{pmatrix}, \quad D^2 = \begin{bmatrix} B^TB \otimes BB^T & 0 \\ 0 & BB^T \otimes B^TB \end{bmatrix}$$

Theorem If A is $m\times n$ & B is $n\times m$,

$$\det(I-AB) = \det(I-BA)$$

Proof.

$$\begin{pmatrix} I & 0 \\ -B & I \end{pmatrix}\begin{pmatrix} I & A \\ B & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I-AB \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ B & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} = \begin{pmatrix} I-AB & A \\ 0 & I \end{pmatrix}$$

Now take determinants. $\square$

Corellary $t^{n-m} \det(tI - AB) = \det(tI - BA)$

If A is $m \times n$ & B is $n \times m$, then

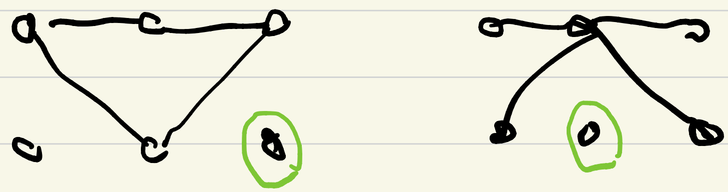
Corellary If A is $m \times n$ and B is $n \times m$, then

$$AB, BA$$

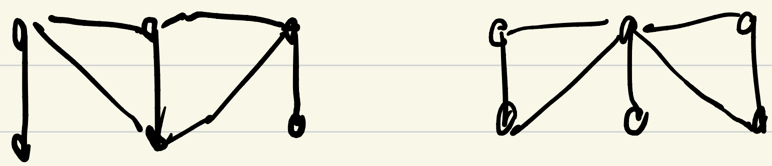
have the same non-zero eigenvalues, with the same multiplicities.

Lemma If X and Y are bipartite graph, the two components ? of $X \times Y$ have the same non-zero eigenvalues, with the same multiplicities

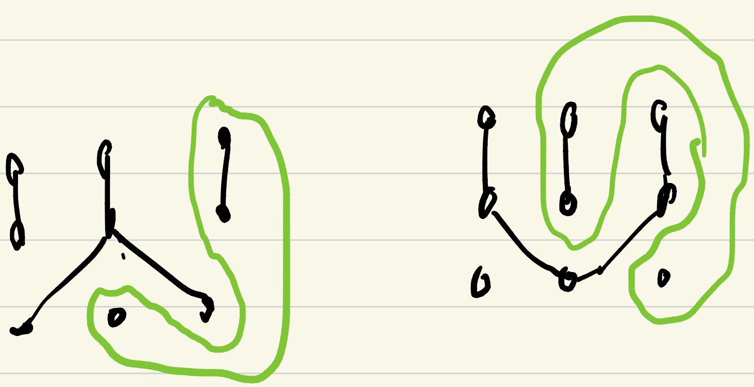
Partitioned tensor product



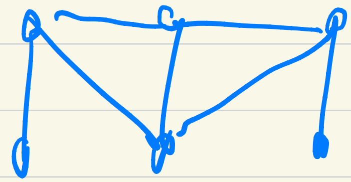
smallest pair of cospectral graphs

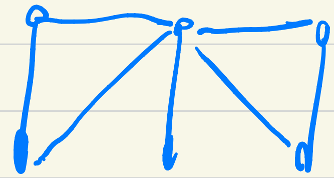


smallest pair of connected cospectral graphs



smallest pair of cospectral forests





$$\begin{pmatrix} A_1 & H \\ H^T & A_2 \end{pmatrix} \qquad \begin{pmatrix} A_2 & H^T \\ H & A_1 \end{pmatrix}$$

$$\begin{bmatrix} I & B \\ B^T & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} I & B \\ B^T & I \end{pmatrix} \times \begin{pmatrix} A_1 & H \\ H^T & A_2 \end{pmatrix} = \begin{pmatrix} I \otimes A_1 & B \otimes H \\ B^T \otimes H^T & I \otimes A_2 \end{pmatrix}$$

partitioned tensor product

$$\begin{pmatrix} I & B \\ B^T & I \end{pmatrix} \times \begin{pmatrix} A_2 & H^T \\ H & A_1 \end{pmatrix} = \begin{pmatrix} I \otimes A_2 & B \otimes H^T \\ B^T \otimes H & I \otimes A_1 \end{pmatrix}$$

Choose Q & R so $Q^TQ = I$, $R^TR = I$ and

$$Q^T B R = \Sigma$$ – diagonal

How?

$$\begin{pmatrix} Q^T\otimes I & 0 \\ 0 & R^T\otimes I \end{pmatrix} \begin{pmatrix} I\otimes A_1 & B\otimes H \\ B^T\otimes H^T & I\otimes A_2 \end{pmatrix} \begin{pmatrix} Q\otimes I & 0 \\ 0 & R\otimes I \end{pmatrix}$$

$$= \begin{pmatrix} I\otimes A_1 & \Sigma\otimes H \\ \Sigma\otimes H^T & I\otimes A_2 \end{pmatrix}$$

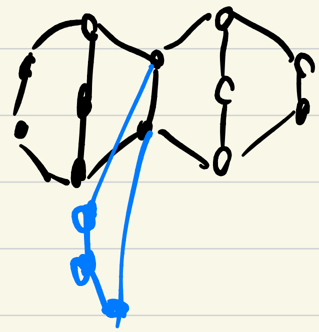
e.g. B is 2×3:

$$\begin{pmatrix} I\otimes A_1 & B\otimes H \\ B^T\otimes H^T & I\otimes A_2 \end{pmatrix} = \left[\begin{array}{cc|ccc} A_1 & 0 & \sigma_1 H & 0 & 0 \\ 0 & A_1 & 0 & \sigma_2 H & 0 \\ \hline \sigma_1 H^T & 0 & A_2 & 0 & 0 \\ 0 & \sigma_2 H^T & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 & A_2 \end{array}\right] \sim \left[\begin{array}{cc|cc|c} A_1 & \sigma_1 H & 0 & 0 & 0 \\ \sigma_1 H^T & A_2 & 0 & 0 & 0 \\ \hline 0 & 0 & A_1 & \sigma_2 H & 0 \\ 0 & 0 & \sigma_2 H^T & A_2 & 0 \\ \hline 0 & 0 & 0 & 0 & A_2 \end{array}\right]$$

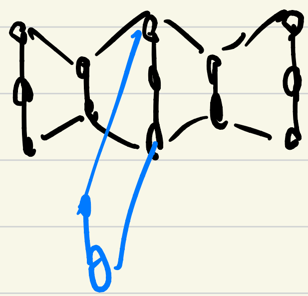
$$\begin{pmatrix} I\otimes A_2 & B\otimes H^T \\ B^T\otimes H & I\otimes A_1 \end{pmatrix} = \begin{pmatrix} A_2 & 0 & \sigma_1 H^T & 0 & 0 \\ 0 & A_2 & 0 & \sigma_2 H^T & 0 \\ \sigma_1 H & 0 & A_1 & 0 & 0 \\ 0 & \sigma_2 H & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & A_1 \end{pmatrix} \sim \left[\begin{array}{cc|cc|c} A_2 & \sigma_1 H^T & 0 & 0 & 0 \\ \sigma_1 H & A_1 & 0 & 0 & 0 \\ \hline 0 & 0 & A_1 & \sigma_2 H^T & 0 \\ 0 & 0 & \sigma_2 H & A_2 & 0 \\ \hline 0 & 0 & 0 & 0 & A_1 \end{array}\right]$$

$$\begin{bmatrix} A_1 & H \\ H^T & A_2 \end{bmatrix} \sim \begin{bmatrix} A_2 & H^T \\ H & A_1 \end{bmatrix}$$

(proof due to Knuth)



$$\begin{tikzcd} G \arrow[r] & H & G \arrow[l] \arrow[r] \arrow[d] & H & G \arrow[l] \\ & & H & & \end{tikzcd}$$



$$\begin{tikzcd} H & G \arrow[l] \arrow[r] & H & G \arrow[l] \arrow[r] & H \\ & & G \arrow[u] & & \end{tikzcd}$$

Singular values

Problem Find the best rank-1 approximation to B.

Assume $\|y\| = \|z\| = 1$. Then

$$\langle B - \lambda y z^T, B - \lambda y z^T \rangle = \langle B, B \rangle - 2\, y^T B z + \lambda^2$$

Minimum occurs when $\lambda = y^T B z$,

value is $\langle B, B \rangle - (y^T B z)^2$.

We need to maximize $(y^T B z)^2$

Suppose $h \in y^\perp$, $\|h\|$ small

$$(y^T B z + h^T B z)^2 \approx (y^T B z)^2 + (y^T B z)(h^T B z)$$

Hence if y maximizes, $h^T B z = 0$ and $h \in (Bz)^\perp$.

So $y^\perp \in (Bz)^\perp$ & $Bz \in \langle y \rangle$

Therefore if yz^T is optimal

$$Bz = \lambda y, \qquad B^T y = \mu z \qquad \Rightarrow y \in \mathrm{col}(B)$$

$$z \in \mathrm{col}(B^T)$$

and

$$\lambda \mu z = \lambda B^T y = B^T B z$$

$$\mu \lambda y = \mu B z = B B^T y$$

Also

$$\lambda = y^T B z ; \qquad \mu = z^T B y \Rightarrow \lambda = \mu$$

Traditionally we set σ equal to λ, μ

Theorem Assume B is $m \times n$ & $rk(B) = k$. Let $\sigma_1 \geq \cdots \geq \sigma_k$ be the square roots of the eigenvalues of BB^T. Then there is an orthonormal basis $y_1, \ldots, y_k$ of $col(B)$ and an orthonormal basis $z_1, \ldots, z_r$ of $col(B^T)$ such that

$$B = \sum_{r=1}^{k} \sigma_r y_r z_r^T$$

$Q := [y_1, \dots, y_k]$, $R := [z_1, \dots, z_k]$, $\Sigma := \mathrm{diag}(\sigma_i)$

$$\boxed{\underbrace{B}_{m \times n} = \underbrace{Q}_{m \times k} \Sigma \underbrace{R^T}_{k \times n}} \qquad Q^T Q = I, \quad R^T R = I$$

singular value decomposition

(So $Q^T B R = \Sigma$.)