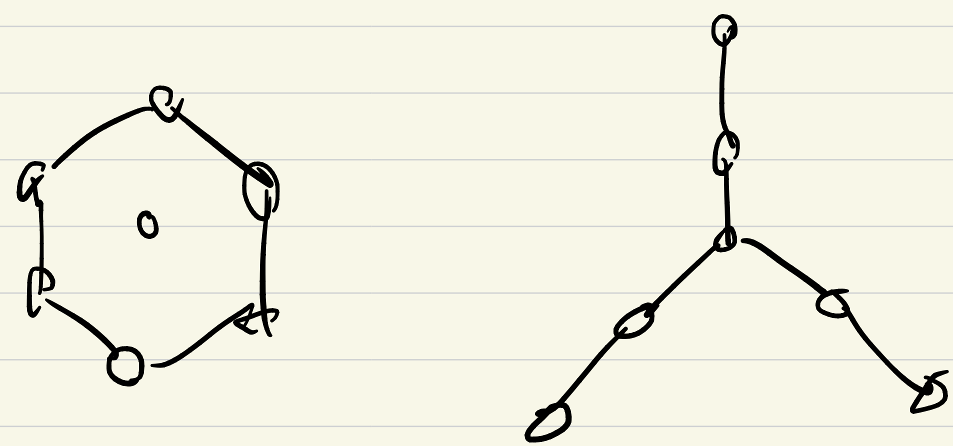
Cospectral Graphs:

Local switching

The smallest pair of cospectral graphs with cospectral complements:



Line graphs

vertex-edge incidence matrix B

(order $|V(X)| \times |E(X)|$).

$$B^T B = 2I + A(L(X))$$

$$BB^T = \Delta + A$$

(unsigned Laplacian)

$$X = K_{1,3} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

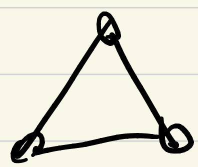
$$B^T B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad BB^T = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

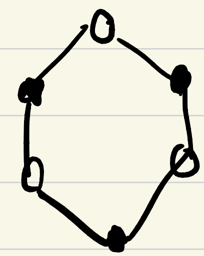
$$X = K_3 \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

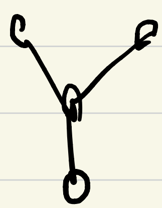
$$B^T B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad BB^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

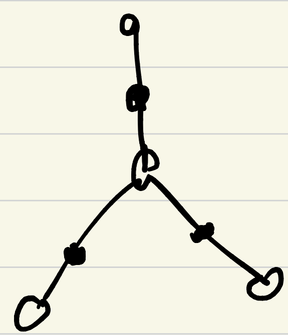
Subdivision graph

$$A(S(X)) = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$$

X 

$S(X)$ 



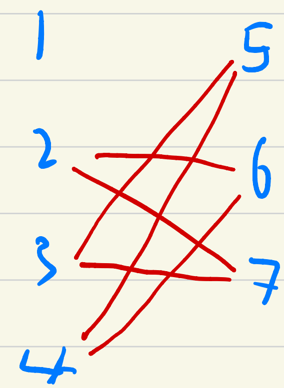


$$A^2 = \begin{bmatrix} B^T B & 0 \\ 0 & BB^T \end{bmatrix} = \begin{bmatrix} 2I + A(L) & 0 \\ 0 & \Delta + A \end{bmatrix}$$

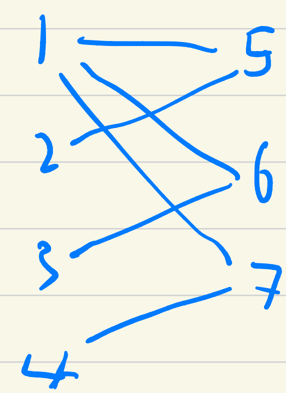
$\Rightarrow$ $C_6 \cup K_1$ & $S(K_{1,3})$ are cospectral

Complements?

$$A(S(K_3) \cup K_1) = \left[\begin{array}{c|c} 0 & \begin{matrix}0&0&0\\0&1&1\\1&0&1\\1&1&0\end{matrix} \\ \hline \begin{matrix}0&0&1&1\\0&1&0&1\\0&1&1&0\end{matrix} & 0 \end{array}\right]$$



$$A(S(K_{1,3})) = \left[\begin{array}{c|c} 0 & \begin{matrix}1&1&1\\1&0&0\\0&1&0\\0&0&1\end{matrix} \\ \hline \begin{matrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{matrix} & 0 \end{array}\right]$$



Set $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\quad Q = \frac{1}{2} J_{4,4} - I \quad (Q^2 = I)$

$$QB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J_{4,3} - B$$

Our subdivision graphs are

$$\begin{bmatrix} O & B \\ B^T & O \end{bmatrix} \qquad \begin{bmatrix} O & J-B \\ J-B^T & O \end{bmatrix}$$

with complements

$$\begin{bmatrix} J_4-I & J-B \\ (J-B)^T & J_3-I \end{bmatrix} \qquad \begin{bmatrix} J_4-I & B \\ B^T & J_3-I \end{bmatrix}$$

$Q(J_4) = J_4$

$Q(J_4-I)Q^T = J_4-I$

Extensions

Choose 01-matrix B, order $2k \times m$ such that each column is $\underset{\sim}{0}$, $\underset{\sim}{1}$ or has exactly half its entries zero

If $Q = \frac{1}{k} J_{2k} - I$ and $x \in \mathbb{R}^{2k}$ has $\mathbf{1}^T x = k$, then

$$Qx = \frac{1}{k} \underset{\sim}{1} \underset{\sim}{1}^T x - x = \underset{\sim}{1} - \underset{\sim}{x}$$

$(Q\underset{\sim}{0} = 0, \; Q\underset{\sim}{1} = \underset{\sim}{1})$

$B_0 \quad B_1 \qquad B_1 \; J-B_1$

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix}\begin{pmatrix} Q^T & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} QA_1Q^T & QB \\ B^TQ^T & 1 \end{pmatrix}$$

So if $QA_1 = A_1Q$ (X, regular)

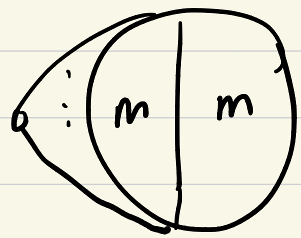
$$\begin{pmatrix} A_1 & QB \\ B^TQ & A_2 \end{pmatrix}, \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix}$$ are cocospectral

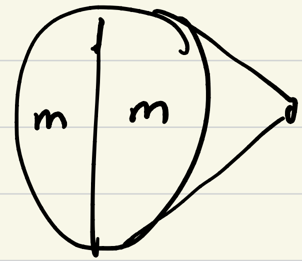
Alternatively:

Given graph X, subset S of $V(X)$ ($|S|$ even). If each vertex not in S is adjacent to all, none, or exactly half the vertices in X.

Examples

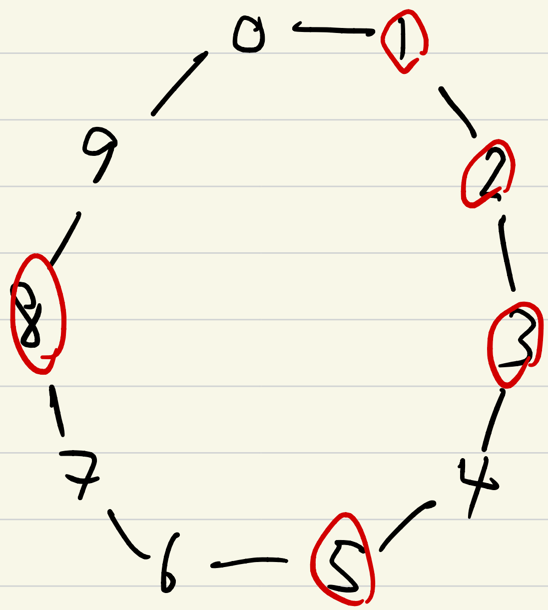
(a) X regular on $2m$ vxs





$$\begin{bmatrix} A & b \\ b^T & 0 \end{bmatrix} \qquad \begin{bmatrix} A & 1-b \\ (1-b)^T & 0 \end{bmatrix}$$

C_{10}



Latin square graphs:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

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011
101
110
$\vdots$

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010
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