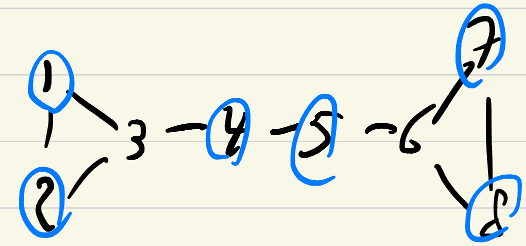
Equitable partitions

partition π (of $V(X)$)



characteristic matrix P

$\pi = \{\{1,2,4,5,7,8\}, \{3,6\}\}$

$\mathcal{A}$: algebra of operators on $V(X)$

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

We say π is equitable relative to $\mathcal{A}$ if $\mathrm{col}(P)$ is $\mathcal{A}$-invariant.

example:

$\mathcal{A}$ = adjacency algebra = $\langle A \rangle$

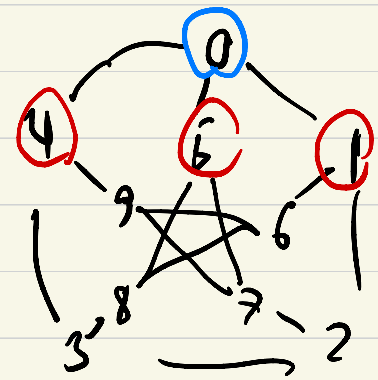
partition π, char. matrix P

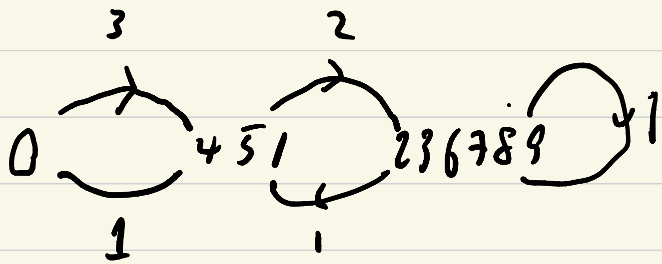
π equitable $\Leftrightarrow$ $\mathrm{col}(P)$ is A-invariant.

$\Leftrightarrow$ there are constants $B_{i,j}$ such that each vx in π_i has exactly $B_{i,j}$ neighbours in π_j

Lemma If $G \leq \mathrm{Aut}(X)$, the orbits of G form an equitable partition.

$B = (B_{i,j})$ is the adjacency matrix of the quotient graph X/π.





$$B = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Lemma $AP = PB$

Corollary $\phi(B,t) \mid \phi(A,t)$

Proof. There exists Q, order $n \times (n-|\pi|)$, such that the columns of $[P\ Q]$ are orthogonal. Hence $[P\ Q]^T[P\ Q]$ is diagonal. As $\mathrm{col}(Q) = \mathrm{col}(P)^\perp$ we see that $AQ = QC$ and

$$A[P\ Q] = [P\ Q]\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}.$$

Therefore A & $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ are similar.

If $Bz = \theta z$, then

$$\theta Pz = PBz = APz$$

and so Pz is an eigenvector for A constant on the cells of π.

If $y^T A = \lambda y^T$, then

$$y^T PB = y^T AP = \lambda y^T P$$

and $y^T P$ is a left eigenvector for B.
(Its entries are the sums of the entries of y over the cells of π.)

Symmetrizing

Set $D = P^T P$. Then $D^{-1/2} P^T P D^{-1/2} = I$

and thus the columns of $D^{-1/2} P$ are orthonormal.

We have

$$A P D^{-1/2} = P B D^{-1/2} = P D^{-1/2} \cdot \underbrace{D^{1/2} B D^{-1/2}}_{\hat{B}}$$

$\hat{B}$ — normalized char matrix

As $A\hat{P} = \hat{P}\hat{B}$ & $\hat{P}^T\hat{P} = I$, we see that

$$\hat{B} = \hat{P}^T A \hat{P}$$

is symmetric. Consequently $\hat{B}\hat{P}^T = \hat{P}^T A$ and

$$A\hat{P}\hat{P}^T = \hat{P}\hat{B}\hat{P}^T = \hat{P}\hat{P}^T A$$

Therefore A & $\hat{P}\hat{P}^T$ commute

orthogonal projection onto col (P)

Function spaces

If P is the char. matrix of π, then the vectors in $\text{col}(P)$ are functions on $V(X)$ constant on cells of π. This space of functions is closed under multiplication; we denote it by $F(\pi)$.

Lemma If $\mathcal{F}$ is a vector space of functions on $V(X)$ then $\mathcal{F} = F(\pi)$ for some partition π if & only if $\mathcal{F}$ is closed under multiplication and contains $\underline{1}$

Lemma If $F(\pi)$ & $F(\sigma)$ are A-invariant, so are

$$F(\pi) \cap F(\sigma), \qquad F(\pi) + F(\sigma)$$

Directed graphs

We have two choices,(1) If π is a partition of $V(D)$, then $F(\pi)$ can be invariant under

(a) $\langle A \rangle$ — the adjacency algebra

(b) $\langle A, A^T \rangle$

The decision is up to you. Note that orbit partitions are equitable relative to $\langle A, A^T \rangle$

(1) at least two

For any partition π, we see that $f(\pi)$ is J-invariant, so we might as well add it to our algebra.

Laplacians

$$Q = \begin{bmatrix} \frac{1}{m_1} J_{m_1} & & \\ & \ddots & \\ & & \frac{1}{m_k} J_k \end{bmatrix}$$

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\Rightarrow J_1 A_{12} = A_{12} J_2$$

$A = (A_{ij})$ $k \times k$ blocks

$$\frac{1}{m_i} J_{m_i} \cdot A_{ij} = A_{ij} \cdot \frac{1}{m_j} J_{m_j}$$

row & column sums of each block are constant.

Since $LJ = JL = 0$, the partition with one cell ($= V(X)$) is equitable.

"almost equitable" partition

D. Cardoso

Distance partitions

Assume X is connected. If $S \subseteq V(X)$, we have the distance partition $\partial_S(X)$. Its i-th cell is the set of vertices at distance i from S.

The maximum distance of a vertex from S is the covering radius of S.

The subset S is completely regular if $\partial_S(X)$ is equitable.

Lemma If S is completely regular, the vertices at maximum distance from S form a completely regular subset. $\square$

Theorem A graph X is distance-regular if it is regular and each vertex is a completely regular subset.

Godsil & Shawe-Taylor