Continuous quantum walks

A state is a positive semidefinite matrix with trace 1, e.g., $e_u e_u^T$

If X is a graph

$$U(t) = \exp(itA)$$

determines a continuous quantum walk. If the initial state is D, then the state $D(t)$ at time t is given by

$$D(t) := U(t) D U(-t)$$

If $A = \sum_r \theta_r E_r$, then

$$D(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s$$

Then

$$\boxed{E_k D(t) E_\ell = e^{it(\theta_k - \theta_\ell)} E_k D E_\ell}$$

If $D = e_a e_a^T$ & $D(t) = e_b e_b^T$, we have perfect state transfer from a to b at time t.

$D_u := e_u e_u^T$

vertex state

If we have ab-pst at time t then,

since D_a and D_b and the idempotents E_r are real, therefore $e^{it(\theta_k - \theta_l)}$ is real, for all k, l such that $E_r D_a E_s \neq 0$.

$\Rightarrow \pm 1$

Now
$$E_r D_a E_s = E_r e_a (E_s a)^T \neq 0$$
if and only if neither $E_r e_a$ nor $E_s e_a$ are zero.

i.e., θ_r & θ_s lie in the eigenvalue support of a (which equals esupp(b)).

If $E_k e_a e_a^T E_k = E_k e_b e_b^T E_k$ then

(a) $E_k e_a$ & $E_k e_b$ are parallel column spaces

(b) $(E_k)_{a,a} = (E_k)_{b,b}$ (traces)

We say vertices a & b are strongly cospectral if they are cospectral and parallel.

If D is a state, define

$$\hat{D} := \sum_r E_r D E_r$$

Note that $\hat{D}_1 = \hat{D}_2$ if & only if

$$E_r D_1 E_r = E_r D_2 E_r, \quad \forall r$$

We observe that

$$(E_r D_a E_r)^2 = E_r e_a \underline{e_a^T E_r E_r e_a} e_a^T E = \underline{(E_r)_{a,a}} E_r D_a E_r$$

and $\hat{D}_a$ has spectral decomposition

$$\hat{D}_a = \sum_r (E_r)_{a,a} \underbrace{\frac{1}{(E_r)_{a,a}} E_r D_a E_r}_{\text{idempotent}}$$

Theorem $\hat{D}$ is the orthogonal projection of D onto the commutant of A.