Perturbing matrices

References

Kato: Perturbation Theory for Linear Operators. (Springer, 1980)

Lancaster & Tismenetsky: The Theory of Matrices (Academic Press, 1985 2nd Edition)

A matrix pencil is the set

$$A + tH, \quad t \in \mathbb{R}.$$

It is Hermitian if A & H are.

We want to understand the behaviour of the eigenvalues of $A+tH$ as t varies.

Why?

Kato (1980) pp 121-122 **Theorem** There is an orthonormal basis of eigenvectors $z_1(\eta), \dots, z_n(\eta)$ for $A + \eta H$ such that each vector is a holomorphic function of η.

Corollary For the eigenvalues $\theta_r(\eta)$, likewise, Kato (1980) p64

Theorem Let A and H be $n \times n$ Hermitian matrices. Then there is an integer m such that, for all but finitely many values of t, the matrix $A + tH$ has exactly m distinct eigenvalues. $\square$

We refer to the "finitely many values of t" as the exceptional points of the pencil.


ϑ_4
ϑ_3
ϑ_2
ϑ_1
$\rightarrow \eta$

example: rank

(a) small changes

(b) $\det(A+\eta H)$

We distinguish three cases for pencils

(a) $AH = HA$

(b) $\mathrm{rk}(H) = 1$

(c) everything else

(a) $AH = HA$

$\mathbb{F} = \mathbb{C}$ (or $\mathbb{R}$)

A matrix A is normal if $AA^* = A^*A$

e.g. any Hermitian matrix; any unitary matrix

If L is unitary, D is diagonal & $A = L^*DL$, then

$$AA^* = L^*DL \cdot L^*\bar{D}L = L^*D\bar{D}L = L^*\bar{D}DL = A^*A$$

So any matrix that is unitarily diagonalizable is normal.

Theorem If A & B are Hermitian and commute, they can be simultaneously, unitarily, diagonalized

Proof. Assume λ is an eigenvalue of A and $U = \ker(A - \lambda I)$. Then

$$0 = B0 = B(A-\lambda I)U = (A-\lambda I)BU$$

and therefore U is B-invariant. $\square$

Theorem If A is normal, it is unitarily diagonalizable.

Proof) The matrices $B = \frac{1}{2}(A + A^*)$ & $C = \frac{i}{2}(-A + A^*)$ are Hermitian. Since A is normal, they commute and

$$A = B + iC, \quad A^* = B - iC$$

Since B & C can be simultaneously unitarily diagonalized, A can be unitarily diagonalized. $\square$

Exer: A normal $\Leftrightarrow$ $\langle Ax, Ax \rangle = \langle A^*x, A^*x \rangle$ $\forall x$

Cor. If x is evec for A, its an eigenvector for A^*

Perturbation: $AH = HA$

We can simultaneously diagonalize A & H.

If $\theta_1, \dots, \theta_n$ and $\lambda_1, \dots, \lambda_m$ are respectively the eigenvalues of A & H, eigenvalues of $A + \eta H$ are of the form

$$\theta_r + \eta \lambda_s, \quad \text{multiplicity } \operatorname{rk}(E_r F_s)$$

(b) $\mathrm{rk}(H) = 1$

rank-1 updates

$$H = hh^*$$

$$(A + \eta hh^*) z(\eta) = \theta(\eta) z(\eta)$$

differentiate

$\rightarrow$ $$hh^* z + (A + \eta hh^*) z' = \theta' z + \theta z'$$

$$(hh^* - \theta') z = (\theta I - A - \eta hh^*) z'$$

$$z^* (hh^* - \theta') z = 0$$

$$\Rightarrow \theta' = |\langle h, z\rangle|^2 \geq 0$$

$$\langle h, z\rangle^2 = tr(E_r hh^T E_r)$$

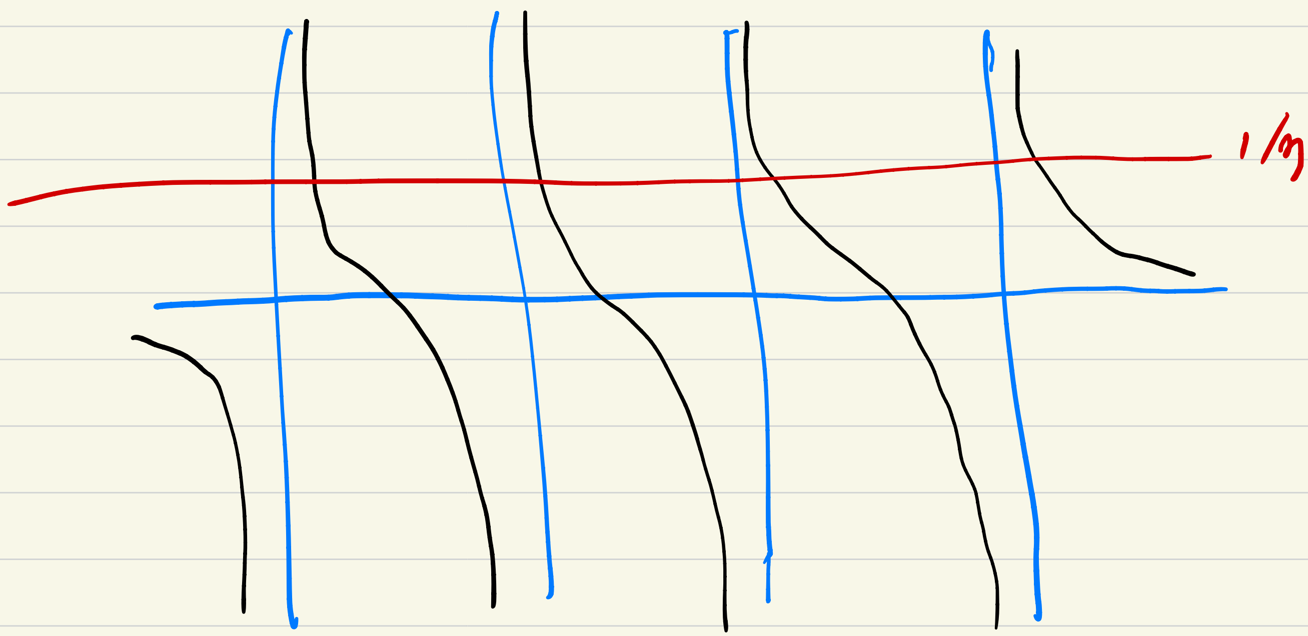
So $\theta' = \langle E_r, H\rangle$; $\theta' = 0$ if θ_r not in esupp(h)

$= (E_r)_{a,a}$ where $H = e_a e_a^T$

$$\det(tI - A - \eta H) = \det(tI - A)\det\left(I - (tI-A)^{-1} h h^*\right)$$

$$= \det(tI-A)\,\det\left(1 - h^*(tI-A)^{-1} h\right)$$

$$\frac{\phi(A+\eta H, t)}{\phi(A, t)} = 1 - \eta \sum_r \frac{h^* E_r h}{t - \theta_r}$$



General case:

$$(A+\eta H)E(\eta) = \theta(\eta)E(\eta)$$

$$HE(\eta) + (A+\eta H)E'(\eta) = \theta'(\eta)E(\eta) + \theta(\eta)E'(\eta)$$

$$(H-\theta'(\eta)I)E(\eta) = (\theta(\eta)I - A - \eta H)E'(\eta)$$

Now $E(\eta)(\theta(\eta)I - A - \eta H) = 0$, so

$$E(\eta)(H-\theta'(\eta)I)E(\eta) = 0$$

$$E(\eta)HE(\eta) - \theta'(\eta)E(\eta) = 0$$

$$uu^*Huu^*u \qquad u^*Hu\cancel{u^*u}$$

$$\overbrace{E(\eta) = U(\eta)}^{n\times m} U(\eta)^*, \quad U(\eta)^* U(\eta) = I_m$$

Hence

$$\boxed{U(\eta)^* H U(\eta) - \theta'(\eta) I = 0}$$

and therefore θ' is an eigenvalue of $U(\eta)^* H U(\eta)$.

The matrices

$$U^* H U, \quad H U U^* = HE = HE^2, \quad EHE$$

have the same non-zero eigenvalues, same mults.

So it's the eigenvalues of

$$\hat{H} = \sum_r{}' E_r H E_r$$

that determine the derivatives θ_r'.

Adding edges

We consider pencils where $H = e_i e_j^T + e_j e_i^T$,

i.e.,

$$H = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right]$$

So we want the eigenvalues of $E_r H E_r$.

The matrices $E_r H E_r$ & $H E_r^2 = H E_r$ has the same nonzero eigenvalues, with the same multiplicities.

If $E_r = (E_{i,j})_{i,j=1}^{n}$, then

$$HE_r = \left[\begin{array}{cc|ccc} E_{21} & E_{22} & E_{23} & \cdots & \\ E_{11} & E_{12} & E_{13} & \cdots & \\ \hline 0 & 0 & 0 & \cdots & \\ \vdots & & & & \\ 0 & 0 & 0 & & \end{array}\right]$$

which has characteristic polynomial

$$t^2 - (E_{12} + E_{21})t + E_{12}E_{21} - E_{11}E_{22}$$

Note that $E_{1,2} = E_{2,1}$ and

$$E_{i,j} = \langle E_r e_i, E_r e_j \rangle$$

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Consequently

$$E_{1,2}^2 - E_{1,1}E_{2,2} = \langle E_r e_1, E_r e_2\rangle^2 - \langle E_r e_1, E_r e_1\rangle\langle E_r e_2, E_r e_2\rangle$$

and by Gram-Schmidt, this is less than or equal to zero, with equality if & only if $E_r e_1$ & $E_r e_2$ are parallel.

Now there are three cases:

(a) $E_{1,2}^2 - E_{1,1}E_{2,2} < 0$, so HE_r has one positive and one negative eigenvalue, their sum is $2E_{1,2}$. (two new eigenvalues)

(b) $E_r e_1$ & $E_r e_2$ are parallel, the eigenvalues of HE_r are 0 and $2E_{1,2}$. (one new eigenvalue)

(c) $E_{1,1} = E_{2,2} = E_{1,2} = 0$. (no change)

ie $E_r e_1$, $E_r e_2$ both zero (and so they are parallel).

Theorem If A is positive semidefinite, there are vectors $x_1, \dots, x_k$ such that

$$A = x_1 x_1^T + \cdots + x_k x_k^T$$

Proof If $A \succeq 0$ and $A \neq 0$, there is x such that $x^T A x > 0$. Set

$$B := A - \frac{1}{x^T A x} A x x^T A$$

I claim that $B \succeq 0$. We have

$$y^T B y = y^T A y - \frac{(y^T A x)^2}{x^T A x}$$

By Gram-Schmidt applied to $A^{1/2} x$ & $A^{1/2} y$, we have

$(y^T A y)(x^T A x) \geq (y^T A x)^2$.

psd rank-1

$$\text{So} \quad A = B + \frac{1}{x^TAx} Axx^TA$$

If $Az = 0$, then $Bz = 0$. As $Bx = 0$ & $Ax = 0$,

$rk(B) < rk(A)$. Now induct.

Let $n^+(M)$ & $n^-(M)$ respectively denote the number of positive & negative eigenvalues of M (assumed Hermitian).

Lemma If $M \succeq 0$, then $n^+(A-M) \le n^+(A)$

Proof If $rk(M) = 1$, this follows by rank-1 perturbation. More induction.

Lemma $n^+(E_r H E_r) \le n^+(H)$; $n^-(E_r H E_r) \le n^-(H)$

Proof From spectral decomposition we have

$$H = H_0 - H_1$$

where H_0 & H_1 are positive semidefinite.

So $E_r H E_r = E_r H_0 E_r - E_r H_1 E_r$ and from the previous lemma $n^+(E_r H E_r) \le n^+(E_r H_0 E_r) = n^+(H)$

Similarly $n^-(E_r H E_r) \le n^-(H)$.