Inertia & Cocliques

(in characteristic zero)

Assume A is real & symmetric, order $n \times n$.

The 1-dimensional subspaces $\langle x \rangle$ such that $x^T A x = 0$ are the points of a ==projective quadric== Q_A.

We define $x^\perp$ to be $\ker(x^T A)$; this is a hyperplane. If W is a subset

$$W^\perp := \bigcap_{x \in W} x^\perp$$

If $w_1, \dots, w_d$ is a basis for W then

$$W^{\perp} = \ker\left(\begin{bmatrix} w_1^T A \\ \vdots \\ w_d^T A \end{bmatrix}\right)$$

Lemma If A is invertible, then

$$\dim(W^{\perp}) = n - \dim(W)$$

and $(W^{\perp})^{\perp} = W$.

An isotropic subspace U is a subspace contained in $\mathcal{Q}_A$.

If $S \subseteq V(X)$, we define $\langle S \rangle = \{e_u : u \in S\}$.

Lemma If $A = A(X)$ and $S \subseteq V(X)$, then $\langle S \rangle$ is an isotropic subspace on $\mathcal{Q}_A$ if & only if S is a coclique

It follows that if A is invertible, $2\alpha(X) \leq |V(X)|$.
But there is a better bound.

If A is a real symmetric matrix, let n^+, n^-, n^0 respectively denote the number of positive, negative and zero eigenvalues of A.

The triple (n^+, n^-, n^0) is called the inertia of A.

Theorem If A is $n \times n$, the maximum dimension of an isotropic subspace on Q_A is $\min\{n-n^+, n-n^-\}$.

Proof. Let $\mathcal{U}$ be an isotropic subspace of $\mathbb{R}^{V(X)}$ of dimension k. Let $W(+)$ be the subspace of $\mathbb{R}^{V(X)}$ spanned by the eigenvectors with positive eigenvalue, define $W(-)$ analogously.

If $w \in W(+)$, then $w^T A w > 0$, hence $\mathcal{U} \cap W(+) = \langle 0 \rangle$. Similarly $\mathcal{U} \cap W(-) = \langle 0 \rangle$ and therefore $k \leq \min\{n - n^+, n - n^-\}$. $\square$

Two matrixes A & B are congruent if there is an invertible matrix M such that $B = M^T A M$.

Remarks:

(a) congruence is an equivalence relation

(b) a real symmetric matrix is congruent to its diagonal matrix of eigenvalues

(c) If A & B are congruent, then the max. dimensions of the isotropic subspace on Q_A & Q_B are equal.

Theorem Two real symmetric matrices are congruent if and only if they have the same inertia.

Proof We show that if A & B are congruent, they have the same inertia. Let $W(+)$ be a subspace of $\mathbb{R}^{V(x)}$ with maximum dimension such that $w^T A w > 0$ for all non-zero w in $\mathbb{R}^{V(x)}$. If $B = M^T A M$, then $M^{-1} W(+)$ is a subspace of $\mathbb{R}^{V(x)}$ on which B is positive.... $\square$

Corollary Let X be a graph on n vertices.
Then $\alpha(X) \leq \min\{n - n^+(A), n - n^-(A) : A \text{ wtd adj mtx}\}$

examples

(a) K_n

perfect graphs, in fact

(b) connected bipartite graphs

(c) Petersen graph: $3^{(1)}, 1^{(5)}, (-2)^{(4)}$

Folded (2d+1)-cubes

We construct a folded (d+1)-cube from the d-cube by adding the edges of a perfect matching joining vertices at distance d.

If d is odd, the folded (d+1)-cube is bipartite, and we are not interested.

triangle-free
srg on 16 vts

examples

(a) $Q_3/2 \cong K_4$ (b) $Q_5/2 \cong$ Clebsch

Problem. What is the chromatic number of the folded $(2d+1)$-cube?

It's at least three. We attempt to compute the maximum size of a 3-colourable subgraph of $Q_d/2$.

Lemma The maximum size of a k-partite subgraph of X is $\alpha(X \square K_k)$.

We use the inertia bound. The eigenvalues of the folded $(2r+1)$-cube are

$$2r+1-4i \qquad (i=0,\dots,r)$$

with multiplicities

$$\binom{2r+1}{2i}.$$

BCN

If $r=5$, we get

eval	$11,$	$7,$	$3,$	$-1,$	$-5,$	-9
mult	1	55	330	462	165	11
	$\binom{11}{0}$	$\binom{11}{2}$	$\binom{11}{4}$	$\binom{11}{5}$	$\binom{11}{3}$	$\binom{11}{1}$

The eigenvalues of the Cartesian product of the folded 11-cube with $-K_3$ are

$$\underbrace{12^{(2)} \quad 9^{(1)} \quad 8^{(110)} \quad 5^{(55)} \quad 4^{(660)} \quad 1^{(330)}} \quad 0^{(924)} \quad \underbrace{-3^{(462)} \quad -4^{(330)} \quad -7^{(165)} \quad -8^{(22)} \quad -11^{(11)}}_{990}$$

So use $-1.1\,K_3 \longrightarrow \alpha_3(X) \leq 990$

For folded 13-cube, bound obtained is 4098

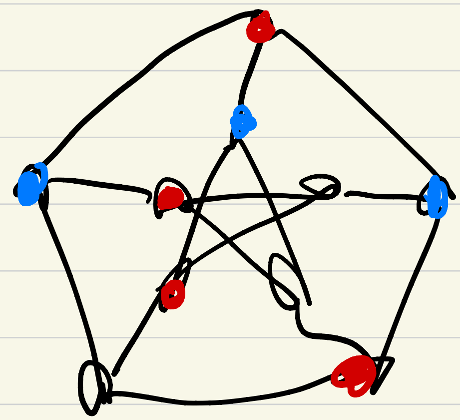
(vs 4096 vertices ☹)

$$\alpha(L(K_5))=2$$

inertia bound : 5!!

1	5	4
3	1	-2
6	-2	1

$L(K_5)$ has a spanning subgraph isomorphic to $K_4 \cup 2K_3$:



(a) edges on 1

(b) edges on 2, not 1

(c) edges not on 1 or 2.

Let C be its adjacency matrix.

Consider

$$\det(C + t(A-C)).$$

Inertia only changes when this is zero.

When $-1 < t < 0.618$, we find that $C + t(A-C)$ has inertia $(2, 13, 0)$.

If x is the characteristic vector of a a subset S of $V(X)$, then $x^T A x = 0$ if and only if S is a coclique. Note that

$$x^T A x = tr(x^T A x) = tr(A x x^T) = sum(A \circ x x^T)$$

and it may be convenient to use $x x^T$ (or $\frac{1}{|S|} x x^T$, a projection) to represent S.

A proper colouring of a graph is a partition π of $V(X)$ into cocliques, which we can represent by its characteristic matrix P.

The i-th cell of π is a coclique if & only if $e_i^T P^T A P e_i = 0$, hence π is a colouring if & only if

$$\operatorname{tr}(P^T A P) = \sum_{i=1}^{|\pi|} e_i^T P^T A P e_i = 0$$

Here

$$\operatorname{tr}(P^T A P) = \operatorname{tr}(A P P^T) = \operatorname{sum}(A \circ P P^T)$$

and PP^T is block diagonal. So

$$\operatorname{tr}(P^T A P) = \sum_i A \circ (P e_i (P e_i)^T)$$

Cocliques & colourings

It might prove convenient to use $\widehat{P}$ in place of P. (Or other weightings.)

If P is the characteristic matrix of a proper colouring, the map

$$u \in V(X) \mapsto e_u^T P$$

embeds $V(X)$ into the unit sphere in $\mathbb{R}^{|\pi|}$, taking adjacent vertices to orthogonal vectors. (An orthogonal representation.)

We can convert the columns of P into $n \times n$ diagonal matrices $Q_1, \dots, Q_{|\pi|}$. These satisfy:

$$Q_i Q_j = \delta_{i,j} Q_i \, ; \qquad \sum_i Q_i = I$$

Thus they form a resolution of the identity.

Lemma Assume π is a partition of $V(X)$ with c cells, and that $Q_1, \dots, Q_c$ is the corresponding resolution of the identity. Then π is a colouring if & only if $\sum_i Q_i A Q_i = 0$

Theorem If X is a k-regular graph on n vertices with least eigenvalue τ, then

$$\alpha(X) \leq \frac{n}{1 - k/\tau}$$

Proof The matrix

$$A - \tau I - \frac{k-\tau}{n} J$$

is positive semidefinite (work out its eigenvalues). So if x is the characteristic vector of a coclique S

$$0 \leq x^T \left(A - \tau I - \frac{k-\tau}{n} J\right) x$$

$$= x^T A x - \tau x^T x - \frac{k-\tau}{n} x^T \underset{\sim}{1}\, \underset{\sim}{1}^T x$$

We have

$$x^T A x = 0, \quad x^T x = |S|, \quad \mathbb{1}^T x = |S|$$

and so

$$0 \leq -\tau |S| - \frac{k-\tau}{n} |S|^2$$

$$\Rightarrow \quad |S| \leq \frac{n}{1 - k/\tau}. \qquad \square$$

If equality holds, $(S, V \setminus S)$ is an equitable partition.