Type-II matrices

We work over $\mathbb{C}$. Denote the Schur-inverse of a matrix M by $M^{(-)}$. We say M is a type-II matrix if M is $n \times n$ and

$$M M^{(-)T} = nI$$

Examples

(a) any Hadamard matrix.

(b) $\theta^n = 1$, $M_{i,j} = \theta^{(i-1)(j-1)}$.

(c) If M & N are type-II, so is $M \otimes N$.

(d) a flat unitary matrix.

(e)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & t & -t \\ 1 & -1 & -t & t \end{bmatrix}, \qquad t \in \mathbb{C} \setminus 0$$

(f) "Potts model"

$$W = (t-1)I + J, \quad W^{(-)} = (t^{-1}-1)I + J$$

$$WW^{(-)} = (t-1)(t^{-1}-1)I + (t+t^{-1}-2+n)J$$

If $t+t^{-1}+n-2=0$, then $(t-1)(t^{-1}-1)=n$. So W is type-II if

$$t = \frac{1}{2}\left(-n+2 \pm \sqrt{n^2-4n}\right)$$

(g) Symmetric designs: $NN^T = \overset{r-\lambda}{n} I + \lambda J$

N is $v \times v$, with $v = 1 + \frac{k^2-k}{\lambda}$. If

$$W = N + t(J-N)$$

$$W^{(-)} = N + t^{-1}(J-N)$$

then we find:

$$WW^{(-)T} = ((1-t)N + tJ)((1-t^{-1})N^T + t^{-1}J)$$

$$= (2-t-t^{-1})NN^T + (t(1-t)JN^T + (1-t)t^{-1}NJ + nJ)$$

$$= (2-t-t^{-1})(nI + \lambda J) + ((1-t)(t+t^{-1})k + n]J$$

$$\vdots$$

A monomial matrix is a product of a permutation matrix with an invertible diagonal matrix. If M is type-II and P, Q are monomial, then PMQ is type-II.

We say M & PMQ are equivalent. If M is type-II, so are M^T & $\bar{M}$, but these are not deemed to be equivalent to M.

A trace characterization

Lemma An $n \times n$ matrix invertible matrix W is type-II if and only if, for any two diagonal matrices D_1 & D_2,

$$\langle D_1 , W^{-1} D_2 W \rangle = \frac{1}{n} \operatorname{tr}(D_1) \operatorname{tr}(D_2)$$

Proof

$$\langle e_i e_i^T, W^{-1} e_j e_j^T W \rangle = \operatorname{tr}(e_i e_i^T W^{-1} e_j e_j^T W) = e_i^T W^{-1} e_j \; e_j^T W e_i$$

$$= (W^{-1})_{i,j} \; W_{j,i}$$

and the lemma holds for $D_1 = e_i e_i^T$ and $D_2 = e_j e_j^T$

if & only if

$$(W^{-1})_{i,j} W_{j,i} = \frac{1}{n}.$$

This holds for all i & j if & only if $W^{-1} = \frac{1}{n} W^{(-)T}$.

Corollary If W is $n \times n$ and type-II and D is diagonal,

$$(W^{-1} D W)_{i,i} = \frac{1}{n} \operatorname{tr}(D).$$

Equivalently, $W^{-1} D W$ has constant diagonal.

A resolution of the identity is a sequence of pairwise orthogonal projections that sum to I.

Exercise If $P_1, \dots, P_k$ are projections and $\sum P_i$ is idempotent, then $P_i P_j = 0$ when $i \neq j$.

Lemma Suppose $P_1, \dots, P_k$ is a resolution of the identity. If W is a $k \times k$ type-II matrix and we define

$$U_i = \frac{1}{\sqrt{n}} \sum_j W_{i,j} P_j$$

then $U_1, \dots, U_k$ are invertible and

$$\sum_{i=1}^{k} P_i \otimes P_i = \sum_{i=1}^{k} U_i \otimes U_i^{-1} .$$

If W is unitary (equiv., flat), then $U_1, \dots, U_k$ are unitary.

$$\underbrace{V \otimes V^* = \mathcal{E}nd(V)}_{\mathrm{Mat}_{n\times n}(\mathbb{C})}$$

$$M \to \sum_r F_r M G_r^*$$

Lemma If $\sum_i A_i \otimes B_i = \sum_j C_j \otimes D_j$, then for all M

$$\sum_i A_i M B_i = \sum_j C_j M D_j$$

Proof The span of $\{A \otimes B : A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})\}$ is an algebra. Define an endomorphism $\tau_{A,B}$ of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ by

$$\tau_{A,B} : M \to AMB^*$$

Then $\tau_{A,B}$ is a homomorphism from $\mathrm{Mat}_{n\times n}(\mathbb{C}) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ to $\mathrm{End}(\mathrm{Mat}_{n\times n}(\mathbb{C}))$.

If $\sum_i A_i \otimes B_i = \sum_j C_j \otimes D_j$, then $\sum_i \tau_{A_i,B_i}$ and $\sum_j \tau_{C_j,D_j}$ agree on M.

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Theorem For any graph X we have $\chi(X) \geq 1 - \frac{\theta_1}{\theta_n}$.

Proof Assume $c = \chi(X)$ and let $\Phi_1, \dots, \Phi_c$ be the resolution of the identity coming from the partition of $V(X)$ into colour classes. Let $U_1, \dots, U_c$ be the unitary matrices constructed from $\Phi_1, \dots, \Phi_c$ using a flat unitary matrix. Then

$$0 = \sum_{i=1}^{c} \Phi_i A \Phi_i = \sum_{i=1}^{c} U_i A U_i^*$$

and accordingly

$$A_1 = -\sum_{i=2}^{c} U_1^* U_i A U_i^* U_1$$

Choose an eigenvector z of A with eigenvalue θ_1, and set $y_i = U_i^* U_1 z$. Then

$$\theta_1 = z^T A z = -\sum_{j=2}^{c} y_j^T A y_j \leq (c-1)(-\theta_n).$$

result due to Hoffman, proof to Elphick & Wocjan

Coherent algebras

A ==coherent algebra== is a matrix algebra that is:

(a) closed under transpose & complex conjugation

(b) contains J

(c) is Schur-closed.

We will focus on the real case.

Bose-Mesner algebras of association schemes provide one class of examples

Lemma The commutant of a set of permutation matrices is Schur-closed

Proof If P is a permutation matrix, then $P(A \circ B) = PA \circ PB$. If P commutes with A and B,

$$PA \circ PB = AP \circ BP = (A \circ B)P \qquad \square$$

It follows that the commutant of a permutation group is a coherent algebra.

Lemma If a subspace of $\mathbb{R}^N$ is closed under Schur multiplication, it has a 01-basis.

Proof If p is a polynomial and $v \in \mathbb{R}^N$, define

$$p \circ v = (p(v_i))_{i=1}^N$$

Choose p_k so $p(v_k)=1$ and $p_k(v_j)=0$ if $j \neq k$. Then $p_k \circ v$ is a 01-vector and if S is the set of entries of v, we have

$$w = \sum_{v_k \in S} v_k \, p_k(v).$$

$\square$

Corollary Any coherent algebra has a basis β of 01-matrices such that

(a) $\sum_{M \in \beta} M = J$

(b) I is a sum of elements of β

(c) β is transpose-closed.

The matrices in β form a coherent configuration (or the directed graphs they represent do).

The orbitals of a permutation group are a coherent configuration.

A coherent algebra is homogeneous if $I \in \beta$.

The commutant of a permutation group is homogeneous if & only if the group is transitive.

Lemma A commutative coherent algebra is is homogeneous.