Type-II matrices ctd

Suppose W is a Schur-invertible matrix

Define vectors $W_{i/j}$ by

$$(W_{i/j})_k = \frac{W_{ki}}{W_{kj}}$$

(ratio of columns)

If we use ∂_i to denote the diagonal matrix formed using the entries of We_i, then ∂_i is invertible and

$$W_{i/j} = \partial_j^{-1} We_i$$

The **Nomura algebra** $\mathcal{N}_W$ of a Schur-invertible matrix W is the set of matrices for which each vector $W_{i/j}$ is an eigenvector. Clearly $I \in \mathcal{N}_W$.

Lemma If W_1 & W_2 are Schur invertible and equivalent, then $\mathcal{N}_{W_1} \cong \mathcal{N}_{W_2}$.

Lemma $\mathcal{N}_{W_1 \otimes W_2} \cong \mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2}$

Lemma Assume W is square and Schur invertible. Then W is a type-II matrix if and only if $J \in \mathcal{N}_W$.

Lemma $W_{k/i}^{T} W_{i/j} = n\delta_{j,k}$

Proof

$$W_{k/i}^{T} W_{i/j} = \sum_{r} \frac{W_{r,k}}{W_{r,i}} \frac{W_{r,i}}{W_{r,j}} = \sum_{r} W_{r,k} \left(W_{r,j}^{-1}\right)$$

$$= \left(W^{(-)} e_j\right)^{T} W e_k$$

$$= \left(W^{(-)T} W\right)_{j,k}$$

$$= \delta_{j,k}\, n$$

Suppose $u_1, \ldots, u_n$ are linearly independent eigenvectors for A. Set $U = [u_1, \ldots, u_n]$. Then there is a diagonal matrix Δ such that $AU = U\Delta$, and then

$$U^{-1}A = \Delta U^{-1}$$

and the rows of U^{-1} are left eigenvectors for A.

Assume $(U^{-1})^T = [v_1, \ldots, v_n]$. As

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} [u_1, \ldots, u_n] = I,$$

also

$$I = [u_1 \cdots u_n] \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = \sum_i u_i v_i^T$$

This is almost a spectral decomposition of A — the matrices $n_i v_i^T$ sum to I, are idempotent & pairwise orthogonal. (But they are not symmetric)

Lemma If W is type-II, then N_W is commutative.

Proof The vectors $W_{1,j}, \ldots, W_{n,j}$ are a basis of $\mathbb{C}^n$. As all matrices in N_W are diagonalized by this basis, it is commutative.

If W is an $n \times n$ type-II matrix

$$Y_{i,j} := \frac{1}{n} W_{i/j} W^{T}_{j/i}$$

Let $\partial_i(W)$ be the $n \times n$ diagonal matrix given by

$$\partial_i(W)_{r,r} = W_{r,i}$$

We abbreviate $\partial_i(W)$ to ∂_i.

We note that $Y_{i,i} = \frac{1}{n} J$ and $Y_{i,j}^{T} = Y_{j,i}$

Remark If W is flat, $Y_{i,j}$ is Hermitian

Lemma $\sum_{i=1}^{n} Y_{i,j} = I, \quad \sum_{i=1}^{n} Y_{j,i} = I$

Proof

$$\sum_{i=1}^{n} Y_{i,j} = [W_{1,j} \cdots W_{n,j}] \begin{bmatrix} W_{j,1}^{T} \\ \vdots \\ W_{j,n}^{T} \end{bmatrix} = \partial_j^{-1} W W^{(-)T} \partial_j = nI.$$

We define $\mathcal{Y}_W$ to be the $n \times n$ matrix over $\mathrm{Mat}_{n\times n}(\mathbb{C})$ with $(\mathcal{Y}_W)_{i,j} = Y_{i,j}$. It is the matrix of idempotents of W.

If W is flat, $\mathcal{Y}$ is unitary and its entries are projections.

Theorem TFAE:

(a) $M \in \mathcal{N}_W$

(b) the vectors $w_{i,j}$ are eigenvectors for M

(c) the vectors $w_{j,i}^T$ are left eigenvectors for M

(d) M commutes with $Y_{i,j}$, for all i & j

Proof

(a) $\Rightarrow$ (b) : definition of $\mathcal{N}_W$

(b) $\Rightarrow$ (c) : page 6

$$\left.\begin{array}{l}\text{(b), (c)} \Rightarrow \text{(d)} \\ \text{(d)} \Rightarrow \text{(a)}\end{array}\right\} \quad Axy^T = xy^TA$$

: look at row & column spaces

Corollary If W is type-II, then $\mathcal{N}_W$ is transpose-closed.

Proof. If $MY_{i,j} = Y_{i,j}M$ then

$$Y_{i,j}^T M^T = M^T Y_{i,j}^T$$

and, since $Y_{i,j}^T = Y_{j,i}$, we have $M^T \in \mathcal{N}_W$.

Quantum permutations & colourings

A quantum permutation is an $n \times n$ matrix over the ring $\mathrm{Mat}_{d\times d}(\mathbb{C})$ such that:

(a) each entry is a projection

(b) each row sums to I_d

(c) each column sums to I_d

examples:

(a) any permutation matrix

(b) the matrix of idempotents of a type-II matrix.

Alat

Lemma If $P_1, \dots, P_m$ are $d \times d$ projections and $Q = \sum_i P_i$ is a projection, then $P_i P_j = \delta_{ij} P_i$.

Proof Assume $U_1, \dots, U_m$ are matrices such that $U_i^* U_i = I$ & $U_i U_i^* = P_i$. Then

$$Q = [U_1, \dots, U_m] \begin{bmatrix} U_1^* \\ \vdots \\ U_m^* \end{bmatrix}$$

and therefore

$$\begin{bmatrix} U_1^* \\ \vdots \\ U_m^* \end{bmatrix} [U_1 \cdots U_m] = \begin{bmatrix} U_1^* U_1 & & \\ & \ddots & \\ & & U_m^* U_m \end{bmatrix}$$

is also a projection. As its diagonal is I, it is equal to I.

Corollary The projections in a row or column of a quantum permutation form a resolution of the identity. Consequently a quantum permutation is a unitary matrix.

(17-11-21)

Quantum colourings

If $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ are resolutions of I_d, we say they are **orthogonal** if $P_i Q_i = 0$ for $i = 1, \ldots, m$. $\left(\Leftrightarrow \sum_i P_i Q_i = 0\right)$

$tr(P_i Q_i) = 0$

example: any two distinct rows of a quantum permutation.

A **quantum m-colouring** of a graph X on n vertices is an $n \times n$ matrix of $d \times d$ projections such that each row is a resolution of the identity and rows corresponding to adjacent vertices are orthogonal.

The characteristic matrix of a classical colouring is a quantum colouring (with $d=1$).

Lemma A quantum n-colouring of K_n is a quantum permutation.

Proof We need to show the each column of the quantum colouring sums to I_d. Since distinct entries in the same column are orthogonal, each column sum is a projection.

Assume M is the quantum n-colouring and

$$Q_j = \sum_{i=1}^{n} M_{ij}.$$

Then Q_j represents orthogonal projection onto a subspace of $\mathbb{C}^n$, whence $tr(Q_j) \le d$

Now $\sum_j tr(P_{ij}) = tr(I_d) = d$ and hence

$$\sum_{i,j} tr(P_{ij}) = nd$$

As $\sum_{i,j} P_{ij} = \sum_j Q_j$, we have

$$nd = \sum_j tr(Q_j) \le nd.$$

Hence $tr(Q_j) = d$ for all j & our quantum colouring is a quantum permutation. $\square$

We saw that a classical m-colouring gave rise to a resolution of the identity with m terms. We can do the same thing for quantum colourings.

If $P = (P_{ij})$ is the matrix of a quantum colouring, we can form a diagonal matrix $\mathcal{P}_i$ from the i-th column of P:

$$\begin{bmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{bmatrix} \longrightarrow \begin{bmatrix} P_{1j} & & \\ & \ddots & \\ & & P_{nj} \end{bmatrix} =: \mathcal{P}_j$$

Here $\mathcal{P}_i$ is a projection.

Further $P_i P_j = 0$ if $i \neq j$ and

$$\sum_i P_i = I$$

so we have a resolution of the identity.

Theorem P is a quantum c-colouring of X if & only if

$$\sum_{i=1}^{c} P_i (A(X) \otimes I_d) P_i = 0.$$

Exercise

So there are unitary matrices $\mathcal{U}_1, \dots, \mathcal{U}_c$ such that

$$\sum_{i=1}^{c} P_i \otimes P_i = \sum_{i=1}^{c} \mathcal{U}_i \otimes \mathcal{U}_i^*$$

and hence

$$\sum_{i=1}^{c} \mathcal{U}_i (A \otimes I) \mathcal{U}_i^* = 0 .$$

It follows that $c \geq 1 - \frac{\theta_1}{\theta_n}$.