Coherent algebra

Coproduct

If P & Q are $n \times n$ quantum permutations, we define $P \star Q$ by

$$(P \star Q)_{i,j} := \sum_r P_{ir} \otimes Q_{rj}$$

This is called the coproduct of P and Q.

Lemma $P \star Q$ is a quantum permutation. $\square$

If the entries of P are $d \times d$ and those of Q are $e \times e$, then the entries of $P \star Q$ are $de \times de$.

Direct sums Assume P & Q are $n \times n$ quantum permutations. Their **direct sum** $P \oplus Q$ is defined by

$$(P \oplus Q)_{ij} = P_{ij} \oplus Q_{ij}$$

Lemma $P \oplus Q$ is a quantum permutation. If $A \otimes I_d$ commutes with P and $B \otimes I_e$ commutes with Q, then $A \otimes I_{d+e}$ commutes with $P \oplus Q$.

Theorem Let P be a quantum permutation. If the entries of P commute, then P is similar to a direct sum of permutation matrices. $\square$

Proof (sketch). If the entries of P commute, we can simultaneously diagonalize them.

Back to type-II matrices:

Recall that if W is $n \times n$ type-II

$$Y_{i,j} := \frac{1}{n} W_{i,j} W^T_{j,i}$$

$$\mathcal{Y} := (Y_{i,j})_{i,j=1}^{n}$$

matrix of idempotents

As $Y_{i,j}^T = Y_{j,i}$, we see that $\mathcal{Y}$ is symmetric.
Each row & column of $\mathcal{Y}$ sums to I, hence if W is flat, $\mathcal{Y}$ is a quantum permutation.

Lemma $\mathcal{Y}_W$ is a type-II matrix.

Proof First, $Y_{i,j}^T = Y_{j,i}$ whence $\mathcal{Y}$ is symmetric.
Also $Y_{i,j}^{(-)} = n W_{j,i} W_{i,j}^T = n^2 Y_{j,i}$. If

$$\mathcal{Y} = \begin{bmatrix} Y_{1,1} & Y_{1,2} & Y_{1,3} \\ Y_{2,1} & Y_{2,2} & Y_{2,3} \\ Y_{3,1} & Y_{3,2} & Y_{3,3} \end{bmatrix}$$

then

$$\mathcal{Y}^{(-)T} = n^2 \begin{bmatrix} Y_{1,1} & Y_{2,1} & Y_{3,1} \\ Y_{1,2} & Y_{2,2} & Y_{3,2} \\ Y_{1,3} & Y_{2,3} & Y_{3,3} \end{bmatrix}$$

and, using orthogonality, $\mathcal{Y}\mathcal{Y}^{(-)T} = n^2 I$.

Remark: if W is a Hadamard matrix, so is $\mathcal{Y}$.

Let S be the permutation operator on $V \otimes V$ that sends $u \otimes v$ to $v \otimes u$ ($\forall u, v$). (So $S^2 = I$ and $S(A \otimes B) = (B \otimes A) S$.)

Lemma If W is type-II, then $\mathcal{Y}_{W^T} = S \mathcal{Y}_W S$.

Proof

$$n (Y_{i,j})_{r,s} = \frac{W_{r,i}}{W_{r,j}} \frac{W_{s,j}}{W_{s,i}} = \frac{W_{r,i}}{W_{s,i}} \frac{W_{s,j}}{W_{r,j}} = \frac{W^T_{i,r}}{W^T_{i,s}} \frac{W^T_{j,s}}{W^T_{j,s}}$$

$$= n (Y_{r,s} (W^T))_{i,j}$$

So the left & right terms are equal respectively to

$$(e_i \otimes e_r)^T \mathcal{Y}_W (e_j \otimes e_s), \quad (e_r \otimes e_i)^T \mathcal{Y}_{W^T} (e_s \otimes e_j)$$

Digression If W is type-II, define

$$\hat{W} := (W \otimes W^{(-)T}) S$$

Then

$$\hat{W}^{(-)T} = \left((W^{(-)} \otimes W^T) S \right)^T$$

$$= S (W^{(-)T} \otimes W)$$

But $S(A \otimes B) = (B \otimes A) S$:

$$(B \otimes A) S \cdot (u \otimes v) = (B \otimes A)(v \otimes u) = Bv \otimes Au = S(Au \otimes Bv)$$

$$= S(A \otimes B)(u \otimes v)$$

Hence $\hat{W}^{(-)T} = \hat{W}$, further

$$\hat{W}_{i,i} = (e_i \otimes e_i)(W \otimes W^{(-)T}) S (e_i \otimes e_i) = W_{i,i} W^{(-)}_{i,i} = 1$$

Thus the diagonal of $\hat{W}$ is constant.

Question: what is the relation between $\hat{w}$ and y_w?

If A and B are $n\times n$ matrices

$$[A,B] := AB - BA.$$

Theorem If W is type-II,

$$\mathcal{N}_W = \{M : [I\otimes M, \mathcal{Y}] = 0\}, \quad \mathcal{N}_{W^T} = \{N : [N\otimes I, \mathcal{Y}] = 0\}$$

Proof $I\otimes M$ commutes with $\mathcal{Y}$ if & only if $MY_{i,j} = Y_{i,j}M$ for all i & j. Hence $M \in \mathcal{N}_W$.

For the second claim:

$$S(N\otimes I)\mathcal{Y}_W S = (I\otimes N) S \mathcal{Y}_W S = (I\otimes N) \mathcal{Y}_{W^T}$$

Theorem Let Z be an $n \times n$ matrix over $\mathrm{Mat}_{d \times d}(\mathbb{C})$. Assume that the entries of Z are idempotent, and the idempotents in each row & column are pairwise orthogonal. Then the set of $n \times n$ matrices M such that $[M \otimes I_d, Z] = 0$ is Schur-closed.

We derive the proof from the following Lemma.

Lemma If Z is as in the theorem, then

$$(M\otimes I)Z \circ (N\otimes I)Z = ((M\circ N)\otimes I)Z,$$

$$Z(M\otimes I) \circ Z(N\otimes I) = Z((M\circ N)\otimes I).$$

Proof

$$((M\otimes I)Z)_{i,j} = \sum_r M_{i,r} Z_{r,j}, \quad ((N\otimes I)Z)_{i,j} = \sum_r N_{i,r} Z_{r,j}$$

$$(Z(M\otimes I))_{i,j} = \sum Z_{j,r} M_{r,i}, \quad (Z(N\otimes I))_{i,j} = \sum_r Z_{j,r} N_{r,i}$$

Now

$$\left(\sum_r M_{i,r} Z_{r,j}\right)\left(\sum_r N_{i,r} Z_{r,j}\right) = \sum_r M_{i,r} N_{i,r} Z_{r,j} = (((M\circ N)\otimes I)Z)_{i,j}$$

$$\left(\sum Z_{j,r} M_{r,i}\right)\left(\sum_r Z_{j,r} N_{r,i}\right) = \sum_r M_{r,i} N_{r,i} Z_{j,r} = (Z((M\circ N)\otimes I))_{i,j}$$

Corollary. If W is type-II, then $\mathcal{N}_W$ is Schur-closed.

Corollary If P is a quantum permutation then

$$\mathcal{C} = \{ M : [M \otimes I, P] = 0 \}$$

is a coherent algebra.

Proof We get Schur closure from the lemma. We have

$$(J\otimes I)P = J\otimes I = P(J\otimes I)$$

and therefore $J \in \mathcal{C}$. Next, if $P(M\otimes I) = (M\otimes I)P$, then since P is unitary,

$$P(M\otimes I) = (M\otimes I)P \Rightarrow (M\otimes I)P^* = P^*(M\otimes I) \Rightarrow P(M^*\otimes I) = (M^*\otimes I)P$$

and thus $M^* \in \mathcal{C}$. (Since $\mathcal{C}$ has a 01-basis, it is closed under complex conj.; since the basis is closed under transpose, $\mathcal{C}$ is closed under transpose.)

Note that $I \otimes M$ commutes with P if & only if M commutes with each entry of P, equivalently if M lies in the algebra $\langle P_{i,j} \rangle$.

Graphs with adjacency matrices A & B are quantum isomorphic if there is a quantum permutation P such that

$$(A \otimes I)P = P(B \otimes I)$$

Accordingly, $B \otimes I = P^*(A \otimes I)P \Rightarrow A$ & B are similar.

In fact, if $(A_1 \otimes I)P = P(B_1 \otimes I)$ & $(A_2 \otimes I)P = P(B_2 \otimes I)$, then

$$((A_1 \otimes I) \circ (A_2 \otimes I))P = (A_1 \otimes I)P \circ (A_2 \otimes I)P = \cdots = P((B_1 \otimes I) \circ (B_2 \otimes I))$$

Hence if X and Y are quantum isomorphic, their coherent algebras are isomorphic.