Resolvents

(Remark: we're following Kato, roughly)

A rational matrix is a matrix whose entries are rational functions over some field $\mathbb{F}$

A rational function p/q is proper if $\deg(p) < \deg(q)$.

Theorem Assume $\mathbb{F}$ is algebraically closed and $p(t)/q(t)$ is a proper rational function over $\mathbb{F}$. Assume $\theta_1, \dots, \theta_k$ are the distinct zeros of $q(t)$, with respective multiplicities $m_1, \dots, m_k$. Then there are polynomials $p_1, \dots, p_k$ with $\deg(p_i) = m_i$, such that

$$\frac{p(t)}{q(t)} = \sum_{r=1}^{k} \frac{p_r(t)}{(t-\theta_r)^{m_r}}$$

partial fraction expansion

A rational matrix is proper if each entry is proper.

From the theorem, if A is a proper rational matrix

$$A = \sum_r (t-\theta_r)^{-m_r} P_r(t)$$

Here the θ_r range over the poles of the entries of A and $\deg(P_r(t)) < m_r$. This is a partial fraction expansion of A.

If θ is a pole of order m, we also have a series expansion

$$A = \sum_{k \geq -m} A_k (t-\theta)^k$$

(for suitable matrices A_k)

For an already familiar example of a partial fraction expansion:

$$(tI - A)^{-1} = \sum_r (t - \theta_r)^{-1} E_r$$

If A is a square matrix, its resolvent $R(z)$ is $(zI - A)^{-1}$.

We seek a partial fraction expansion of $R(z)$, for general matrices.

Lemma $R(z) - R(w) = -(z-w)R(z)R(w)$

Proof

$$(zI-A)\big(R(z)-R(w)\big)(wI-A)$$

$$= (wI-A) - (zI-A)$$

$$= (w-z)I$$

and so

$$R(z)-R(w) = -(z-w)(zI-A)^{-1}(wI-A)^{-1}. \qquad \square$$

Corollary: $\frac{d}{dz}R(z) = -R(z)^2$

We present one application.

Lemma If A is symmetric, all poles of $R(z)$ are simple.

Proof Assume θ is an eigenvalue of A. If $Au=\theta u$ then $(zI-A)^{-1}u = \frac{1}{z-\theta}u$ and therefore θ is a pole of $R(z)$. So

$$R(z) = \sum_{r\geq -m} A_r (z-\theta)^r$$

where $m \geq 1$, and we assume without loss that $A_{-m} \neq 0$.

Then $R(z)^T R(z)$ is equal to $\underbrace{A_{-m}^T A_{-m}}_{\text{not zero}} (z-\theta)^{-2m}$, plus terms of higher order. As A is symmetric, $R(z)^T = R(z)$.

By the previous lemma, $R'(z) = -R(z)^2$ and the lowest order term of $R'(z)$ is $-mA_{-m}(z-\theta)^{-m-1}$.

Therefore $m+1 = 2m$ & $m = 1$. $\square$

We derive information about the coefficients in the series expansion of $R(z)$.

Lemma. If θ is an eigenvalue of A with multiplicity m & $R(z) = \sum_{r \geq -m}' A_r (z-\theta)^r$, then

$$A_r A_s = \begin{cases} -A_{r+s+1}, & r, s \geq 0; \\ A_{r+s+1}, & r, s \leq -1, \quad r+s \geq -m; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Assume first that $\theta = 0$. Since

$$R(z) - R(w) = -(z-w)R(z)R(w)$$

we have

$$\sum_{r \geq -m} A_r \underbrace{\frac{z^r - w^r}{z-w}}_{z^{r-1} + wz^{r-2} + \cdots + w^{r-1}} = -\sum_{r,s \geq -m} A_r A_s z^{r+s}$$

Comparing coefficients of $z^i w^j$ yields

$$A_{i+j+1} = \begin{cases} -A_i A_j, & i,j \geq 0 \\ A_i A_j, & -m+1 \leq i+j;\ i,j \leq -1 \\ 0, & \text{otherwise.} \end{cases}$$

$$A_r = (-1)^r A_0^{r+1} \qquad A_{-1}^2 = A_{-1}. \quad A_{-1} A_r = -A_r \quad (r \geq 0)$$

$$A_{-2} A_{-r} = A_{-1-r} \Rightarrow A_{-s} = (A_{-2})^{s-1}$$

To make this clearer, suppose $m=3$. Then LHS is

$$A_{-3}\frac{z^{-3}-w^{-3}}{z-w}+A_{-2}\frac{z^{-2}-w^{-2}}{z-w}+A_{-1}\frac{z^{-1}-w^{-1}}{z-w}+A_1+A_2(z+w)+A_3(z^2+wz+w^2)+\cdots$$

$$=-\frac{z^2+wz+w^2}{z^3w^3}A_{-3}-\frac{z+w}{z^2w^2}A_{-2}-\frac{1}{zw}A_{-1}+A_1+A_2(z+w)+\cdots$$

The coefficient of z^iw^j on the RHS

is $-A_iA_j$.

Since

$$A_{i+j+1} = \begin{cases} -A_i A_j, & i,j \geq 0 \\ A_i A_j, & -m+1 \leq i+j;\ i,j \leq -1 \\ 0, & \text{otherwise.} \end{cases}$$

we get

$$\boxed{A_r = (-1)^r A_0^{r+1} \qquad (r \geq 0)}$$

$$A_{-1} A_{-r} = A_{-r} \qquad (r \geq 1) \qquad \Rightarrow \boxed{A_{-1}^2 = A_{-1}}$$

$$A_{-2} A_{-r} = A_{-r-1} \qquad (r \geq 2) \qquad \Rightarrow \boxed{A_{-r} = (A_{-2})^{r-1}}$$

$$(A_{-2})^m = 0$$

Hence series is determined by A_0, A_{-1} & A_{-2}.

Note that A_{-1} is idempotent & A_{-2} is nilpotent.

Set $E_\theta = A_{-1}$, $N_\theta = A_{-2}$.

Now

$$(tI-A)R(z) = ((t-z)I + zI - A)R(z)$$

$$= (t-z)R(z) + I$$

and setting $t=\theta$ yields

$$\sum_{r\geq -m} (\theta I-A)A_r(z-\theta)^r = I - \sum_{r\geq -m} A_r(z-\theta)^{r+1}$$

$$\Rightarrow \begin{cases} (\theta I-A)A_r = A_{r-1} & (r\neq 0) \\ (\theta I-A)A_0 = -A_{-1} - I \\ (\theta I-A)A_{-m} = 0 \end{cases} \Rightarrow \boxed{\begin{array}{c} (\theta I-A)E_\theta = N_\theta \ \& \\ (\theta I-A)^r E_\theta = N_\theta^r \end{array}}$$

Principal parts

We define the **principal part** $P_\theta(z)$ of $R(z)$ by

$$P_\theta(z) = \sum_{r=1}^{m} A_{-r}(z-\theta)^{-r}$$

So

$$P_\theta(z) = (z-\theta)^{-1}E_\theta + \sum_{r=1}^{m-1} N_\theta^r (z-\theta)^{-r-1}$$

$$= (z-\theta)^{-1}\sum_{r=0}^{m-1} (\theta I - A)^r E_\theta (z-\theta)^{-r}$$

We also have $E_\theta R(z) = P_\theta(z)$

Theorem $R(z) = \sum_\theta P_\theta(z)$

Proof The entries of $R(z)$ and $P_\theta(z)$ are proper rational functions, and therefore each entry of

$$R(z) - \sum_\theta P_\theta(z)$$

is a proper rational function. As each pole of $R(z)$ occurs at an eigenvalue of A, the above difference has no poles. Therefore it is a polynomial but, since $R(z)$ & $\sum_\theta P_\theta(z)$ go to zero as $z \to \infty$, it is zero. $\square$

Lemma E_θ is a polynomial in A; $\sum_\theta E_\theta = I$

Proof

1) $E_\theta = \int_C (zI-A)^{-1}\, dz$

2) $\int_C (zI-A)^{-1} dz$ is a limit of weighted sums of matrices $(z_r I - A)^{-1}$

3) Any matrix that commutes with A commutes with E_θ

4) E_θ is a polynomial in A. (Theorem 4.8.1)

For the second claim,

$$(zI-A)^{-1} = z^{-1}(I - z^{-1}A)$$

and so

$$\int_C (zI-A)^{-1}\,dz = I$$

Order of poles of $R(z)$

Assume θ has multiplicity m. Then $A_{r} = 0$ when $r < -m$, equivalently when $(\theta I - A)^m E_\theta = 0$. So the order of the pole of $R(z)$ at θ is at most m.

Theorem The order of the pole of $R(z)$ at θ equals the multiplicity of θ as a zero of the minimum polynomial of A.

Corollary Any square matrix A is the sum of a diagonalizable matrix and a nilpotent matrix, each a polynomial in A