Algebras

What algebras have we met:

(a) $\mathbb{F}$

(b) $\mathbb{F}(t)$,

(c) adjacency algebra: $\mathbb{F}[A] \cong \mathbb{F}[t]/(\psi_A(t))$

(d) normal matrices: $\langle A, A^* \rangle$

(e) commutants, coherent algebras

(f) $\langle A, zz^* \rangle$ (with A Hermitian)

These are all $*$-closed (except (c), for directed graphs)

Burnside's theorem

Burnside

Theorem Assume $\mathcal{A}$ is a subalgebra of $Mat_{n\times n}(\mathbb{C})$. If $\mathcal{A}$ does not fix a non-zero proper subspace, then $\mathcal{A} = Mat_{n\times n}(\mathbb{C})$.

you can use your favorite alg. closed field.

The proof requires a number of steps.

We say an algebra $\mathcal{A}$ acting on a vector space V is transitive if, given non-zero u & v in V, there is an element A in $\mathcal{A}$ such that $Au = v$

Claim An algebra acting on V is transitive if & only if it has no proper non-zero invariant subspace.

Proof Suppose $0 < U < V$ and U is $\mathcal{A}$-invariant. If $u \in U \setminus 0$ and $v \in V \setminus U$, no element of $\mathcal{A}$ maps u to v.

decomposable $\Rightarrow$ not transitive

Conversely, assume no proper non-zero invariant subspace. Choose $u \neq 0$ in V. Then the $\mathcal{A}$-module generated by u is $\mathcal{A}$-invariant, hence equals V.

Claim If the action of $\mathcal{A}$ on V is indecomposable, so is the action of $\mathcal{A}^*$ on V^*.

Proof If $\mathcal{A}$ fixes U, then $\mathcal{A}^*$ fixes $U^\perp$.

The rest is an exercise.

Claim If the action of $\mathcal{A}$ is indecomposable, then $\mathcal{A}$ contains a rank-1 matrix.

Proof. Choose T in $\mathcal{A}$ with minimal rank. Assume by way of contradiction that $rk(T) \geq 2$. Then there are vectors u & v such that Tu & Tv are linearly independent.

By transitivity, there is A in $\mathcal{A}$ such that $ATu = v$. Accordingly $TATu = Tv$ & Tu are linearly independent.

Next, $\operatorname{im}(T)$ is invariant under TA and therefore is an eigenvalue λ of TA (acting on $\operatorname{im} T$). Hence $\operatorname{rk}(TA - \lambda I) \leq \dim(TA) \leq \dim(T) = \operatorname{rk}(T)$. Since we chose T with minimal rank, $TAT - \lambda T = 0$. Then $Tv = TATu = \lambda Tu$, so Tu & Tv are linearly independent — a contradiction.

We complete the proof by showing that $\mathcal{A}$ contains all rank-1 matrices.

Assume $xy^* \in \mathcal{A}$. Then $Axy^*B \in \mathcal{A}$ for all A, B in $\mathcal{A}$. By transitivity, if $u, v \in V \setminus 0$, we can find A, B so $Ax = u$, $v^*B = y^*$. $\square$

We offer one application. A flag in the vector space V is a sequence of subspaces:

$$U_0 < U_1 < \cdots < U_k .$$

A maximal flag has length $k = \dim(V)$.

Each maximal flag corresponds to an ordered basis $u_1, \dots, u_k$ —— $U_r := \operatorname{span}\{u_1, \dots, u_r\}$.

Claim Let $\beta = (v_1, \dots, v_n)$ be an ordered basis. L in $\mathrm{End}(V)$ fixes each subspace in the maximal flag determined by the basis if & only if the matrix representing L is triangular.

Theorem If $\mathcal{A}$ is a commutative subalgebra of $\mathrm{Mat}_{n\times n}(\mathbb{C})$, then $\mathcal{A}$ fixes a maximal flag.

Proof Assume $n = \dim(V)$. If $n=1$, there is nothing to prove. If $n \geq 2$, then $\mathrm{End}(V)$ is not commutative and therefore $\mathcal{A}$ fixes a proper non-zero subspace, U say. If $\dim(U)$ is minimal then $\mathcal{A}|U \cong \mathrm{End}(U)$. As $\mathcal{A}$ is commutative, $\dim(U) = 1$.

Now $\mathcal{A}$ acts on V/U and we can use induction to claim that $\mathcal{A}$ fixes a maximal flag in V/U. $\square$

Corollary. Any matrix in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is similar to a triangular matrix.