Graph eigenvalues

An algebraic integer is totally real if all roots of its minimal polynomial are real.

An algebraic integer is totally positive if all conjugates are positive reals. The square of a totally real algebraic integer is totally positive (and conversely). The totally real algebraic integers form a ring.

adjacency matrix

Graph eigenvalues are totally real. Laplacian eigenvalues are totally positive.

Question (Hoffman, 1973) Is every totally real algebraic integer a graph eigenvalue?

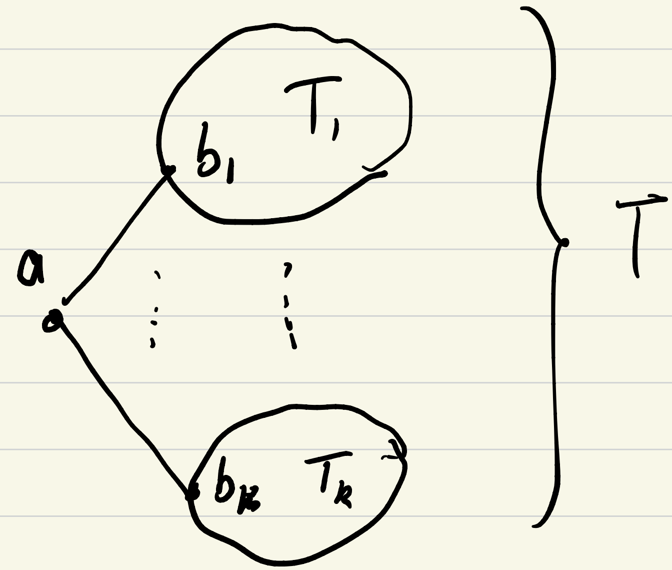
Answer 1 (Estes, 1992) Yes.

Answer 2 (Salez, 2015) Every totally real algebraic integer is an eigenvalue of a tree.

Salez: totally real

$\Rightarrow$ tree eigenvalue

$$\frac{\phi(S,t)}{\prod \phi(T_i,t)} = t - \sum_i \frac{\phi(T_i \setminus b_i, t)}{\phi(T_i,t)}$$



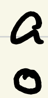
The generating function for closed walks at the vertex a in X is

$$C_a(X,t) = \frac{t^{-1}\phi(X \setminus a, t^{-1})}{\phi(X, t^{-1})}$$

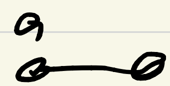
We use $C_a^{(1)}(X,t)$ to denote the generating function for closed walks that start at a and return exactly once.

constant term is zero

examples



$$C_a^{(1)}(K_1,t)=0$$



$$C_a^{(1)}(K_2,t)=t^2$$

Lemma

$$C_a(X,t) = \frac{1}{1-C_a^{(1)}(X,t)}$$

$$C_a(X,t) = \frac{t^{-1}\phi(X\setminus a,t^{-1})}{\phi(X,t^{-1})}$$

$$C_a^{(1)}(X,t) = 1-\frac{1}{C_a(X,t)}$$

example $X = K_2$, $C_a^{(1)}(K_2, t) = t^2$

$$C_a(K_2, t) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots$$

We rewrite the formula

$$\frac{\phi(T, t)}{\prod \phi(T_i, t)} = t - \sum_i \frac{\phi(T_i \setminus b_i, t)}{\phi(T_i, t)}$$

as

$$\frac{\phi(T, t^{-1})}{\phi(T \setminus a, t^{-1})} = t^{-1} - \sum_i \frac{\phi(T_i \setminus b_i, t^{-1})}{\phi(T_i, t^{-1})}$$

$$\underbrace{\frac{\phi(T, t^{-1})}{t^{-1}\phi(T \setminus a, t^{-1})} = 1}_{} - \underbrace{t^2 \sum_i \frac{t^{-1}\phi(T_i \setminus b_i, t^{-1})}{\phi(T_i, t^{-1})}}_{}$$

Lemma $C_a^{(1)}(T) = \sum_i \frac{t^2}{1-C_{b_i}^{(1)}(T)}$

Lemma $C_a^{(1)}(T,\lambda^{-1}) = 1 \iff \phi(T,\lambda)=0$

Let λ be a totally real algebraic integer and define the rational function

$$\psi(t) = \frac{1}{\lambda^2(1-t)}$$

Theorem [Salez] Let $\mathcal{F}$ be the smallest subset of $\mathbb{R}$ that contains 0 such that if $\alpha_1,\dots,\alpha_d \in \mathcal{F}\setminus\{1\}$ then $\psi(\alpha_1)+\dots+\psi(\alpha_d) \in \mathcal{F}$. Then $\mathcal{F} = \mathbb{Q}(\lambda^2)$.

You're referred to Salez (https://arxiv.org/abs/1302.4423) for the proof.

Tridiagonal matrices

Lemma Assume the polynomials f & g are monic and coprime and $\deg(g) = \deg(f) - 1$. There there are $a, b \in \mathbb{R}$ and a monic polynomial h such that

$$f(t) = (t-a)\, g(t) - b\, h(t).$$

If g interlaces f, then $b > 0$ and h interlaces g. $\square$

Recursively, we get a sequence of pair (a_i, b_i)

Define

$$T = \begin{bmatrix} a_0 & b_0 & & \\ 1 & a_1 & b_1 & \\ & & \ddots & \\ 0 & \cdots & 0 \; 1 & a_n \end{bmatrix}$$

Then

$$\det(tI-T) = (t-a_0)\,g(t) - b_0\,h(t)$$

and, if g interlaces f, then $b_0, \dots, b_{n-1} > 0$.
If f is even, $a_0 = \cdots = a_n = 0$.

Conversely if T is tridiagonal, as given, then its eigenvalues are real & interlaced.

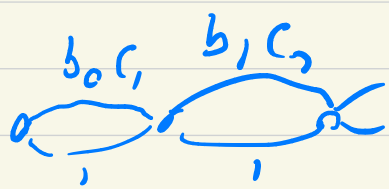
Theorem If θ is a totally real algebraic integer, then some integer multiple of θ is an eigenvalue of a tree.

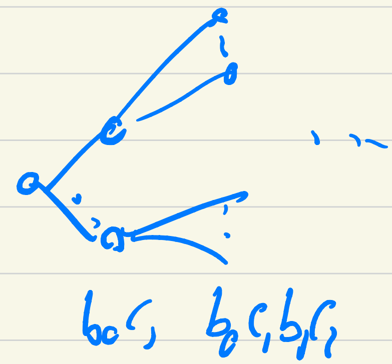
Proof Let $\psi(t)$ be the minimal polynomial of θ. If $\psi(t)$ is even, set $g=\psi$. Otherwise $\psi(-t)$ & $\psi(t)$ are coprime [Exercise] and $\psi(t)\psi(-t)$ has only simple zeros and we set $g(t)=\psi(t)\psi(-t)$. The zeros of $g'(t)$ interlace the zeros of g and we can construct a tridiagonal matrix T with zero diagonal & with θ as an eigenvalue.

The entries of T are rational, so there is an integer m such that $S=mT$ is an integer matrix.

$$D^{-1}\begin{bmatrix} 0 & b_0 & 0 \\ c_1 & 0 & b_1 \\ 0 & c_2 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & b_0c_1 & 0 \\ 1 & 0 & b_1c_2 \\ & 1 & 0 \end{bmatrix}$$

Then S is diagonally similar to an integer tridiagonal matrix R with $R_{i,i-1} = 1$ for all i.





R is a quotient of the adjacency matrix of a tree

Laplacian eigenvalues

The Laplacian $\Delta - A$ of X is positive semidefinite, so all its ei envalues are non-negative.

Lemma The eigenvalues of the Laplacian of X are totally positive algebraic integers.

Question Is the converse true?

We note some properties of Laplacian eigenvalues

(1) Any Laplacian eigenvalue is a graph eigenvalue.

(2) We have

$$L(X) \square L(Y) = L(X \square Y)$$

and thus the sum of two Laplacian eigenvalues is a Laplacian eigenvalue

(3) An eigenvalue of a graph with maximum valency k is an eigenvalue of a k-regular graph. If X is k-regular, $kI - A(X) = L(X)$ and so if θ is an eigenvalue of a graph with maximum valency k, then $k-\theta$ is a Laplacian eigenvalue.

Since $-\theta$ is an eigenvalue of $X \times K_2$, we deduce that $k+\theta$ is a Laplacian ~~eigenvalue~~ eigenvalue too.

An algebraic integer θ is a Perron value if, for each conjugate ρ of θ, we have $|\rho| < \theta$.

Theorem (Lind) Every Perron number is the spectral radius of a non-negative integer matrix.

Hence each Perron number is the spectral radius of a digraph.